Global solutions for some reaction-diffusion systems: L^∞, L^p, L^1 and L^2-strategies

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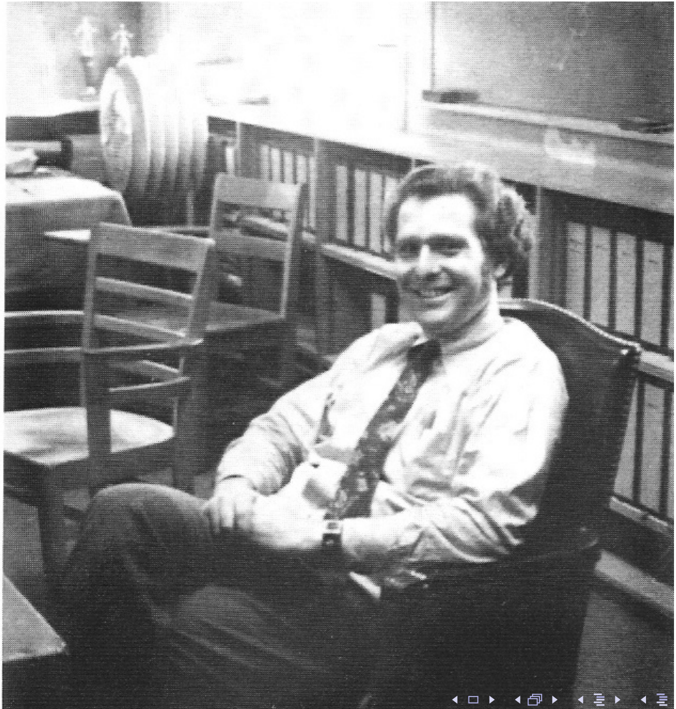

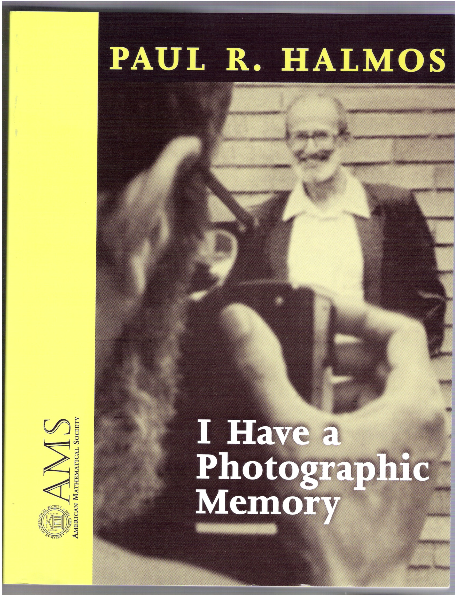
PAUL R. HALMOS
I Have a Photographic Memory
AMS
American Mathematical Society


MARINE

An easy O.D.E.

$$\begin{cases} u' = -u\, v^{\beta}, \\ v' = \quad u\, v^{\beta} \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0, \\ u_0, v_0 \textit{ given in } [0, \infty), \end{cases}$$

where $u, v : [0, T) \to \boldsymbol{R}$ are the unknown functions. Here $\beta \geq 1$. Local existence of a nonnegative unique solution on a maximal interval $[0, T^*)$ is well-known due to the C^1-property of $(u, v) \to uv^{\beta}$. Moreover $u \geq 0, v \geq 0$ and

$$(u+v)'(t) = 0 \Rightarrow (u+v)(t) = u_0 + v_0,$$

so that: $\sup_{t \in [0, T^*)} |u(t)| + |v(t)| < +\infty$,

and therefore

$$T^* = +\infty$$

What happens when diffusion is added?

