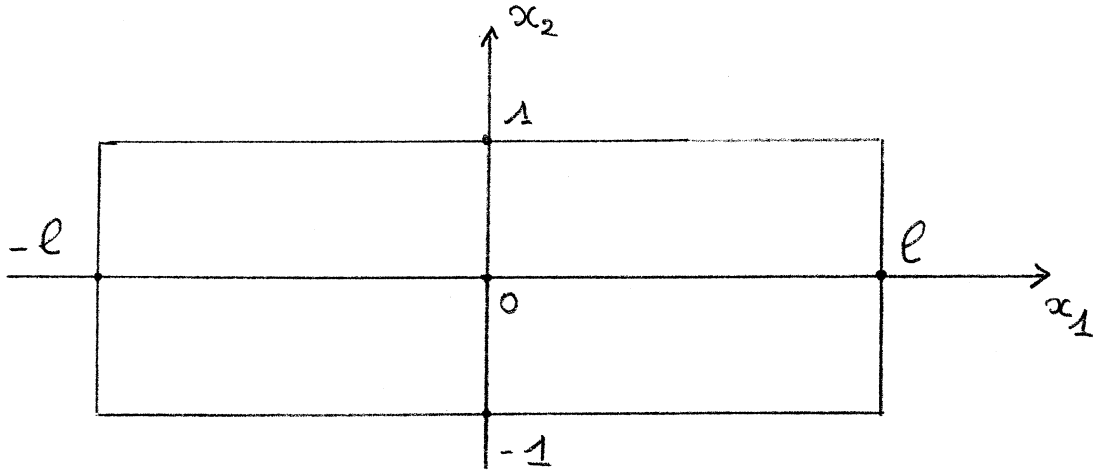
Joint work with K. Yeressian

1. Introduction



$$\Omega_\ell = (-\ell, \ell) \times (-1, 1)$$
$$= \ell(-1,1) \times (-1,1)$$

$$f = f(x_2) \in L^2(-1,1)$$

$H_0^1(O)$ the usual Sobolev space

$\omega = (-1,1)$, $\ell\omega = (-\ell, \ell)$

$$K_\ell = \{ v \in H_0^1(\Omega_\ell) \mid v \geq 0 \text{ a.e. in } \Omega_\ell \}.$$

$$\boxed{\begin{cases} u_\ell \in K_\ell, \\ \displaystyle\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla (v - u_\ell)\, dx \geq \int_{\Omega_\ell} f\,(v - u_\ell) \quad \forall v \in K_\ell . \end{cases} \quad (\ell)}$$

$$u_\ell = u_\ell(x_1, x_2)$$

$$u_\ell \to \ ? \qquad \text{when } \ell \to +\infty$$

$$K_\infty = \{ v \in H_0^1(\omega) \mid v \geq 0 \text{ a.e. in } (-1,1) \}.$$

$$\boxed{\begin{cases} u_\infty \in K_\infty, \\ \displaystyle\int_{\omega} \partial_{x_2} u_\infty \cdot \partial_{x_2} (v - u_\infty)\, dx_2 \geq \int_{\omega} f\,(v - u_\infty)\, dx_2 \quad \forall v \in K_\infty \end{cases} \quad (\infty)}$$

$$u_\infty = u_\infty(x_2)$$

$$u_\ell \to u_\infty \quad \text{when } \ell \to +\infty$$

with an <u>exponential rate of convergence</u>.

2. A first convergence result

Theorem 2.1: $\exists$ C, α positive constants independent of ℓ such that

$$\int_{\Omega_{\ell/2}} |\nabla(u_\ell - u_\infty)|^2 dx \leq C e^{-\alpha\ell} |f|^2_{2,\omega}$$

($\omega = (-1,1)$, $|\ \ |_{2,\omega}$ is the usual $L^2(\omega)$-norm).

