Stochastic calculus via regularization in Banach spaces with financial motivations.

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Some recent references

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Available preprints:
http://www-math.math.univ-paris13.fr/$\sim$ russo/

1 Motivations

Let W be the real Brownian motion equipped with its canonical filtration $(\mathcal{F}_t)$. $\langle W \rangle_t = t$.

- If $h \in L^2(\Omega)$, the martingale representation theorem states the existence of a predictable process $\xi \in L^2(\Omega \times [0, T])$ such that

$$h = \mathbb{E}[h] + \int_0^T \xi_s dW_s$$

- If $h \in \mathbb{D}^{1,2}$ in the sense of Malliavin, Clark-Ocone formula implies that $\xi_s = \mathbb{E}\left[D^m h | \mathcal{F}_s\right]$, so that

$$h = \mathbb{E}[h] + \int_0^T \mathbb{E}\left[D^m h | \mathcal{F}_s\right] dW_s \tag{1}$$

where D^m is the Malliavin gradient.

- We suppose that the law of $X = W$ is not anymore the Wiener measure but X is still a finite quadratic variation process but not necessarily a semimartingale.
- Are there reasonable classes of random variable which can be represented in the form

$$h = H_0 + \text{"} \int_0^T \xi_s dX_s \text{"} ?$$

$H_0 \in \mathbb{R}$, ξ adapted?

Examples of processes with finite quadratic variation

1) S is an $(\mathcal{F}_t)$-**semimartingale** with decomposition $S = M + V$, M $(\mathcal{F}_t)$-local martingale and V bounded variation process. So $[S] = [M]$.

2) D is a $(\mathcal{F}_t)$-**Dirichlet** process with decomposition $D = M + A$, M $(\mathcal{F}_t)$-local martingale and A an $(\mathcal{F}_t)$-adapted zero quadratic variation process. $[D] = [M]$. Föllmer (1981).

3) D is a $(\mathcal{F}_t)$-**weak-Dirichlet** process with decomposition $D = M + A$, M $(\mathcal{F}_t)$-local martingale and A such that $[A, N] = 0$ for any continuous $(\mathcal{F}_t)$-local martingale N. Errami-Russo (2003), Gozzi-Russo (2005).

a) In general D does not have finite quadratic variation

b) If A is a finite quadratic variation process
$[D] = [M] + [A]$

c) There are finite quadratic variation weak Dirichlet processes which are not Dirichlet processes.

4) (Houdré-Villa, Russo-Tudor)

$B^{H,K}$ bifractional Brownian motion with parameters $H \in]0,1[$, $K \in]0,1]$ such that $HK \geq 1/2$

- If $HK > 1/2$, $[B^{H,K}] = 0$.
- If $HK = 1/2$, then
  - $[B^{H,K}]_t = 2^{1-K} t$
  - If $K = 1$ and if $H = 1/2$, $B^{H,K}$ is a Brownian motion
  - If $K \neq 1$, $B^{H,K}$ is not a semimartingale (not even a Dirichlet with respect to its own filtration).

5) Skorohod integrals. If (u_t) is in $L^{1,2}$, under reasonable conditions on Du, $[\int_0^t u_s \delta W_s]_t = \int_0^t u_s^2 ds$.

6) For fixed $k \geq 1$, Föllmer Wu Yor construct a weak k-order Brownian motion X, which in general is not even Gaussian.
X is a **weak k-order Brownian motion** if for every $0 \leq t_1 \leq \cdots \leq t_k < +\infty$, $(X_{t_1}, \cdots, X_{t_k})$ is distributed as $(W_{t_1}, \cdots, W_{t_k})$. If $k \geq 4$ then $[X]_t = t$.

Definition 1 *Let* $T > 0$ *and* $X = (X_t)_{t\in[0,T]}$ *be a real continuous process prolongated by continuity.*
Process $X(\cdot)$ *defined by*

$$X(\cdot) = \{X_t(u) := X_{t+u}; u \in [-T, 0]\}$$

will be called ***window process****.*

- $X(\cdot)$ is a $C([-T, 0])$-valued stochastic process.
- $C([-T, 0])$ is a typical non-reflexive Banach space.

The representation problem

We suppose $X_0 = 0$ and $[X]_t = t$.

Which are the classes of functionals

$$H : C([-T, 0]) \longrightarrow \mathbb{R}$$

such that the r.v.

$$h := H(X_T(\cdot))$$

admits a representation of the type

$$h = H_0 + \text{"} \int_0^T \xi_s dX_s \text{"}$$

- In that case we look for an explicit expressions for
  - $H_0 \in \mathbb{R}$
  - ξ adapted process with respect to the canonical filtration of X

Idea: **Representation of $h = H(X_T(\cdot))$**

We express $h = H(X_T(\cdot))$ as

$$h = H(X_T(\cdot)) = \lim_{t \uparrow T} u(t, X_t(\cdot))$$

where $u \in C^{1,2}\left([0, T[\times C([-T, 0])\right)$ solves an infinite dimensional PDE.

We have

$$h = u(0, X_0(\cdot)) + \int_0^T D^{\delta_0} u(s, X_s(\cdot)) d^- X_s \qquad (2)$$

where $D^{\delta_0} u(s,\eta) = D\,u(s,\eta)(\{0\})$. We recall that $D\,u : [0,T] \times C([-T,0]) \longrightarrow C^*([-T,0]) = \mathcal{M}([-T,0])$.