$$\begin{cases} \partial_t u - d_1 \Delta u = -uv^\beta \ in \ Q_T = (0,T) \times \Omega \\ \partial_t v - d_2 \Delta v = \ \ uv^\beta \ in \ Q_T = (0,T) \times \Omega \\ \partial_\nu u = \partial_\nu v = 0 \ on \ \Sigma_T = (0,T) \times \partial\Omega, \\ u(0) = u_0 \geq 0, \ \ v(0) = v_0 \geq 0. \end{cases}$$

Here $\Omega \subset \boldsymbol{R}^N$, regular. The total mass is preserved:

$$\int_\Omega \partial_t (u+v) - \int_\Omega \Delta(d_1 u + d_2 v) = 0.$$

$$\partial_\nu (d_1 u + d_2 v) = 0 \ on \ \partial\Omega \Rightarrow \int_\Omega \Delta(d_1 u + d_2 v) = 0.$$

$$\int_\Omega (u+v)(t) = \int_\Omega u_0 + v_0$$

Insufficient for global existence!

Local existence for reaction-diffusion systems with L^∞-data

$$(S)\begin{cases} \partial_t u - d_1 \Delta u = -uv^\beta \text{ on } Q_T \\ \partial_t v - d_2 \Delta v = uv^\beta \text{ on } Q_T \\ \partial_\nu u = \partial_\nu v = 0 \text{ on } \Sigma_T, \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0. \end{cases}$$

Theorem: Let $u_0, v_0 \in L^\infty(\Omega), u_0 \geq 0, v_0 \geq 0$. Then, there exist a maximum time $T^* > 0$ and (u, v) unique classical nonnegative solution of (S) on $[0, T^*[$. Moreover,

$$\sup_{t\in[0,T^*[} \left\{\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)}\right\} < +\infty \Rightarrow [T^* + \infty].$$

Remark: here $\int_\Omega (u+v)(t) = \int_\Omega u_0 + v_0$, that is

$$\sup_{t\in[0,T^*[} \left\{\|u(t)\|_{L^1(\Omega)} + \|v(t)\|_{L^1(\Omega)}\right\} \leq \|u_0\|_{L^1(\Omega)} + \|v_0\|_{L^1(\Omega)}.$$

How does this estimate help for global existence? Very frequent situation in applications !

Same question for the general family of systems:

$$\begin{cases} \forall i = 1, ..., m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_T \\ \partial_\nu u_i = 0 & \text{on } \Sigma_T \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0. \end{cases}$$

$d_i > 0,\ f_i : [0, \infty)^m \to \boldsymbol{R}$ of class C^1 where

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- **(P):** Positivity (nonnegativity) is preserved
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- **(M')** $\forall r \in [0, \infty[^m,\ \sum_{1 \leq i \leq m} a_i f_i(r) \leq C[1 + \sum_{1 \leq i \leq m} r_i]$ for some $a_i > 0$

$$
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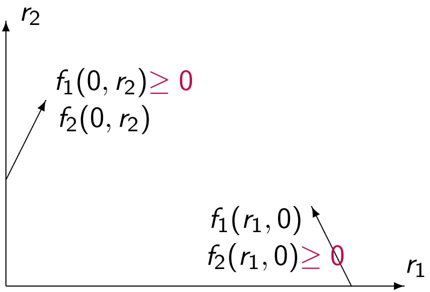
- **(P) Preservation of Positivity:** $\forall i = 1, ..., m$
  $\forall r = (r_1, ..., r_m) \in [0, \infty[^m,\ f_i(r_1, ..., r_{i-1}, 0, r_{i+1}, ..., r_m) \geq 0.$



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- **(M):** $\sum_{1 \leq i \leq m} f_i(r_1, ..., r_m) \leq 0 \Rightarrow$ **'Control of the Total Mass':**

$$\forall t \geq 0, \quad \int_\Omega \sum_{1 \leq i \leq r} u_i(t, x) dx \leq \int_\Omega \sum_{1 \leq i \leq r} u_i^0(x) dx.$$