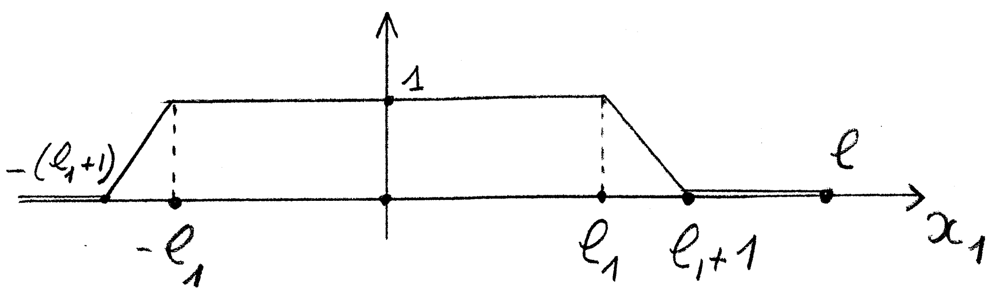
Remark: For ℓ_0 fixed, $\ell_0 \leq \ell/2$

$$|u_\ell - u_\infty|_{H^1(\Omega_{\ell_0})} \longrightarrow 0$$

with an exponential rate of convergence.

Proof: Fix $\ell_1 \leq \ell - 1$.

Choose $\rho = \rho(x_1)$ as follows



$$0 \le \rho \le 1 , \quad \rho = 1 \text{ on } \ell_1 \omega , \quad \rho = 0 \text{ on } \mathbb{R} \setminus (\ell_1 + 1)\omega$$

$$| \partial_{x_1} \rho | \le 1$$

Then

$$u_\ell - \rho^2 (u_\ell - u_\infty) \in K_\ell .$$

From

$$\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla (v - u_\ell) \, dx \ge \int_{\Omega_\ell} f \, (v - u_\ell) \, dx$$

$\Rightarrow$

$$\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla \{ -\rho^2 (u_\ell - u_\infty) \} \, dx \ge \int_{\Omega_\ell} f \{ -\rho^2 (u_\ell - u_\infty) \} \, dx$$

$$u_\infty + \rho^2 (u_\ell (x_1, \cdot) - u_\infty) \in K_\infty \quad \text{a.e. } x_1 \in \ell \omega$$

From

$$\int_\omega \partial_{x_2} u_\infty \cdot \partial_{x_2} (v - u_\infty) \, dx_2 \ge \int_\omega f \, (v - u_\infty) \, dx_2$$

$\Rightarrow$

$$\int_\omega \partial_{x_2} u_\infty \, \partial_{x_2} \{ \rho^2 (u_\ell (x_1, \cdot) - u_\infty) \} \, dx_2$$

$$\ge \int_\omega f \{ \rho^2 (u_\ell (x_1, \cdot) - u_\infty) \} \, dx_2$$

$$\partial_{x_2} u_\infty \cdot \partial_{x_2} w = \nabla u_\infty \cdot \nabla w$$

$$\int_{\Omega_\ell} \nabla u_\infty \cdot \nabla \{\rho^2 (u_\ell - u_\infty)\}\, dx \geq \int_{\Omega_\ell} f \{\rho^2 (u_\ell - u_\infty)\} dx .$$

Summing up

$$\int_{\Omega_\ell} \nabla (u_\ell - u_\infty) \nabla \{\rho^2 (u_\ell - u_\infty)\}\, dx \leq 0$$

$$\int_{\Omega_\ell} |\nabla (u_\ell - u_\infty)|^2 \rho^2 \, dx \leq -2 \int_{\Omega_\ell} \partial_{x_1} (u_\ell - u_\infty)\, \rho\, \partial_{x_1} \rho\, (u_\ell - u_\infty)\, dx$$

$$\leq 2 \left\{ \int_{\Omega_\ell} \partial_{x_1} (u_\ell - u_\infty)^2 \rho^2 \, dx \right\}^{\frac{1}{2}} \left\{ \int_{\Omega_\ell} (\partial_{x_1} \rho)^2 (u_\ell - u_\infty)^2 \, dx \right\}^{\frac{1}{2}}$$

$$\int_{\Omega_\ell} |\nabla (u_\ell - u_\infty)|^2 \rho^2 \, dx \leq 4 \int_{\Omega_\ell} (\partial_{x_1} \rho)^2 (u_\ell - u_\infty)^2 \, dx$$

$\Rightarrow$

$$\int_{\Omega_{\ell_1}} |\nabla (u_\ell - u_\infty)|^2 \, dx \leq 4 \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} (u_\ell - u_\infty)^2 \, dx .$$

