Homogenization with iterated function system

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$\mathcal{A}^\epsilon$ partial differential operator, $\epsilon < 1$

$$\mathcal{A}^\epsilon u^\epsilon = f,$$

boundary initial conditions

The homogenization theory study the following issues:

- $\lim_{\epsilon\to 0} u^\epsilon$. What kind of convergence? What is the convergence rate?
- Explicit analytical construction of $\mathcal{A}$.
- Properties of the limiting equation:

$$\mathcal{A}u = f$$

Nguetseng [8] and Allaire [2]: two scale convergence to capture the two-scale structure of the homogenization problem.

Classic homogenization: periodic structure

We will quit the periodicity assumption!!!

—The coefficients are generated by iterated function system.

—We will define two scale convergence for IFS

—Usually the solutions of the limiting equations live on fractals.

1 Setting of the problem

Iterated function system(IFS)

$\varphi_1, ..., \varphi_m : \mathbb{R}^n \to \mathbb{R}^n$ similarities, $r_1, ..., r_n \in]0, 1[$,

$$d(\varphi_i(x), \varphi_i(y)) = r_i d(x, y), \quad \forall x, y \in \mathbb{R}^n$$

Moran's open set condition:
$\exists\ O \subset \mathbb{R}^n$ open, bounded set,

$$\varphi_i(O) \subset O \text{ and } \varphi_i(O) \cap \varphi_j(O) = \emptyset\, (i \neq j)$$

$A \subseteq \mathbb{R}^n$:

$$F(A) := \varphi_1(A) \cup ... \cup \varphi_m(A),$$

$F^1 := F$, $F^k := F \circ F^{k-1}$ $(k \in \mathbb{N})$, and $D := \cap_{k \geq 1} F^k(\overline{O})$.

Theorem 1.1 *(Hutchinson)*
F is a contraction, D is the unique nonempty compact set which is invariant under F:

$$F(D) = D \subseteq \overline{O}$$

and

$$\lim_{k \to \infty} F^k(A) = D,$$

for every compact subset $A \subset \mathbb{R}^n$. Moreover, the Hausdorff dimension of D is the solution of

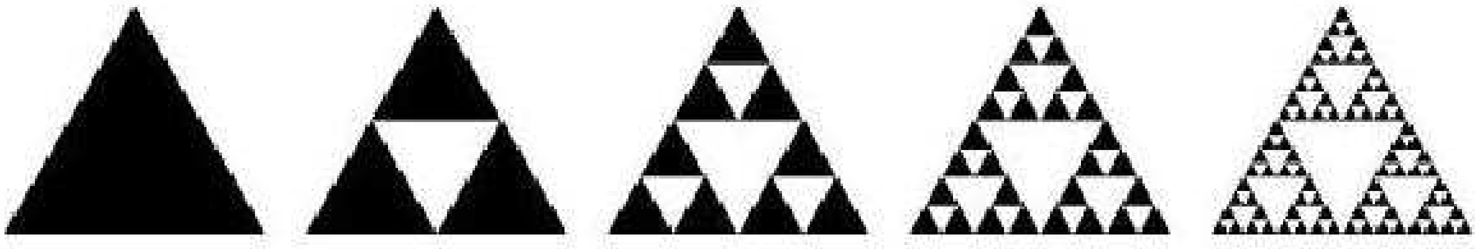
$$\sum_{i=1}^{m} r_i^s = 1.$$

Example:
q_1, q_2, q_3 the vertices of an equilateral tiangle.

$\varphi_i : \mathbb{R}^2 \to \mathbb{R}^2$

$$\varphi_i(x) = \frac{1}{2}(x - q_i) + q_i,\ i = 1, 2, 3.$$

$r_1 = r_2 = r_3 = \frac{1}{2}$ and $O := Int(q_1 q_2 q_3)$.



$$\mathcal{M} : = \{\mu, \text{positiv, Borel regular measure on } \mathbb{R}^n, \text{ bounded support and finite mass}\}$$

$$\mathcal{M}^1 := \{\mu \in \mathcal{M} :\ \mu(\mathbb{R}^n) = 1\}.$$

Let

$$C(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{R} :\ f \text{ is continuous}\}.$$