Add up, integrate on Ω, use $\int_\Omega \Delta u_i = \int_{\partial\Omega} \partial_\nu u_i = 0$:

$$\int_\Omega \partial_t [\sum u_i(t)] dx = \int_\Omega \sum_i f_i(u) dx \leq 0.$$

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- Remark: same with **(M')**

QUESTION:

What about Global Existence of solutions

under assumption **(P)+(M)***??*

or more generally **(P)+ (M')** *??*

Explicit examples with property **(P)+(M) ou (M')**

"Chemical morphogenetic process ("Brusselator", Prigogine)

$$\begin{cases} \partial_t u - d_1 \Delta u = -uv^2 + b\,v \\ \partial_t v - d_2 \Delta v = uv^2 - (b+1)\,v + a \\ u_{|\partial\Omega} = b/a, \;\; v_{|\partial\Omega} = a, \\ a, b, d_1, d_2 > 0. \end{cases}$$

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See also: Glycolosis model–Gray-Scott models

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- **Exothermic combustion in a gas**

$$\begin{cases} \partial_t Y - \mu \Delta Y = -H(Y,T) \\ \partial_t T - \lambda \Delta T = q\, H(Y,T), \end{cases}$$

$Y = \text{concentration of a reactant}, \; T = \text{temperature},$

Explicit examples with **(P)+(M)**

- **Lotka-Volterra Systems (Leung,...)**

$$\forall i = 1...m, \;\; \partial_t u_i - d_i \Delta u_i = e_i u_i + u_i \sum_{1\leq j\leq m} p_{ij} u_j,$$

with $e_i, p_{ij} \in \boldsymbol{R}$ such that for some $a_i > 0$.

$$\forall w \in R^m, \;\; \sum_{i,j=1}^{m} a_i w_i p_{ij} w_j \leq 0,$$

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- **Model of diffusive calcium dynamics: H.G. Othmer**

$$\begin{cases} \partial_t u_1 = d_1 \Delta u_1 + \lambda(\gamma_0 + \gamma_1 u_4)(1 - u_1) - \frac{p_1 u_1^4}{p_2^4 + u_1^4} \\ \partial_t u_2 - d_2 \Delta u_2 = -k_1 u_2 + k_1' u_3 \\ \partial_t u_3 - d_3 \Delta u_3 = -k_1' u_3 - k_2 u_1 u_3 + k_1 u_2 + k_2' u_4 \\ \partial_t u_4 - d_4 \Delta u_4 = k_2 u_1 u_3 + k_3' u_5 - k_2' u_4 - k_3 u_1 u_4 \\ \partial_t u_5 - d_5 \Delta u_5 = k_3 u_1 u_4 - k_3' u_5. \end{cases}$$

Elementary chemical reactions: a simple example

- $$U_1 + U_2 \underset{k^-}{\overset{k^+}{\rightleftharpoons}} U_3$$

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- Fick's diffusion law: $\mathbf{V_1} = -d_1 \nabla u_1 \Rightarrow \nabla(u_1 \mathbf{V_1}) = -d_1 \Delta u_1$

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- $$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -k^+ u_1 u_2 + k^- u_3 \\ \partial_t u_2 - d_2 \Delta u_2 = -k^+ u_1 u_2 + k^- u_3 \\ \partial_t u_3 - d_3 \Delta u_3 = \;\; k^+ u_1 u_2 - k^- u_3 \end{cases}$$

Note :$f_1 + f_2 + 2f_3 = 0$ and positivity is preserved.

A quadratic model

$$U_1 + U_2 \underset{k^-}{\overset{k^+}{\rightleftharpoons}} U_3 + U_4$$

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -k^+ u_1 u_2 + k^- u_3 u_4 \\ \partial_t u_2 - d_2 \Delta u_2 = -k^+ u_1 u_2 + k^- u_3 u_4 \\ \partial_t u_3 - d_3 \Delta u_3 = k^+ u_1 u_2 - k^- u_3 u_4 \\ \partial_t u_4 - d_4 \Delta u_4 = k^+ u_1 u_2 - k^- u_3 u_4 \end{cases}$$

Note: $f_1 + f_2 + f_3 + f_4 = 0$ and positivity is preserved.