Poincaré on the section

$$\int_\omega (u_\ell(x_1, \cdot) - u_\infty)^2 dx_2 \leq C_p^2 \int_\omega \partial_{x_2} (u_\ell - u_\infty)^2 \, dx_2$$

$$\Rightarrow \quad \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} (u_\ell - u_\infty)^2 dx \leq C_p^2 \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} |\nabla (u_\ell - u_\infty)|^2 dx$$

$$\int_{\Omega_{\ell_1}} |\nabla (u_\ell - u_\infty)|^2 dx \leq 4 C_p^2 \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} |\nabla (u_\ell - u_\infty)|^2 dx$$

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 dx \le 4C_p^2 \int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_\infty)|^2 dx - 4C_p^2 \int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 dx$$

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 dx \le a \int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_\infty)|^2 dx$$

$$a = 4C_p^2/(1+4C_p^2) < 1.$$

Starting from $\ell_1 = \frac{\ell}{2}$ one iterates $\left[\frac{\ell}{2}\right]$ times

$$\int_{\Omega_{\ell/2}} |\nabla(u_\ell - u_\infty)|^2 dx \le a^{\left[\frac{\ell}{2}\right]} \int_{\Omega_{\frac{\ell}{2}+\left[\frac{\ell}{2}\right]}} |\nabla(u_\ell - u_\infty)|^2 dx$$

$$\frac{\ell}{2} - 1 < \left[\frac{\ell}{2}\right] \le \frac{\ell}{2}$$

$$\int_{\Omega_{\ell/2}} |\nabla(u_\ell - u_\infty)|^2 dx \le a^{\frac{\ell}{2}-1} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx$$

$$= \frac{1}{a} e^{-\frac{\ell}{2}\ln\frac{1}{a}} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx$$

$$= \frac{1}{a} e^{-\beta\ell} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx$$

with $\beta = \frac{1}{2}\ln\frac{1}{a} > 0$.

- <u>Estimate of u_ℓ, u_∞</u>

$$0 \in K_\ell, K_\infty$$

$$\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla (v - u_\ell)\, dx \geq \int_{\Omega_\ell} f (v - u_\ell)\, dx$$

$$\int_{\omega} \partial_{x_2} u_\infty\, \partial_{x_2} (v - u_\infty)\, dx_2 \geq \int_{\omega} f (v - u_\infty)\, dx_2$$

$$\int_{\Omega_\ell} |\nabla u_\ell|^2\, dx \leq \int_{\Omega_\ell} f \cdot u_\ell\, dx$$

$\Rightarrow$

$$\int_{\Omega_\ell} |\nabla u_\ell|^2 dx \leq \left\{ \int_{\Omega_\ell} f^2 dx \right\}^{1/2} \left\{ \int_{\Omega_\ell} u_\ell^2 dx \right\}^{1/2}$$

$$\leq C_p \left\{ \int_{\Omega_\ell} f^2 dx \right\}^{1/2} \left\{ \int_{\Omega_\ell} |\nabla u_\ell|^2 dx \right\}^{1/2}$$

$$\int_{\Omega_\ell} |\nabla u_\ell|^2 dx \leq C_p^2 \int_{\Omega_\ell} f^2 dx = C_p^2 \int_{(-\ell,\ell)} \int_{\omega} f^2 dx$$

$$= 2\ell C_p^2 |f|_{2,\omega}^2 .$$

Similarly

$$\int_{\omega} (\partial_{x_2} u_\infty)^2 dx_2 \leq C_p^2 |f|_{2,\omega}^2$$

$$\int_{\Omega_\ell} |\nabla u_\infty|^2 dx = \int_{(-\ell,\ell)} \int_{\omega} (\partial_{x_2} u_\infty)^2 dx \leq 2\ell C_p^2 |f|_{2,\omega}^2 .$$

$$\Rightarrow \int_{\Omega_{\ell/2}} |\nabla (u_\ell - u_\infty)|^2 dx \leq \frac{1}{a} e^{-\beta \ell} \int_{\Omega_\ell} |\nabla (u_\ell - u_\infty)|^2 dx$$

$$\leq \frac{2}{a} e^{-\beta \ell} \left\{ \int_{\Omega_\ell} |\nabla u_\ell|^2 dx + \int_{\Omega_\ell} |\nabla u_\infty|^2 dx \right\} \leq \frac{8 C_p^2}{a} e^{-\beta \ell} \ell\, |f|_{2,\omega}^2$$