For $\mu \in \mathcal{M}$, $\psi \in C(\mathbb{R}^n)$,

$$\mu(\psi) := \int_{\mathbb{R}^n} \psi d\mu.$$

If $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, **push forward measure** $\varphi_\# : \mathcal{M} \to \mathcal{M}$:

$$\varphi_\# \mu(A) := \mu(\varphi^{-1}(A)),\ A \subseteq \mathbb{R}^n,$$

equivalently

$$\varphi_{\#}\mu(\psi) := \mu(\psi \circ \varphi), \ \psi \in \mathbb{C}(\mathbb{R}^n).$$

For $\mu, \nu \in \mathcal{M}^1$, **Monge-Kantorovits metric**

$$L(\mu, \nu) := \sup\{\mu(\varphi) - \nu(\varphi) : \varphi \in C(\mathbb{R}^n), \ Lip\varphi \leq 1\}.$$

$(\mathcal{M}^1, L)$ is complete metric space.

If $\nu \in \mathcal{M}^1$,

$$G(\nu) := \sum_{i=1}^{m} r_i^s \varphi_{i\#}\nu.$$

Theorem 1.2 *(Hutchinson)*

$$G(\nu)(\psi) = \sum_{i=1}^{m} r_i^s \nu(\psi \circ \varphi_i), \, \psi \in C(\mathbb{R}^n),$$

and $G : \mathcal{M}^1 \to \mathcal{M}^1$ *is a contraction map. Consequently, there exists a unique measure* $\overline{\mu} \in \mathcal{M}^1$*, the invariant measure on* K*, such that*

$$G(\overline{\mu}) = \overline{\mu} \quad \text{and} \quad spt\overline{\mu} = K.$$

For $\nu \in \mathcal{M}^1$ put

$$G^1(\nu) := G(\nu), \ G^k(\nu) = G \circ G^{k-1}(\nu), \ k \in \mathbb{N}.$$

$$G^k(\nu) = \sum_{|\sigma|=k} r_\sigma^s \varphi_{\sigma\#}(\nu),$$

and

$$\overline{\mu} = \lim_{k\to\infty} G^k(\nu)$$

in the sence of L metric.

$\Omega \subset \mathbb{R}^n$ nonempty bounded, connected open domain with smooth boundary, O be an open set, $\Omega \subset O$. ν be the normalized restriction of the Lebesgue measure on Ω, i.e.

$$spt(\nu) = \overline{\Omega} \text{ and } \nu(\Omega) = 1,$$

dx the Lebesque measure on O.

For $i_1, ..., i_k \in \{1, ..., m\}$

σ the word $i_1...i_k$, $|\sigma| = k$ be the length of σ. Let $\varphi_\sigma = \varphi_{i_1} \circ ... \circ \varphi_{i_k}$ and $r_\sigma = r_{i_1}...r_{i_k}$.

$$\Omega_\sigma := \varphi_\sigma(\Omega).$$

By the open set condition, for $\sigma \neq \sigma'$ with $|\sigma| = |\sigma'|$, we have:

$$\bigcup_{|\sigma|=k} \Omega_\sigma = \Omega.$$

$$\Omega_\sigma \cap \Omega_{\sigma'} = \emptyset.$$

Dirichlet problem

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\left(\varphi_\sigma^{-1}(x)\right)\frac{\partial u^k(x)}{\partial x_j}\right) = f(x), \quad \text{for } x \in \Omega_\sigma,\ |\sigma| = k$$

$$u^k(x) = 0, \quad \text{for } x \in \partial\Omega_\sigma. \quad (1.1)$$

Assume the coefficients $a_{ij} : \Omega \to \mathbb{R}$ are from $L^\infty(\Omega)$ and uniformly elliptic on Ω, i.e. there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(y)\xi_i\xi_j \geq \alpha \parallel \xi \parallel^2, \text{ for all } y \in \Omega,\ \xi \in \mathbb{R}^n. \quad (1.2)$$

Suppose that there exists $\beta > 0$ such that

$$\parallel \sum_{i=1}^{n} a_{ij}(y)\xi_i \parallel \leq \beta \parallel \xi \parallel \text{ for all } y \in \Omega,\ \xi \in \mathbb{R}^n$$

a.e. on Ω. f is sufficiently smooth and independent of σ.