L. Desvillettes, K. Fellner, M.P., J. Vovelle, Th. Goudon, A. Vasseur

Superquadratic reaction-diffusion systems.

- A general chemical reaction:

$$p_1 U_1 + p_2 U_2 + \dots + p_m U_m \underset{k^-}{\overset{k^+}{\rightleftharpoons}} q_1 U_1 + q_2 U_2 + \dots + q_m U_m,$$

p_i, q_i =nonnegative integers.

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$$\partial_t u_i - d_i \Delta u_i = (p_i - q_i)\left(k^- \Pi_{j=1}^m u_j^{q_j} - k^+ \Pi_{j=1}^m u_j^{p_j}\right), \forall i = 1...m.$$

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- Here $\sum_i m_i p_i = \sum_i m_i q_i$ for some $m_i \in (0, \infty), i = 1 \ldots m$. This implies **(M')**: $\sum_{i=1}^m m_i f_i = 0$.

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- Global existence completely open in this general situation!

More examples

- **Diffusive epidemic models: SIR**
  S=Susceptibles= can be infected
  I=Infectives=infected and transmit disease
  R=Removed=immune; $P = S + I + R$

$$\begin{cases} S_t - \nabla \cdot d_1(x)\nabla S = bP - (m + kP)S - g(S, I) \\ I_t - \nabla \cdot d_2(x)\nabla I = -(m + kP)I + g(S, I) - \lambda I \\ R_t - \nabla \cdot d_3(x)\nabla R = -(m + kP)R + \lambda I \end{cases}$$

May be coupled with an extra variable: $S = S(t, x, age)$....

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  May be coupled with an extra variable: $S = S(t, x, age)$....

- **Nickel-Iron alloy electrodeposition : N. Alaa, A. Tounsi et al.** : $\forall i = 1...5$

$$\begin{cases} \partial_t w_i - d_i(w_i)_{xx} + b(x)(w_i)_x - m_i[w_i\Phi_x]_x = S_i(w) \\ S_1 = S_2 = 0, \ S_3(w) = S_4(w) = -S_5(w) \\ -[\Phi]_{xx} = \sum_{i=1}^{5} z_i w_i, \quad z_i \in R, \ +\text{bdy cond.} \end{cases}$$

Explicit examples

- **Modelization of pollutants transfer in atmospher ($N = 3$): R. Texier-Picard, MP,(degenerate diffusion)**:

$$\left\{ \begin{array}{l} \partial_t \phi_i = d_i \, \partial^2_{zz} \phi_i + \omega \cdot \nabla \phi_i + f_i(\phi) + g_i, \ \forall i = 1...20, \\ + \text{ Bdy and initial conditions} \end{array} \right.$$