3. A more general setting

$$x \in \mathbb{R}^n \quad , \quad x = (X_1, X_2)$$

$$X_1 = (x_1, \dots, x_p) \in \mathbb{R}^p \quad , \quad X_2 = (x_{p+1}, \dots, x_n) \in \mathbb{R}^{n-p}$$

$$\nabla_{X_1} = (\partial x_1, \dots, \partial x_p) \quad , \quad \nabla_{X_2} = (\partial x_{p+1}, \dots, \partial x_n)$$

ω_1 a bounded, convex open subset of $\mathbb{R}^p$

$$0 \in \omega_1 \qquad [\omega_1 = (-1, 1) = \omega, \ n = 2]$$

ω_2 a bounded open subset of $\mathbb{R}^{n-p}$

$$[\omega_2 = \omega, \ n = 2]$$

$$\Omega_\ell = \ell\omega_1 \times \omega_2$$

$$f = f(X_2) \in L^2(\omega_2)$$

$K(x_2)$ = a closed interval containing 0

$$K_e = \{ v \in H_0^1(\Omega_e) \mid v(x_1, x_2) \in K(x_2) \text{ a.e. } x \in \Omega_e \}$$

$$\boxed{\begin{array}{l} u_e \in K_e, \\ \displaystyle\int_{\Omega_e} \nabla u_e \cdot \nabla (v - u_e)\, dx \geq \int_{\Omega_e} f\,(v - u_e)\, dx \quad \forall v \in K_e. \end{array}} \quad (e)$$

$$K_\infty = \{ v \in H_0^1(\omega_2) \mid v(x_2) \in K(x_2) \text{ a.e. } x_2 \in \omega_2 \}$$

$$\boxed{\begin{array}{l} u_\infty \in K_\infty, \\ \displaystyle\int_{\omega_2} \nabla_{x_2} u_\infty \cdot \nabla_{x_2} (v - u_\infty)\, dx_2 \geq \int_{\omega_2} f\,(v - u_\infty)\, dx_2 \quad \forall v \in K_\infty. \end{array}} \quad (\infty)$$

$$K(x_2) = [0, +\infty) \quad , \quad n = 2$$

Remark: One could consider a more general elliptic operator, a more general f...

Theorem: $\exists\, C, \alpha$ positive constants independent of ℓ s.t.

$$\int_{\Omega_{\ell/2}} |\nabla(u_\ell - u_\infty)|^2 dx \leq C e^{-\alpha \ell} |f|^2_{2,\omega_2}$$

($| \ |_{2,\omega_2}$ the usual $L^2(\omega_2)$-norm).

Remark: For $\ell_0 \leq \frac{\ell}{2}$ fixed $\Rightarrow$

$$|u_\ell - u_\infty|_{H^1(\Omega_{\ell_0})} \longrightarrow 0$$

with an exponential rate of convergence.

Proof: Fix $\ell_1 \leq \ell - 1$.

Choose $\rho = \rho(x_1)$ such that

$$0 \leq \rho \leq 1 \ , \quad \rho \equiv 1 \text{ on } \ell_1\omega_1 \ , \quad \rho \equiv 0 \text{ on } \mathbb{R}^p \setminus (\ell_1 + 1)\omega_1$$

$$|\nabla_{X_1} \rho| \leq C$$

$$u_\ell - \rho^2(u_\ell - u_\infty) \in K_\ell \ \ldots\ldots$$

Applications

- $K(x_2) = \mathbb{R}$

$u_\ell \in K_\ell = H_0^1(\Omega_\ell)$, $-\Delta u_\ell = f = f(x_2)$ in Ω_ℓ

$u_\infty \in K_\infty = H_0^1(\omega_2)$, $-\Delta u_\infty = f$ in ω_2

$u_\ell \to u_\infty$ exponentially quickly

- $K(x_2) = [\varphi(x_2), +\infty)$

one obstacle problem.

- $K(x_2) = [\varphi(x_2), \psi(x_2)]$

two obstacle problem.

Remark : One can consider $f = f(x) = f(x_1, x_2)$.

