On iterated integrated tail

Gheorghiță Zbăganu

F = distribution on $[0,\infty)$
$\underline{F}(x) := F((x,\infty)) \to$ right tail

$$\mu = \int_0^\infty \underline{F}(x)\,\mathrm{d}x \quad .$$

The integrated tail

$$F_I(x) = \frac{\int_0^x \underline{F}(y)\,\mathrm{d}y}{\int_0^\infty \underline{F}(y)\,\mathrm{d}y} \Leftrightarrow \underline{F}_I(x) = \frac{1}{\mu}\int_x^\infty \underline{F}(y)\,\mathrm{d}y$$

Compare to

Lorenz curve: $L(F)(y) = \frac{1}{\mu}\int_0^y F^{-1}(t)\,\mathrm{d}t$

Ignatov's Theorem

As $n \to \infty$, $L^n(F) \Rightarrow F^*$, $\mathrm{d}F^*/\mathrm{d}\lambda\,(x) = x^p$, $p = \frac{1+\sqrt{5}}{2}$

Ignatov's Conjecture:

Let $T(F) = F_I$. Then $\lim_{n\to\infty} T^n(F) = \text{Exponential}(\lambda_F)$.

False in general.

We prove that it is true in some cases if we agree to add Dirac's measure δ_0 and the null measure δ_∞ to the family of exponential distributions under the name Exp(∞) and Exp(0).

Definitions.

Hazard rate: $\underline{F}(x) = e^{-\int_0^x \lambda(y)dy} \Leftrightarrow \lambda_F(x) = \dfrac{f_F(x)}{\underline{F}(x)}$

$\boldsymbol{m(t) = m_F(t)} = \int e^{tx}\, dF(x)$ is the m.g.f. of F

$t^*(F) = \sup\{\, t \mid \boldsymbol{m(t)} < \infty\}$

F is short tailed $\Leftrightarrow$ ***t*(F) = ∞***

F is medium $\Leftrightarrow$ **0 <** ***t*(F)*** **< ∞**

F is long tailed $\Leftrightarrow$ ***t*(F)*** **= 0**

THEOREM A. *Consider the operator* $T: M_{\mathbf{ac}} \to M_{\mathrm{ac}}$ defined by $TF = F_I$. *Suppose that the limit* $\lambda_F(\infty)$ *does exist. Then*

(i). $\lambda_F(\infty) = t^*(F)$;

(ii).
- *If F is medium tailed, then* $T^nF \Rightarrow \mathrm{Exp}(\lambda_F(\infty))$;
- *If F is short tailed, then* $T^nF \Rightarrow \delta_0$;
- *If F is long tailed, then* T^nF *does not converge at all*: the mass of T^nF vanishes at infinity: $(\underline{T^nF})(x) \to 1$ as $n \to \infty$.

THEOREM B. *In the same conditions as in THEOREM A, the sequence* $\left(\dfrac{\int x^{n+1}dF(x)}{(n+1)\int x^n dF(x)}\right)_n$ *has a limit*

and $$\lim_{n\to\infty} \frac{\int x^{n+1}dF(x)}{(n+1)\int x^n dF(x)} = \frac{1}{\lambda_F(\infty)}$$

Alternative proof: Ivan (Cluj)

COUNTEREXAMPLE C. (PĂLTĂNEA). *There exist medium sized distributions for which the limits in Theorems A and B do not exist. Their* *hazard rate has no Cesaro limit at infinity, meaning*

that $\lim_{x \to \infty} \dfrac{\int_0^x \lambda_F(x)dx}{x}$ ***does not exist.***

Ideas of the proof of The. A,B: **use properties of $TF = F_I$**

Stochastic domination: $F \prec_{\mathrm{st}} G \Leftrightarrow \underline{F} \leq \underline{G}$

Hazard rate stochastic domination: $F \prec_{\mathrm{HR}} G \Leftrightarrow \lambda_G \leq \lambda_F$
IFR (DFR) distributions: λ_F increasing (decreasing)
Monotonous operator: $F \prec_{\mathrm{st}} G \Rightarrow TF \leq TG$
HR-monotonous operator: $F \prec_{\mathrm{HR}} G \Rightarrow TF \leq TG$

Continuity properties.