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- **The reaction terms:**

$$\begin{cases}
f_1(\phi) &=& -k_1\phi_1 + k_{22}\phi_{19} + k_{25}\phi_{20} + k_{11}\phi_{13} + k_9\phi_{11}\phi_2 + k_3\phi_5\phi_2 \\
&+& k_2\phi_2\phi_4 - k_{23}\phi_1\phi_4 - k_{14}\phi_1\phi_6 + k_{12}\phi_{10}\phi_2 - k_{10}\phi_{11}\phi_1 - k_{24}\phi_{19}\phi_1, \\
f_2(\phi) &=& k_1\phi_1 + k_{21}\phi_{19} - k_9\phi_{11}\phi_2 - k_3\phi_5\phi_2 - k_2\phi_2\phi_4 - k_{12}\phi_{10}\phi_2 \\
f_3(\phi) &=& k_1\phi_1 + k_{17}\phi_4 + k_{19}\phi_{16} + k_{22}\phi_{19} - k_{15}\phi_3 \\
f_4(\phi) &=& -k_{17}\phi_4 + k_{15}\phi_3 - k_{16}\phi_4 - k_2\phi_2\phi_4 - k_{23}\phi_1\phi_4 \\
f_5(\phi) &=& 2k_4\phi_7 + k_7\phi_9 + k_{13}\phi_{14} + k_6\phi_7\phi_6 - k_3\phi_5\phi_2 + k_{20}\phi_{17}\phi_6 \\
f_6(\phi) &=& 2k_{18}\phi_{16} - k_8\phi_9\phi_6 - k_6\phi_7\phi_6 + k_3\phi_5\phi_2 - k_{20}\phi_{17}\phi_6 - k_{14}\phi_1\phi_6 \\
f_7(\phi) &=& -k_4\phi_7 - k_5\phi_7 + k_{13}\phi_{14} - k_6\phi_7\phi_6 \\
f_8(\phi) &=& k_4\phi_7 + k_5\phi_7 + k_7\phi_9 + k_6\phi_7\phi_6 \\
f_9(\phi) &=& -k_7\phi_9 - k_8\phi_9\phi_6 \\
f_{10}(\phi) &=& k_7\phi_9 + k_9\phi_{11}\phi_2 - k_{12}\phi_{10}\phi_2 \\
f_{11}(\phi) &=& k_{11}\phi_{13} - k_9\phi_{11}\phi_2 + k_8\phi_9\phi_6 - k_{10}\phi_{11}\phi_1 \\
f_{12}(\phi) &=& k_9\phi_{11}\phi_2 \\
f_{13}(\phi) &=& -k_{11}\phi_{13} + k_{10}\phi_{11}\phi_1 \\
f_{14}(\phi) &=& -k_{13}\phi_{14} + k_{12}\phi_{10}\phi_2 \\
f_{15}(\phi) &=& k_{14}\phi_1\phi_6 \\
f_{16}(\phi) &=& -k_{19}\phi_{16} - k_{18}\phi_{16} + k_{16}\phi_4 \\
f_{17}(\phi) &=& -k_{20}\phi_{17}\phi_6 \\
f_{18}(\phi) &=& k_{20}\phi_{17}\phi_6 \\
f_{19}(\phi) &=& -k_{21}\phi_{19} - k_{22}\phi_{19} + k_{25}\phi_{20} + k_{23}\phi_1\phi_4 - k_{24}\phi_{19}\phi_1 \\
f_{20}(\phi) &=& -k_{25}\phi_{20} + k_{24}\phi_{19}\phi_1.
\end{cases}$$

Back to the model example: what about L^∞-estimates?

- $$(S)\begin{cases} \partial_t u - d_1\Delta u = -uv^\beta \text{ on } Q_T \\ \partial_t v - d_2\Delta v = uv^\beta \text{ on } Q_T \\ \partial_\nu u = \partial_\nu v = 0 \text{ on } \Sigma_T, \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0. \end{cases}$$

  By maximum principle

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  - What happens when $d_1 \neq d_2$?

A general L^p-approach

- $$(S)\begin{cases} \partial_t u - d_1\Delta u = -uv^\beta \text{ on } Q_T \\ \partial_t v - d_2\Delta v = uv^\beta \text{ on } Q_T \\ \partial_\nu u = \partial_\nu v = 0 \text{ on } \Sigma_T, \\ u(0) = u_0 \geq 0, \;\; v(0) = v_0 \geq 0. \end{cases}$$

$$\partial_t v - d_2\Delta v = -[\partial_t u - d_1\Delta u], \;\; u \in L^\infty(Q_{T^*}).$$

$$v = -[\partial_t - d_2\Delta]^{-1}(\partial_t - d_1\Delta)\,u\,(= \mathcal{A}u).$$

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- **Lemma:** the operator $\mathcal{A}$ is continuous from $L^p(Q_T)$ into $L^p(Q_T)$ for all $p \in]1, \infty[$ and all $T > 0$.
  [dual statement of the L^p-regularity theory for the heat operator]

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- Next : $\|v\|_{L^\infty(Q_{T^*})} \leq C \|uv^\beta\|_{L^q(Q_{T^*})}$ *si* $q > (d+1)/2$
- Therefore $\|v\|_{L^\infty(Q_{T^*})} < +\infty$ *et* $T^* = +\infty$.