Example 1.1 *(Homogenization for periodic structures)* $Y = [0,1]^n$, a_{ij}

$$a_{ij}(y + e_k) = a_{ij}(y),\ i,j,k = 1,...,n,\ y \in \mathbb{R}^n.$$

$\varphi_1(x) = \frac{x}{2}$.

For $m = 2^n$ and $i \in \{2.,,,.m\}$ we define the functions $\varphi_i(x)$ as translations of φ_1 by sumes of half of unit vectors such that $Y = \cup_{i=1}^m \varphi_i(Y)$. We choose $O =]0,1[^n$.

for $|\sigma| = k$ and $\epsilon = \frac{1}{2^k}$, we have $a_{ij}(\varphi_\sigma^{-1}(x)) = a_{ij}(\frac{x}{\epsilon})$.

$\to$ periodic structure

Example 1.2 *(Homogenization on a selfsimilar carpet) $n = 2$, $m = 13$, and q_1, q_2, q_3, q_4 be the vertices of a square. The functions φ_i, $i = \overline{1, 13}$, are defined such that it maps the square q_1, q_2, q_3, q_4 in the 13 squares on Figure 1.*

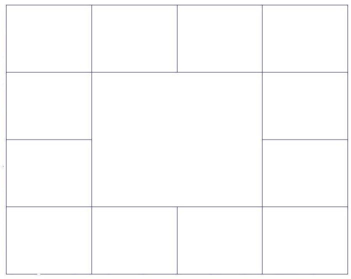


Figure 1

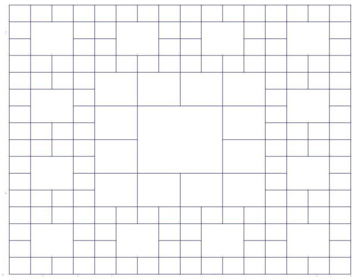


Figure 2

2 Two scale convergence

Assume the solution u^k is of the form

$$u^k(x) = u_0(x, \varphi_\sigma^{-1}(x)) + r_\sigma u_1(x, \varphi_\sigma^{-1}(x)) + r_\sigma^2 u_2(x, \varphi_\sigma^{-1}(x)) + ..., \text{ for } x \in \Omega_\sigma \quad (2.3)$$

$C_b(\mathbb{R}^n)$ bounded and continuous real functions on $\mathbb{R}^n$.
$L^2(O, C_b(\mathbb{R}^n)) = \{u : O \to C_b(\mathbb{R}^n) : \| u \| \in L^2(O)\}$.

The norm of this space is

$$\| u \|_{L^2(O, C_b(\mathbb{R}^n))} = \left[\int_O | \sup_{y \in \mathbb{R}^n} u(x, y)|^2 dx \right]^{\frac{1}{2}}.$$

Oscillations lemma:

Theorem 2.1 *If $\Phi \in L^2(O, C_b(\mathbb{R}^n))$ and ν is the normalized restriction of the Lebesgue measure on Ω, then* *the following convergence result holds:*

$$\lim_{k\to\infty}\sum_{|\sigma|=k} r_\sigma^s \int_\Omega \Phi(\varphi_\sigma(z),z)d\nu(z) = \int_D\left[\int_\Omega \Phi(x,y)d\nu(y)\right]d\mu(x), \tag{2.4}$$

where D and μ are the invariant set and the invariant measure, respectively of the iterated function system $\{\varphi_1, ..., \varphi_m\}$.

$L^2_{\mu\otimes\nu}(D\times\Omega)$ by $L^2(D\times\Omega)$.

Theorem 2.2 *Let $u\in L^2(O, C_b(\mathbb{R}^n))$. Then*

(i) $u(\varphi_\sigma(\cdot),\cdot)\in L^2(\Omega)$, *and*
$\| u(\varphi_\sigma(\cdot),\cdot)\|_{L^2(\Omega)}\leq \| u(\cdot,\cdot)\|_{L^2(O,C_b(\mathbb{R}^n))}$, *for all* σ

(ii) $\displaystyle\lim_{k\to\infty}\sum_{|\sigma|=k} r_\sigma^s \int_\Omega u(\varphi_\sigma(z),z)\psi(z)d\nu(z) = \int_D \mu(x)\int_\Omega u(x,y)\psi(y)d\nu(y),$
$\psi\in L^2(C_b(\mathbb{R}^n))$

(iii) $\displaystyle\sum_{|\sigma|=k} r_\sigma^s \| u(\varphi_\sigma(\cdot),\cdot)\|^2_{L^2(\Omega)}\to \| u(\cdot,\cdot)\|^2_{L^2(D\times\Omega)}$.