(i). *The mapping* $TF = F_I$ *is not continuous in the weak topology.*

(ii) *The mapping T is monotonously continuous in the sense that*
$\underline{F}_n \uparrow \underline{F}$ **or** $\underline{F}_n \downarrow \underline{F} \Rightarrow T(F_n) \Rightarrow T(F)$

(iii). *If* $T(F) = F$ *then either* $F = \delta_0$ or $F = \text{Exponential}\left(\frac{1}{\mu(F)}\right)$

Monotonicity properties.

(i). *The mapping T is not monotonous, but it is* HR-*monotonous*
(ii). *The mapping T is* IFR *and* DFR *preserving*
$F \in \text{IFR} \Rightarrow F_I \in \text{IFR}$ and $F \in \text{DFR} \Rightarrow F_I \in \text{DFR}$
(iii): $F \in \text{IFR} \Rightarrow T(F) \prec_{\mathrm{HR}} F$ *and* $F \in \text{DFR} \Rightarrow F \prec_{\mathrm{HR}} T(F)$

COROLLARY

(i). *If* $F \in$ IFR then the sequence $F_n = T^n(F)$ is HR-decreasing and if $F \in$ DFR is HR-increasing.
(ii). If $F \in$ IFR. *Then* $T^n(F)$ *has a limit, G. If* $F \in$ DFR *then the sequence of non-increasing right continuous functions* $(\underline{T^n(F)})_n$ *has a limit, too,* $\underline{G}$ *. If* $\underline{G}(\infty) = 0$*, then* $T^n(F)$ *weakly converges to G.*

CONJECTURE.

If $\lim_{x\to\infty} \frac{\int_0^x \lambda_F(x)dx}{x} = \lambda^*$ *does exist, then* $T^n F \to \mathrm{Exp}(\lambda^*)$

We prove it if λ_F is periodic

Theorem 1.

If λ *is periodic and has the period* $T > 0$, *then* $F_n \Rightarrow \mathrm{Exp}\left(\frac{\int_0^T \lambda(x)\,\mathrm{d}x}{T}\right)$.

The hazard rates λ_{F_n} *converge uniformly to* $\frac{\int_0^T \lambda(x)\,\mathrm{d}x}{T}$

Corollary 2.

If F has a periodic hazard rate, then the sequence $\left(\frac{\mu_{n+1}}{(n+1)\mu_n}\right)_n$ is convergent and its limit is $\frac{T}{\int_0^T \lambda(x)\mathrm{d}x}$.

PLAN OF THE PROOF

Proposition 2.1.

Let $\lambda:[0,\infty) \to [0,\infty)$ *be measurable having the period* $T > 0$.

Let $\Lambda(x) = \int_0^x \lambda(y)\mathrm{d}y$ *and* $\underline{F}(x) = e^{-\Lambda(x)}$. *Then*

$$\underline{F}(x) = q^{\left[\frac{x}{T}\right]} h\left(T\left\{\frac{x}{T}\right\}\right) \text{ with } q = e^{-\Lambda(T)} \text{ and } h(t) = e^{-\Lambda(t)} \tag{2.1}$$

and

$$\underline{F}_I(x)=q^{\left[\frac{x}{T}\right]}\left(1-p\frac{H(T\left\{\frac{x}{T}\right\})}{H(T)}\right) \textit{ with } p=1-q \textit{ and } H(t)=\int_0^t h(x)\mathrm{d}x \tag{2.2}$$

Moreover, the hazard rate of F_I is

$$\lambda_I(x)=\frac{ph(T\left\{\frac{x}{T}\right\})}{H(T)-pH(T\left\{\frac{x}{T}\right\})} \tag{2.3}$$

Corollary 2.2. *Let $(F_n)_n$ be the sequence given by the recurrence* (1.3). *Then*

$$\underline{F_n}(x)=q^{\left[\frac{x}{T}\right]}h_n(T\left\{\frac{x}{T}\right\}) \textit{ with } q=e^{-\Lambda(T)} \textit{ and } h(t)=e^{-\Lambda(t)} \tag{2.4}$$

The functions $h_n:[0,T)\to\Re_+$ have the properties

$$h_n(0)=1,\ h_n(T)=q,\ h_{n+1}(x)=1-p\frac{\int_0^x h_n(y)\mathrm{d}y}{\int_0^T h_n(y)\mathrm{d}y} \tag{2.5}$$

Main result.