In this case one can show (cf. MC - S. Mardare)

- u_ℓ is a Cauchy-sequence,
- u_ℓ gives at the limit the existence of a solution of the inequation in an unbounded domain.

4. Constraints on the gradient

$$\Omega_\ell = \ell\omega \times \omega \,, \qquad \omega = (-1,1)$$

$$f = f(x_2) \in L^2(\omega)$$

$$K_\ell = \{ v \in H_0^1(\Omega_\ell) \mid |\nabla v(x)| \leq 1 \text{ a.e. } x \in \Omega_\ell \}$$

$$\begin{cases} u_\ell \in K_\ell, \\ \displaystyle\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla (v - u_\ell)\, dx \geq \int_{\Omega_\ell} f (v - u_\ell)\, dx \quad \forall v \in K_\ell . \end{cases} \tag{ℓ}$$

$$K_\infty = \{ v \in H_0^1(\omega) \mid |\partial_{x_2} v(x_2)| \leq 1 \text{ a.e. } x_2 \in \omega \}$$

$$\begin{cases} u_\infty \in K_\infty, \\ \displaystyle\int_{\omega} \partial_{x_2} u_\infty \cdot \partial_{x_2} (v - u_\infty)\, dx_2 \geq \int_{\omega} f (v - u_\infty)\, dx_2 \quad \forall v \in K_\infty . \end{cases} \tag{∞}$$

$$u_\ell \overset{?}{\longrightarrow} u_\infty$$

Speed of convergence ?

$$u_\ell - \rho^2 (u_\ell - u_\infty) \notin K_\ell$$

Theorem 4.1. Suppose that $f \geq 0$ then

$$u_\ell \nearrow u_\infty \quad \text{when } \ell \to +\infty .$$

Proof: (i) $u_\ell, u_\infty \geq 0$

$$\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla(v - u_\ell)\, dx \geq \int_{\Omega_\ell} f(v - u_\ell)\, dx$$

$v = u_\ell^+$

$$\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla u_\ell^- \, dx \geq \int_{\Omega_\ell} f\, u_\ell^- \, dx \geq 0$$

$$\int_{\Omega_\ell} |\nabla u_\ell^-|^2 \, dx \leq 0$$

idem for u_∞

(ii) $u_\ell \leq u_\infty$

$$(u_\ell - u_\infty)^+ \in H_0^1(\Omega_\ell)$$

$$u_\ell - (u_\ell - u_\infty)^+ = u_\ell \text{ or } u_\infty \in K_\ell$$

$$\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla\{-(u_\ell - u_\infty)^+\}\, dx \geq \int_{\Omega_\ell} f\{-(u_\ell - u_\infty)^+\}\, dx$$

$$u_\infty + (u_\ell(x_1, \cdot) - u_\infty)^+ \in K_\infty \quad \text{a.e. } x_1 \in \omega$$

$$\int_\omega \partial_{x_2} u_\infty \, \partial_{x_2} (v - u_\infty) \, dx_2 \geq \int_\omega f \, (v - u_\infty) \, dx_2$$

$\Rightarrow$

$$\int_\omega \partial_{x_2} u_\infty \cdot \partial_{x_2} (u_e - u_\infty)^+ dx_2 \geq \int_\omega f \, (u_e - u_\infty)^+ dx_2$$

$\Rightarrow$

$$\int_{\Omega_e} \nabla u_\infty \cdot \nabla (u_e - u_\infty)^+ dx \geq \int_{\Omega_e} f \, (u_e - u_\infty)^+ dx_2$$

Summing up

$$\int_{\Omega_e} \nabla (u_e - u_\infty) \cdot \nabla (u_e - u_\infty)^+ dx \leq 0$$