Definition 2.1 *Let u^k be a sequence in $L^2(O)$. We say that u^k* **two-scale converges** *to $u_0\in L^2(O\times\Omega)$ and write $u^k \overset{2}{\rightharpoonup} u_0$ if for every function $\Phi\in L^2(O, C_b(\mathbb{R}^n))$ we have*

$$\lim_{k\to\infty}\sum_{|\sigma|=k} r_\sigma^s \int_\Omega u^k(\varphi_\sigma(z))\Phi(\varphi_\sigma(z),z)d\nu(z) = \int_D\int_\Omega u_0(x,y)\Phi(x,y)d\nu(y)d\mu(x) \tag{2.5}$$

Two-scale convergence implies a kind of weak convergence in $L^2(O)$:

Remark 2.1 *Let* u^k *be in* $L^2(O)$ *which two-scale converges to* u_0. *Then, for all* $\Phi \in L^2(O)$

$$\lim_{k\to\infty} \sum_{|\sigma|=k} r_\sigma^s \int_\Omega u^k(\varphi_\sigma(z))\Phi(\varphi_\sigma(z)d\nu(z) = \int_D \left(\int_\Omega u_0(x,y)d\nu(y) \right) \Phi(x)d\mu(x) \quad (2.6)$$

The left side of (2.6) can be written as

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} u^k(y)\Phi(y)dG^k(\nu).$$

Definition 2.2 *The sequence* $(u^k)_{k\in\mathbb{N}} \in L^2(O)$ **diagonally weakly converges to** $u^0 \in L^2(D)$, *denoted by* $u^k \stackrel{d}{\rightharpoonup} u^0$, *if*

$$\lim_{k\to\infty} \sum_{|\sigma|=k} r_\sigma^s \int_\Omega u^k(\varphi_\sigma(z))\Phi(\varphi_\sigma(z)d\nu(z) = \int_D u^0(x)\Phi(x)d\mu(x)$$

for all $\Phi \in L^2(O)$.

Lemma 2.1 *If the sequence* $(u^k)_{k\in N}$ *weakly converges to* $\overline{u}$ *in* $L^2(O)$ *and* $s = n$ *then it diagonally weakly converges to* $\frac{\overline{u}}{|\Omega|}$, *where* $|\Omega|$ *denote the Lebesgue measure of* Ω.

Lemma 2.2 *Let* $u^k \in L^2(O)$ *be a function which admits the two-scale expansion*

$$u^k(x) = u_0(x, \varphi_\sigma^{-1}(x)) + r_\sigma u_1(x, \varphi_\sigma^{-1}(x)) + ..., \text{ for all } \sigma \quad |\sigma| = k, \text{ and } x \in \Omega_\sigma$$

where $u_j \in L^2(O, C_b(\mathbb{R}^n))$, $j \in \{0, 1, ...\}$. *Then* $u^k \stackrel{2}{\rightharpoonup} u_0$.

The following compactness result provides a criteria which enables us to conclude that a given sequence is two-scale convergent.

Theorem 2.3 *Let* $u^k \in L^2(O)$*,* $k \in \mathbb{N}$*, and let*

$$|||u^k||| := \sum_{|\sigma|=k} r_\sigma^s \left[\int_\Omega [u^k \circ \varphi_\sigma(z)]^2 d\nu(z)\right]^{\frac{1}{2}}.$$

If the sequence $(|||u^k|||)_{k\in\mathbb{N}}$ *is bounded then there exists a subsequence of* (u^k) *which two-scale converges to a function* $u_0 \in L^2(D \times \Omega)$*.*

3 Some convergence results

In this section we will prove some convergence results needed in the proof of the main result.

Theorem 3.1 *If* $1 < p < \infty$ *and* $(a_k)_{k\in\mathbb{N}}$ *is a sequence in* $L^p(O)$*, then the following assertions are equivalent*

(a) $(a_k)_{k\in\mathbb{N}}$ *converges weakly*

(b) $\| a_n \|_{L^p(O)} \leq C$*, independently of* n*, and* $\int_{\Omega_\sigma} a_n(x) d\nu_\sigma(x) \to \int_{\Omega_\sigma} a(x) d\nu_\sigma(x)$*, for all* σ*.*

The weak limit of a sequence obtained by iterating function system.

