The Hirota-Satsuma Equation. A model for surface water waves.

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June 10, 2011

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$u = u(x,t) \in \mathbb{R}$, $x, t \in \mathbb{R}$, and ϵ is a small parameter that is a measure of both the small wave amplitude and the long wavelength.

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2– Existence and stability of solitary-wave solutions.

3– Comparison between Hirota-Satsuma and BBM.

Equation in quasilinear form

Observing that

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Define the set

$$\Omega_1 := \{ \phi \in H^1(\mathbb{R}) : -1 \notin \mathrm{spec}(-\partial_x^2 - \phi) \}.$$

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Lemma

Ω_1 is an open subspace of $H^1(\mathbb{R})$.

Local well-posedness

For any $\phi \in \Omega_1$, denote

$$r_j(\phi) := \|\partial_x^j(-\partial_x^2 - \phi + 1)^{-1}\|_{\mathcal{B}(L^2)}. \quad j = 0, 1, 2,$$

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Theorem

Let $u_0 \in \Omega_1$. Then $\exists\ T = T(u_0) > 0$ and $\exists!\ u \in C([0, T]; H^1(\mathbb{R}))$ a solution of (HS) such that $u(\cdot, 0) = u_0$. Moreover the flow map $u_0 \mapsto u$ is smooth.

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Observation: $T = T(\|u_0\|_{L^2}, r_j(u_0),\ j = 0, 1, 2)$.

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- the best constant $C^\star$ in the Sobolev embedding $\|\phi\|_{L^3} \leq C^\star \|\phi\|_{H^1}$, is attained for $\varphi^\star$;

  $\Rightarrow E^\star := E(\varphi^\star) = \frac{1}{3}\|\varphi^\star\|_{H^1}^2$ and $C^\star = \|\varphi^\star\|_{H^1}^{-\frac{1}{3}}$ *where*

$$E(\phi) = \int_{\mathbb{R}} \left(\frac{1}{2}\phi(x)^2 + \frac{1}{2}\phi'(x)^2 - \frac{1}{6}\phi(x)^3 \right) dx,$$

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Proof. Let $\phi \in H^1(\mathbb{R}) \setminus \Omega_1$. Then there exists $\psi \in H^2(\mathbb{R})$ such that $-\psi'' + \psi - \phi\psi = 0$.

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$$\|\psi\|_{H^1}^2 = \int_{\mathbb{R}} \phi\psi^2 dx \leq \|\phi\|_{L^3} \|\psi\|_{L^3}^2 \leq {C^\star}^3 \|\phi\|_{H^1} \|\psi\|_{H^1}^2.$$

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However, because $C^\star = \|\varphi^\star\|_{H^1}^{-\frac{1}{3}}$, this is a contradiction.

Global well-posedness

Theorem

Let $u_0 \in B(0, \|\varphi^\star\|_{H^1})$ be such that $E(u_0) < E(\varphi^\star)$. Then the local solution u of (HS) extends globally in time.

Solitary waves

- For $c > 0$, (HS) admits special solutions of the form

$$u_c(x,t) = \phi_\mu(x - (1+c)t), \quad \text{with} \quad \lim_{|x|\to+\infty} \phi_\mu(x) = 0,$$

where ϕ_μ is solution to

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- $\|\phi_\mu\|_{H^1} < \|\varphi^\star\|_{H^1}$ and $E(\phi_\mu) < E(\varphi^\star)$.
- $\Rightarrow$ *Question:* Are these solitary waves stable?

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Observe that

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This cannot happen since for all $R>0$,

$$u_c \underset{c\to+\infty}{\longrightarrow} 0 \quad \text{in} \quad L^\infty((-R,R)\times(0,T^\star]).$$

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$\forall\, \epsilon > 0$, $\exists\, \delta > 0$, *such that if*

$$\|u_0 - \phi_\mu\|_{H^2} < \delta,$$

then $\forall\, t > 0$, $\exists\, \gamma = \gamma(t) \in \mathbb{R}$ *satisfying*

$$\|u(\cdot, t) - \phi_\mu(\cdot + \gamma)\|_{H^2} < \epsilon,$$

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Moreover, γ *can be chosen as a* C^1 *function satisfying* $|\gamma'(t) + (1 + c)| \leq C\epsilon$, $\forall\, t > 0$.