Extensions and limits of the L^p-approach

- The same approach provides global existence
  - for the "Brusselator", for the epidemic models SIR
  - for the 3×3 system

$$U_1 + U_2 \underset{k^-}{\overset{k^+}{\rightleftharpoons}} U_3 \; : \left\{ \begin{array}{l} \partial_t u_1 - d_1 \Delta u_1 = -k^+ u_1 u_2 + k^- u_3 \\ \partial_t u_2 - d_2 \Delta u_2 = -k^+ u_1 u_2 + k^- u_3 \\ \partial_t u_3 - d_3 \Delta u_3 = \;\; k^+ u_1 u_2 - k^- u_3 \end{array} \right.$$

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- More generally it applies to $m \times m$ systems if there exists a triangular invertible matrix Q with nonnegative entries such that

$$\forall r \in [0, \infty)^m, \ Q\, f(r) \leq [1 + \sum_{1 \leq i \leq m} r_i]\mathbf{b},$$

for some $\mathbf{b} \in \boldsymbol{R}^m, f = (f_1, ..., f_m)^t$.

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- and even not to the "better" system with $\lambda \in [0, 1[$

$$\begin{cases} \partial_t u - d_1 \Delta u = \lambda u^3 v^2 - u^2 v^3 \ in \ Q_T, \\ \partial_t v - d_2 \Delta v = u^2 v^3 - u^3 v^2 \ in \ Q_T \end{cases}$$

where $f(u, v) + g(u, v) \leq 0$ et also $f(u, v) + \lambda g(u, v) \leq 0$

Finite time L^∞-blow up may appear!

- $$\begin{cases} \partial_t u - d_1 \Delta u = f(u,v) \text{ in } Q_T, \\ \partial_t v - d_2 \Delta v = g(u,v) \text{ in } Q_T \end{cases}$$

  Theorem: (D. Schmitt, MP) One can find polynomial nonlinearities f, g satisfying **(P)** and

  $$\textbf{(M)}\ f + g \leq 0, \ \textit{and also} : \exists \lambda \in [0,1[, f + \lambda g \leq 0,$$

  and for which there exists $T^* < +\infty$ with

  $$\lim_{t \to T^*} \|u(t)\|_{L^\infty(\Omega)} = \lim_{t \to T^*} \|v(t)\|_{L^\infty(\Omega)} = +\infty.$$

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- The blow up is similar to $u(t,x) = \frac{1}{(T^* - t)^2 + |x|^2}$ The solution goes out of $L^\infty(\Omega)$ at $t = T^*$, but still exists for $t > T^*$.$---->$ Incomplete blow up !

CONCLUSION at this stage:

Look rather for *weak solutions* which are allowed to go out of $L^\infty(\Omega)$ from time to time or even often.

We ask the nonlinearities to be at least in $L^1(Q_T)$.

$$f_i(u) \in L^1(Q_T) \;\; ?$$

An L^1-approach

$$
(S)\begin{cases}
\forall i = 1, ..., m \\
\partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \textit{in } Q_T \\
\partial_\nu u_i = 0 & \textit{on } \Sigma_T \\
u_i(0, \cdot) = u_i^0(\cdot) \geq 0.
\end{cases}
$$

L^1-**Theorem.** Assume the two conditions **(P)+ (M')** hold. Assume moreover that the following a priori estimate holds:

$$
\forall i = 1, ..., m, \int_{Q_T} |f_i(u)| \leq C.
$$

Then, there exists a global weak solution for System (S).