$\Rightarrow \; (u_e - u_\infty)^+ = 0$

(iii) $u_e \leq u_{e'}$ for $e \leq e'$

We suppose u_e extended by 0 outside Ω_e

$$(u_e - u_{e'})^+ \in H_0^1(\Omega_e)$$

$$u_e - (u_e - u_{e'})^+ = u_e \text{ or } u_{e'} \in K_e$$

$$\int_{\Omega_e} \nabla u_e \cdot \nabla \{-(u_e - u_{e'})^+\} dx \geq \int_{\Omega_e} f \{-(u_e - u_{e'})^+\} dx$$

$$u_{e'} + (u_e - u_{e'})^+ \in K_{e'}$$

$$\int_{\Omega_{e'}} \nabla u_{e'} \cdot \nabla (u_e - u_{e'})^+ dx \geq \int_{\Omega_{e'}} f \, (u_e - u_{e'})^+ dx$$

$$\Rightarrow \quad \int_{\Omega_\ell} \nabla(u_\ell - u_{\ell'}) \cdot \nabla(u_\ell - u_{\ell'})^+ \, dx \le 0$$

$$\Rightarrow \quad (u_\ell - u_{\ell'})^+ = 0.$$

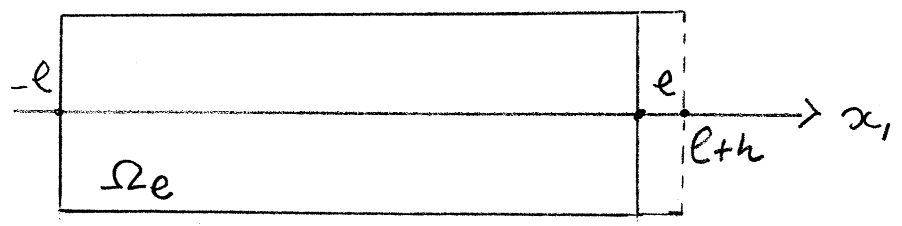
Thus we have

$$0 \le u_\ell \le u_{\ell'} \le u_\infty \qquad \forall\, \ell \le \ell'.$$

The pointwise limit of u_ℓ exists in $\mathbb{R} \times \omega$

$$u_\ell \to \tilde{u}_\infty \quad \text{a.e.}$$

(iv) $\tilde{u}_\infty$ is independent of x_1



$$\tau_h u_\ell(x_1, x_2) = u_\ell(x_1 - h, x_2), \quad h > 0.$$

Claim: $\tau_h u_\ell \le u_{\ell+h}$

The functions are supposed to be extended by 0.

$$\tau_h u_\ell \vee u_{\ell+h} \in K_{\ell+h}$$

$$\tau_h u_\ell \vee u_{\ell+h} - u_{\ell+h} = (\tau_h u_\ell - u_{\ell+h})^+$$

$$\int_{\Omega_{e+h}} \nabla u_{e+h} \cdot \nabla (\tau_h u_e - u_{e+h})^+ dx \geq \int_{\Omega_{e+h}} f\,(\tau_h u_e - u_{e+h})^+ dx$$

$$\tau_{-h}(\tau_h u_e \wedge u_{e+h}) \in K_e$$

$$\int_{\Omega_e} \nabla u_e \cdot \nabla \{ \tau_{-h}(\tau_h u_e \wedge u_{e+h}) - u_e \} dx$$
$$\geq \int_{\Omega_e} f \cdot \{ \tau_{-h}(\tau_h u_e \wedge u_{e+h}) - u_e \} dx$$

$$\Leftrightarrow \int_{\Omega_e} \nabla u_e \cdot \nabla \{ \tau_{-h}(\tau_h u_e \wedge u_{e+h} - \tau_h u_e) \} dx$$
$$\geq \int_{\Omega_e} f \{ \tau_{-h}(\tau_h u_e \wedge u_{e+h} - \tau_h u_e) dx .$$

$$\int v(\tau_{-h} w) = \int (\tau_h v)\, w$$

$$\int_{\Omega_{e+h}} \nabla \tau_h u_e \cdot \nabla \{ -(\tau_h u_e - u_{e+h})^+ \} dx \geq \int_{\Omega_{e+h}} f \{ -(\tau_h u_e - u_{e+h})^+ \} dx$$

$$\Rightarrow \quad \int_{\Omega_{e+h}} \nabla (\tau_h u_e - u_{e+h}) \cdot \nabla (\tau_h u_e - u_{e+h})^+ dx \leq 0$$

$$\Rightarrow \quad \tau_h u_e \leq u_{e+h}$$

This proves the claim i.e.