Theorem 3.2 *Let* $1 \leq p \leq \infty$ *and let* $a \in L^p(\Omega)$*. Set*

$$a_k(x) = a(\varphi_\sigma^{-1}(x)),\ x \in \Omega_\sigma.$$

Then

if $p < \infty$ *as* $|\sigma| = k \to \infty$

$$a_k \rightharpoonup \int_\Omega a(z) d\nu(z) \text{ weakly in } L^p(O),$$

if $p = \infty$

$$a_k \rightharpoonup \int_\Omega a(z) d\nu(z) \text{ weakly}^* \text{in } L^\infty(\mathbb{R}^n).$$

Theorem 3.3 *Let $(u^k)_{k\in\mathbb{N}}$ be a sequence of functions in $H_0^1(\cup_{|\sigma|=k}\Omega_\sigma)$ such that u^k converges diagonally weakly to u^0 in $H^1(D)$. Then $(u^k)_{k\in\mathbb{N}}$ two scale converges to u^0 and there exists a subsequence $(u^{k'})_{k\in\mathbb{N}}$ of u^k and an $u_1 \in L^2(D, H_0^1(\Omega))$ such that*

$$\nabla u^{k'} \stackrel{2}{\rightharpoonup} \nabla_x u^0 + \nabla_y u_1.$$

4 The main result

Theorem 4.1 *Let $f \in H^{-1}(O)$ and let A^k be the matrix $(a_{ij}^k)_{i,j=\overline{1,n}}$, where*

$$a_{ij}^k(x) = a_{ij}(\varphi_\sigma^{-1}(x)),\ k = |\sigma|,\ x \in \Omega_\sigma,\ i,j = \overline{1,n}.$$

Denote by u^k the solution of

$$\begin{cases} -div(A^k \nabla u^k) = f & in\ \Omega_\sigma \\ u^k = 0 & on\ \partial\Omega_\sigma. \end{cases} \tag{4.7}$$

Then

$$\begin{aligned} u^k &\stackrel{d}{\rightharpoonup} u^0 &\quad in \quad & H_0^1(D) \\ A^k \nabla u^k &\stackrel{d}{\rightharpoonup} A^0 \nabla u^0 &\quad in \quad & (L^2(D))^n, \end{aligned}$$

where u^0 is the solution of the homogenized problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^n a_{ij}^0 \frac{\partial^2 u^0}{\partial x_i \partial x_j} = f & in\ D \\ u^0 = 0 & on\ \partial D, \end{cases} \tag{4.8}$$

A^0 is the constant matrix $(a_{i,j}^0)_{i,j=\overline{1,n}}$,

$$a_{ij}^0 := \int_\Omega \left\{ \sum_{l=1}^n \left[a_{il}(y)\frac{\partial \chi^j(y)}{\partial y_l} + \chi^j(y)\frac{\partial a_{li}(y)}{\partial y_l} + a_{li}(y)\frac{\partial \chi^j(y)}{\partial y_l} \right] - a_{ij}(y) \right\} d\nu(y$$

χ^j, $j=\overline{1,n}$ is the solution of the equation

$$-div_y(A(y)\nabla_y\chi^j(y)) = \sum_{i=1}^{n} \frac{\partial a_{ij}(y)}{\partial y_i} \quad in \ \Omega, \qquad (4.9)$$

and $A(y)$ is the matrix $(a_{ij}(y))_{i,j=\overline{1,n}}$.

Remark 4.1 *For periodic structures (see Example 1.1) we have*

$$\sum_{i,j=1}^{n} \frac{\partial^2 u^0(x)}{\partial x_i \partial x_j} \int_Y \sum_{l=1}^{n} \left[\chi^j(y)\frac{\partial a_{li}(y)}{\partial y_l} + a_{li}(y)\frac{\partial \chi^j(y)}{\partial y_l}\right] d\nu(y) = 0,$$

hence

$$a^0_{i,j} = -a_{ij}(y) + \int_\Omega \sum_{l=1}^{n} a_{il}(y)\frac{\partial \chi^j(y)}{\partial y_l} d\nu(y).$$

Remark 4.2 *Often the invariant set D is a fractal*

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