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- Variational problem:

$$(V_\lambda) \begin{cases} \text{Minimize } F(\phi) \text{ on the admissible set of functions} \\ I_\lambda = \{\phi \in H^2(\mathbb{R}) \mid E(\phi) = \lambda \text{ and } \|\phi\|_{H^1} < \|\varphi^\star\|_{H^1}\}. \end{cases}$$

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How to prevent a minimizing sequence from dichotomizing?

Lopes theorem

Theorem (Lopes)

Let E, $F : H^2(\mathbb{R}) \to \mathbb{R}$ be translation invariant, C^2-functionals satisfying the technical condition:

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- *Then for any minimizing sequence $\{\phi_n\}$ for (V_λ), if*

$$\phi_n \rightharpoonup \phi \neq 0 \text{ in } H^2 \quad \text{then} \quad \phi_n \to \phi \text{ in } W^{1,p},\ 2 < p \leq \infty.$$

Moreover, the limit ϕ is a non-trivial solution to the Euler-Lagrange equation $\Lambda'(\phi) = 0$.

Technical results

Lemma

Let $\{\phi_n\}$ a bounded sequence of H^2. Then

$$(\phi_n(\cdot + c_n) \rightharpoonup 0 \text{ in } H^2, \ \forall (c_n) \subset \mathbb{R}) \ \Rightarrow \ (\phi_n \to 0 \text{ in } W^{1,p}, \ 2 < p \leq \infty).$$

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Lemma

If $\{\phi_n\}$ is a minimizing sequence for (V_λ) such that $\phi_n \rightharpoonup \phi$ in $H^1(\mathbb{R})$, then $\|\phi\|_{H^1} < \|\varphi^\star\|_{H^1}$.

Technical results

Lemma (Monotonicity)

(i) $e : (0,1) \to (0, e^\star)$, $\mu \mapsto E(\phi_\mu)$ *is a strictly increasing bijection.*

(ii) $f : (0,1) \to (f^\star, 0)$, $\mu \mapsto F(\phi_\mu)$ *is a strictly decreasing bijection.*

where $e^\star := E(\varphi^\star)$ *and* $f^\star := F(\varphi^\star)$.

Existence of global minimizers

Proposition

Let $\lambda \in (0, e^\star)$, *and let* $\{\phi_n\}$ *be a minimizing sequence for* (V_λ). *Then,* $\exists$ a sequence $\{c_n\} \subset \mathbb{R}$, $\tau \in \mathbb{R}$ *s.t.*

$$\phi_n(\cdot + c_n) \to \phi_\mu(\cdot + \tau), \ \textit{in } H^2(\mathbb{R}),$$

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Proof

- Let $\{\phi_n\}$ a minimizing sequence for (V_λ). Then $\{\phi_n\}$ is bounded in $H^2(\mathbb{R})$.

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- $\{\psi_n\}$ is still a minimizing sequence. Lopes theorem $\Rightarrow$ $\psi_n \to \phi$ in $W^{1,p}$, $2 < p \leq \infty$, and $\phi = \phi_\alpha$ for $\alpha \in (0, 1)$.

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- Then
  $$\alpha = \mu,\ F(\phi_\mu) = F_\lambda,\ E(\phi_\mu) = \lambda,\ \text{and } \|\psi_n\|_{H^2} \underset{n\to+\infty}{\longrightarrow} \|\phi_\mu\|_{H^2}.$$

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- $\exists\ f \neq 0 \in H^2(\mathbb{R})$ such that $f_n \underset{n\to+\infty}{\rightharpoonup} f$ in H^2 and $\exists\ h \in C_0^\infty(\mathbb{R})$ such that $E'(f)h \neq 0$.

Proof of the theorem

- Consider the polynomial

$$P_n(t) = E(f_n + th) = a_n + b_n t + c_n t^2 + dt^3.$$

Then $a_n \longrightarrow \lambda$, $b_n \longrightarrow E'(f)h \neq 0$ and $c_n \longrightarrow \frac{1}{2}E''(f)(h,h)$.

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Contradiction $\square$

Comparison between model equations

Happy Birthday!!!

Happy Birthday, Jerry!!!

References

J. L. Bona, D. Pilod, *Stability of solitary-wave solutions to the Hirota-Satsuma equation*, to appear in Disc. Cont. Dyn. Syst. (2010).

R. Iório, D. Pilod, *Well-Posedness for Hirota-Satsuma's Equation*, Diff. Int. Equations **21** (2008), 1177-1192.

O. Lopes, *Nonlocal variational problems arising in long wave propagation*, ESAIM: Control Optim. Calc. Var. **5** (2000), 501–528.