$$u_e(x_1 - h, x_2) \leq u_{e+h}(x_1, x_2)$$

By a shift on the other side we get

$$u_\ell(x_1+h, x_2) \le u_{\ell+h}(x_1, x_2)$$

$$\Rightarrow \qquad \tilde{u}_\infty(x_1-h, x_2) \le \tilde{u}_\infty(x_1, x_2) \le \tilde{u}_\infty(x_1-h, x_2)$$

for a.e. x $\Rightarrow$ $\tilde{u}_\infty$ is independent of x_1

(v) $\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx \le C$ where C is independent of ℓ.

$$d_\ell(x) = \ell - x_1 \wedge x_1 + \ell \quad \text{in } \Omega_\ell$$

Due to the constraint

$$0 \le u_\ell, u_\infty \le 1$$

$\Rightarrow$

$$0 \le \frac{u_\ell + u_\infty}{2} \le 1$$

$$v = \frac{u_\ell + u_\infty}{2} \wedge d(x_1) \in K_\ell$$

Moreover

$$v = \frac{u_\ell + u_\infty}{2} \text{ on } \Omega_{\ell-1}$$

$$\int_{\Omega_{\ell-1}} \nabla u_\ell \cdot \nabla \frac{u_\infty - u_\ell}{2} dx + \int_{\Omega_\ell \setminus \Omega_{\ell-1}} \nabla u_\ell \cdot \nabla(v - u_\ell) dx$$

$$\ge \int_{\Omega_{\ell-1}} f \, \frac{u_\infty - u_\ell}{2} dx + \int_{\Omega_\ell \setminus \Omega_{\ell-1}} f \; v - u_\ell \, dx$$

$$v = \frac{u_\ell + u_\infty}{2} \wedge d(x_1) \in K_\infty \quad \text{a.e. } x_1$$

$$\int_{(\ell-1)\omega} \nabla u_\infty . \nabla \frac{u_\ell - u_\infty}{2}\, dx_2 + \int_{\ell\omega\setminus(\ell-1)\omega} \nabla u_\infty . \nabla (v - u_\infty)\, dx_2$$

$$\geq \int_{(\ell-1)\omega} f \frac{u_\ell - u_\infty}{2}\, dx_2 + \int_{\ell\omega\setminus(\ell-1)\omega} f . v - u_\infty \, dx_2$$

Integrating in x_1

$$\int_{\Omega_{\ell-1}} \nabla u_\infty . \nabla \frac{u_\ell - u_\infty}{2}\, dx + \int_{\Omega_\ell\setminus\Omega_{\ell-1}} \nabla u_\infty . \nabla (v - u_\infty)$$

$$\geq \int_{\Omega_{\ell-1}} f \frac{u_\ell - u_\infty}{2}\, dx + \int_{\Omega_\ell\setminus\Omega_{\ell-1}} f\, (v - u_\infty)\, dx$$

Recall

$$\int_{\Omega_{\ell-1}} \nabla u_\ell . \nabla \frac{u_\infty - u_\ell}{2}\, dx + \int_{\Omega_\ell\setminus\Omega_{\ell-1}} \nabla u_\ell \nabla (v - u_\ell)\, dx$$

$$\geq \int_{\Omega_{\ell-1}} f \frac{u_\infty - u_\ell}{2}\, dx + \int_{\Omega_\ell\setminus\Omega_{\ell-1}} f\, (v - u_\ell)\, dx$$

We add the two equations to get

$$\int_{\Omega_{\ell-1}} |\nabla(u_\ell - u_\infty)|^2 dx \leq 4\, |\Omega_\ell \setminus \Omega_{\ell-1}|$$

$$+ \int_{\Omega_\ell\setminus\Omega_{\ell-1}} f\, (u_\infty + u_\ell - 2v)$$

$$\leq C$$

(VI) $\tilde{u}_\infty = u_\infty$

Using again the Poincaré inequality

$$\int_{\Omega_\ell} (u_\infty - \tilde{u}_\infty)^2 \, dx \le \int_{\Omega_\ell} (u_\infty - u_\ell)^2 \, dx$$

$$\le C_P^2 \int_{\Omega_\ell} |\nabla (u_\ell - u_\infty)|^2 \, dx \le C$$

$$\Rightarrow \qquad 2\ell \int_\omega (u_\infty - \tilde{u}_\infty)^2 \, dx_2 \le C$$

$\ell \to +\infty \quad \Rightarrow \quad \tilde{u}_\infty = u_\infty .$