Functional analysis for Helmholtz equation in the framework of domain decomposition

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Abstract

The aim of the lecture is to describe a Domain Decomposition algorithm for the Helmholtz equation.

Given a bounded open set $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ where the two open sets Ω_1 and Ω_2 are not overlapping, and Γ a common subset of their boundary (called the fictitious boundary), and given a (global) solution u of the Helmholtz equation

$$\Delta u + k^2 u = f \in L^2(\Omega) \quad \text{and} \quad u \in H_0^1(\Omega)$$

the aim is to understand the dynamics of the sequence $(v_1^n, v_2^n)_{n \in \mathbb{N}}$ solving separatly the Helmholtz equations on Ω_1 and Ω_2, when equating the fluxes through Γ: ($m = 1, 2$ resp. $m' = 2, 1$)

$$\frac{\partial v_m^n}{\partial n_m} - i\gamma v_m^n = -\frac{\partial v_{m'}^{n-1}}{\partial n_{m'}} - i\gamma v_{m'}^{n-1} \quad \text{on} \quad \Gamma$$

The ultimate aim is to prove convergence to $(u_{|\Omega_1}, u_{|\Omega_2})$ of the sequence $(u_1^n, u_2^n)_{n \in \mathbb{N}}$ solving the Helmholtz equations on Ω_1 and Ω_2 with a relaxation on the boundary Γ added, namely:

$$\frac{\partial u_m^n}{\partial n_m} - i\gamma u_m^n = \theta[\frac{\partial u_m^{n-1}}{\partial n_m} - i\gamma u_{n-1}^2] - (1-\theta)[\frac{\partial u_1^{n-1}}{\partial n_{m'}} + i\gamma u_{m'}^{n-1}] \quad \text{on} \quad \Gamma$$

For this sake, the geometry of the set of solutions of the Helmholtz equation on $\Omega_1 \times \Omega_2$ with equated energy fluxes is studied, through the study of the coupling operator defined on $L^2(\Gamma) \times L^2(\Gamma)$ which intertwins the fluxes. It turns out that the key for understanding the convergence of the sequence $(u_1^n, u_2^n)_{n \in \mathbb{N}}$ is the analysis of the spectral properties of the intertwinnig operator.

Using these tools, one can prove that the Domain Decomposition setting for the Helmholtz equation leads to an ill–posed problem. Nevertheless, one can prove that if a solution exists, it is unique, and that the algorithm do converge to the solution.

Convergence of the relaxed algorithm is proven and numerical tests for solving the Helmholtz equation through this domain decomposition algorithm are given.