Theorem. *Let $(E,\mathcal{E})$ measurable space and μ finite measure on it.*
Let $K:E\times E\to[a,\infty)$, $a>0$ be a measurable bounded function and

$$K_\mu f=\int K(x,y)f(y)d\mu(y) \tag{3.4}$$

Let also B be the mapping defined for bounded measurable functions as

$$(Bf)(x)=\frac{(K_\mu f)(x)}{(K_\mu f)(y)} \tag{3.5}$$

where $y\in E$ is fixed.
Suppose that there exists a bounded measurable function h: $\mathrm{E}\to(0,\infty)$, *$m>0$ and a positive constant $c>0$ such that*

$$cK_\mu h=h \tag{3.6}$$

Then the sequence $(B^n f)_n$ *converges to* $f_\infty(x) = \frac{h(x)}{h(y)}$. *The limit does not depend on f.*

Proposition 3.4. *Let F be a continuous distribution on* $[0,\infty)$ *and let* $p \le 1$, $q = 1 - p \ge 0$. *Let B be the mapping* $Bf(x) = 1 - p\frac{\int_0^x f(y)\mathrm{d}F(y)}{\int_0^\infty f(y)\mathrm{d}F(y)}$, *defined for measurable bounded positive functions. Then* $B^n f(x)$ *converges to* $q^{F(x)}$. *As a particular case, if* $F = U(0,T)$ *is the uniform distribution on* $[0,T]$, *then the limit is* $q^{\frac{x}{T}}$, *with the convention that* $0^0 = 1$.

Moreover, if $q < 1$, *the convergence is uniform.*

Remark and open problem. Is it necessary that the probability F from Proposition 3.4 be continuous? We think that the operator $B^n f$ **always has a limit which depends only on *F*, not on the chosen *f*. The limit is not necessarily the exponential** $q^{F(x)}$. The main problem is to decide if the operator B defined in Proposition 3.4 has a fixed point h, i.e. if there exists h such that $Bh = h$.

ANOTHER APPROACH TO THEOREM 3.3 AND ITS APPLICATION

The kernel considered in Theorem is a very particular case in the R-theory for irreducible kernels, theory developed, in Nummelin.

Let h be the C-invariant function for K_μ and let m, M denote its lower (upper) bound. Note the following relations, direct consequence of the assumptions in Theorem 3.3:

$$K_\mu 1_A(x) \ge a\mu(A) K_\mu 1_A(x) \qquad \forall x \in E, A \in \mathcal{E} \qquad 4.1$$

and

$$K_\mu 1_A(x) \ge \frac{a}{M} h(x)\mu(A) \qquad \forall x \in E, A \in \mathcal{E} \qquad 4.2$$

Relation (4.1) implies that K_μ is μ-irreducible, (i.e. $\mu(A) > 0 \Rightarrow K_\mu 1_A(x) > 0 \;\; \forall\, x \in E$), aperiodic and the whole space is a *small set.* Also relation (4.2) implies that h is a *small function.*

We summarize below the properties of K_μ which are relevant in our context>

Proposition 4.1.

(i) *The convergence parameter R of the kernel* K_μ *is c.*

(ii) *There exists a c-invariant measure* π *satisfying* $\pi(E) < \infty$, i.e. *the kernel is c – positive recurrent.*

(iii) *The kernel* K_μ *is c – uniformly ergodic*, i.e.

$$\lim_{n\to\infty} \sup_{x\in E} \sup_{f:|f|\le h} \left| \frac{c^n}{h(x)} K_\mu{}^n f(x) - \pi(f) \right| = 0 \tag{4.2}$$