L^1-Theorem applies to many situations

$$\begin{cases} \partial_t u - d_1 \Delta u = -u e^{v^2} \ in \ Q_T \\ \partial_t v - d_2 \Delta v = u e^{v^2} \ in \ Q_T \end{cases}$$

$$\int_\Omega u(T) + \int_{Q_T} u e^{v^2} = \int_\Omega u_0,$$

whence the $L^1(Q_T)$-estimate of the nonlinearity.

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- $$\int_\Omega u(T) + \int_{Q_T} u^2 v^3 = \lambda \int_{Q_T} u^3 v^2 + \int_\Omega u_0.$$
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- $$For\ \lambda < 1\ :\Rightarrow \int_{Q_T} u^3 v^2 < +\infty, \int_{Q_T} u^2 v^3 < +\infty$$

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- Open problem if $\lambda = 1$: L^1-estimate of the nonlinearity??

L^1-Theorem applies to many situations

More generally it applies if there exists an invertible matrix Q with nonnegative entries such that

$$\forall r \in [0,\infty)^m,\ Q\,f(r) \le [1 + \sum_{1\le i\le m} r_i]\mathbf{b},$$

for some $\mathbf{b} \in \boldsymbol{R}^m$, $f = (f_1, ..., f_m)^t$.

A surprising a priori L^2-estimate for these systems

- $$(S)\begin{cases} \forall i = 1, ..., m & \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_T \\ \partial_\nu u_i = 0 & \text{on } \Sigma_T \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0. & \end{cases}$$

L^2-**Theorem.** Assume **(P)+(M')**. Then, the following a priori estimate holds for the solutions of (S):

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- **Corollary of the L^1- and L^2-Theorems**: Assume **(P),(M')** and f_i is at most quadratic. Then, System (S) has a global weak solution.
- Recall that nonlinearities are quadratic in many examples.

Application to the quadratic chemical reaction:

- $$U_1 + U_2 \underset{k^-}{\overset{k^+}{\rightleftharpoons}} U_3 + U_4$$

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -k^+ u_1 u_2 + k^- u_3 u_4 \\ \partial_t u_2 - d_2 \Delta u_2 = -k^+ u_1 u_2 + k^- u_3 u_4 \\ \partial_t u_3 - d_3 \Delta u_3 = \;\; k^+ u_1 u_2 - k^- u_3 u_4 \\ \partial_t u_4 - d_4 \Delta u_4 = \;\; k^+ u_1 u_2 - k^- u_3 u_4 \end{cases}$$

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- Open problem: does the solution blow up in $L^\infty(\Omega)$ in finite time or not??

Idea of the proof of the L^2-estimate

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- $$0 \leq \min_i d_i \leq a = \frac{\sum_i d_i u_i}{\sum_i u_i} \leq \max_i d_i < +\infty$$
- The operator $W \to \partial_t W - \Delta(aW)$ is not of divergence form and a is not continuous, but bounded from above and from below so that the operator is parabolic and, at least:
$$\|W\|_{L^2(Q_T)} \leq C \|W_0\|_{L^2(\Omega)}.$$

Idea of the proof of the L^2-estimate

- $$\partial_t \left(\sum_i u_i \right) - \Delta \left(\sum_i d_i u_i \right) \le 0.$$
- $$\partial_t W - \Delta (a\, W) \le 0, \quad W = \sum_i u_i \quad a = \frac{\sum_i d_i u_i}{\sum_i u_i}$$
- $$0 \le \min_i d_i \le a = \frac{\sum_i d_i u_i}{\sum_i u_i} \le \max_i d_i < +\infty$$
- The operator $W \to \partial_t W - \Delta(aW)$ is not of divergence form and a is not continuous, but bounded from above and from below so that the operator is parabolic and, at least:
$$\|W\|_{L^2(Q_T)} \le C \|W_0\|_{L^2(\Omega)}.$$
- We may even show that the mapping
$W_0 \in L^2(\Omega) \to W \in L^2(Q_T)$ is compact
where $\partial_t W - \Delta(aW) = 0$, $W(0) = W_0$.

An application of the L^2-compactness

- $$U_1 + U_2 \underset{k_1}{\overset{1}{\rightleftharpoons}} C \underset{1}{\overset{k_2}{\rightleftharpoons}} U_3 + U_4$$

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$$\left\{ \begin{array}{l} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2 + k_1 c \\ \partial_t u_2 - d_2 \Delta u_2 = -u_1 u_2 + k_1 c \\ \partial_t c - d_c \Delta c = u_1 u_2 - (k_1 + k_2) c + u_3 u_4 \\ \partial_t u_3 - d_3 \Delta u_3 = -u_3 u_4 + k_2 c \\ \partial_t u_4 - d_4 \Delta u_4 = -u_3 u_4 + k_2 c, \end{array} \right\} \text{ on } Q_T$$

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- The L^p-approach applies to this system so that global existence of classical solutions holds!

The limit system

Theorem. The solution $(u_1^k, u_2^k, c^k, u_3^k, u_4^k)$, $k = (k_1, k_2)$ of the previous system converges as $k_1 + k_2 \to +\infty$ in $L^2(Q_T)^5$ for all $T > 0$ to $(u_1, u_2, 0, u_3, u_4)$ solution of

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -\alpha u_1 u_2 + \beta u_3 u_4 \\ \partial_t u_2 - d_2 \Delta u_2 = -\alpha u_1 u_2 + \beta u_3 u_4 \\ \partial_t u_3 - d_3 \Delta u_3 = \alpha u_1 u_2 - \beta u_3 u_4 \\ \partial_t u_4 - d_4 \Delta u_4 = \alpha u_1 u_2 - \beta u_3 u_4, \end{cases}$$

where $\alpha = \lim_{k_1+k_2\to\infty} \frac{k_2}{k_1+k_2}, \beta = 1 - \alpha$. The chemical reaction

$$U_1 + U_2 \ \underset{k_1}{\overset{1}{\rightleftharpoons}} \ C \ \underset{1}{\overset{k_2}{\rightleftharpoons}} \ U_3 + U_4$$

"tends" to the limit chemical reaction:

$$U_1 + U_2 \ \underset{1-\alpha}{\overset{\alpha}{\rightleftharpoons}} \ U_3 + U_4$$

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- Work in progress with Jerry, Gisele + M. Meyries (Karlsruhe) on nonlinear boundary conditions, including Wentzell type, like

$$\left\{ \begin{array}{l} \partial_t u_i - d_i \Delta u_i = f_i(u) \text{ in } Q_T \\ \sigma \partial_t u_i + d_i \partial_\nu u_i - \delta_i \Delta_{\partial\Omega} u_i = g_i(u) \text{ on } \Sigma_T \end{array} \right.$$

with $\sigma, \delta_i \geq 0$.

More open problems

- atmospheric vertical diffusion/transport of pollutants (partial results with assumptions on the transport).
- Uniqueness of weak solutions...or more precisely, what is the way to select the "best" solution, since for instance there is even not uniqueness for $\partial_t u - \Delta u = u^3$.
- how to define renormalized solutions for systems???
- what about initial data in $L^1(\Omega)$?
- what about nonlinear diffusions??

$$(E)\begin{cases} \partial_t u - \Delta u^m = f(u,v) & \textit{on } Q_T \\ \partial_t v - \Delta v^p = g(u,v) & \textit{on } Q_T, \end{cases}$$

- Same questions with general Lyapounov type of structure

$$h_1'(u)f(u,v) + h_2'(v)g(u,v) \leq 0,\ h_1, h_2 \textit{ convex}.$$

- same questions for elliptic systems.