2 Finite dimensional calculus via regularization

Definition 2 *Let* X *(resp.* Y*) be a continuous (resp. locally integrable) process.*
Suppose that the random variables

$$\int_0^t Y_s d^- X_s := \lim_{\epsilon \to 0} \int_0^t Y_s \frac{X_{s+\epsilon} - X_s}{\epsilon} ds$$

exists in probability for every $t \in [0, T]$*.*
If the limiting random function admits a continuous modification, it is denoted by $\int_0^\cdot Y d^- X$ *and called* ***(proper) forward integral of*** Y ***with respect to*** X

Covariation of real valued processes

Definition 3 *The* ***covariation of X and Y*** *is defined by*

$$[X,Y]_t = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^t (X_{s+\epsilon} - X_s)(Y_{s+\epsilon} - Y_s) ds$$

if the limit exists in the ucp sense with respect to t.
Obviously $[X,Y] = [Y,X]$.
If $X = Y$, X is said to be ***finite quadratic variation process*** *and $[X] := [X,X]$.*

Connections with semimartingales

1. Let S^1, S^2 be $(\mathcal{F}_t)$-semimartingales with decomposition $S^i = M^i + V^i$, $i = 1, 2$ where M^i $(\mathcal{F}_t)$-local continuous martingale and V^i continuous bounded variation processes. Then
   - $[S^i]$ classical bracket and $[S^i] = \langle M^i \rangle$.
   - $[S^1, S^2]$ classical bracket and $[S^1, S^2] = \langle M^1, M^2 \rangle$.
   - If S semimartingale and Y cadlag and predictable

$$\int_0^{\cdot} Y d^- S = \int_0^{\cdot} Y dS \quad \text{(Itô)}$$

Itô formula for finite quadratic variation processes

Theorem 4 *Let* $F : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ *such that* $F \in C^{1,2}\left([0,T[\times\mathbb{R}\right)$ *and* X *be a finite quadratic variation process. Then*

$$\int_0^t \partial_x F(s, X_s) d^- X_s$$

exists and equals

$$F(t, X_t) - F(0, X_0) - \int_0^t \partial_s F(s, X_s) ds - \frac{1}{2} \int_0^t \partial_{x\,x} F(s, X_s) d[X]_s$$

Stochastic calculus via regularization

- A theory for non-semimartingales.
- Integrands may be anticipative.
- A simple formulation.
- Close to pathwise approach (as rough paths) but still probabilistic.
- An efficient formulation when the integrator is a finite quadratic variation, but it extends to more general cases.

3 About infinite dimensional classical stochastic calculus

We fix now in a general (infinite dimensional) framework. Let

- B general Banach space
- $\mathbb{X}$ a B-valued process
- $F : B \longrightarrow \mathbb{R}$ be of class C^2 in Fréchet sense.

An Ito formula for B-valued processes.

Aim: An Itô type expansion of $F(\mathbb{X})$, available also for $B = C([-T, 0])$-valued processes, as window processes, i.e. when $\mathbb{X} = X(\cdot)$.

The literature does not apply: several problems appear even in the simple case $W(\cdot)$!

Fréchet derivative and tensor product of Banach spaces

$F : B \longrightarrow \mathbb{R}$ be of class C^2 in Fréchet sense, then

- $DF : B \longrightarrow L(B; \mathbb{R}) := B^*$;
- $D^2F : B \longrightarrow L(B; B^*) \cong \mathcal{B}(B \times B) \cong (B\hat{\otimes}_\pi B)^*$

where

- $\mathcal{B}(B, B)$ Banach space of real valued bounded bilinear forms on $B \times B$
- $(B\hat{\otimes}_\pi B)^*$ dual of the tensor projective tensor product of B with B.
- $B\hat{\otimes}_\pi B$ fails to be Hilbert even if B is a Hilbert space (is not even a reflexive space).

A first attempt to an Itô type expansion of $F(\mathbb{X})$

$$\begin{aligned} F(\mathbb{X}_t) &= F(\mathbb{X}_0) +'' \int_0^t {}_{B^*}\langle DF(\mathbb{X}_s), d\mathbb{X}_s \rangle_B \, '' \\ &+ \frac{1}{2}'' \int_0^t {}_{(B\hat{\otimes}_\pi B)^*}\langle D^2 F(\mathbb{X}_s), d[\mathbb{X}]_s \rangle_{B\hat{\otimes}_\pi B} \, '' \end{aligned}$$

A formal proof

$$\int_0^t \frac{F(\mathbb{X}_{s+\epsilon}) - F(\mathbb{X}_s)}{\epsilon} ds \xrightarrow[\epsilon \to 0]{ucp} F(\mathbb{X}_t) - F(\mathbb{X}_0)$$

By a Taylor's expansion the left-hand side equals the sum of

$$\int_0^t {}_{B^*}\langle DF(\mathbb{X}_s), \frac{\mathbb{X}_{s+\epsilon} - \mathbb{X}_s}{\epsilon} \rangle_B ds +$$

$$\int_0^t {}_{(B\hat{\otimes}_\pi B)^*}\langle D^2F(\mathbb{X}_s), \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)^{\otimes 2}}{\epsilon} \rangle_{B\hat{\otimes}_\pi B} ds + R(\epsilon, t)$$

Problems related to integration in Banach space

Let B be a Banach space and $\mathbb{X}$ a B-valued process.

1. Stochastic integration with respect to an integrator $\mathbb{X}$.
2. Quadratic variation of $\mathbb{X}$.

Among the principal references about the subject there are:

1. Da Prato G. and Zabczyk J. Stochastic Equations in Infinite Dimensions. Cambridge University Press, 1992.
2. Métivier M. and Pellaumail J. Stochastic Integration. New York, 1980.
3. Dinculeanu N. Vector Integration and Stochastic Integration in Banach spaces. Wiley-Interscience, New York, 2000.

Da Prato - Zabczyk

- B separable Hilbert space.
- $\mathbb{X}$ B-valued Itô process.

but...

- $C([-T,0])$ is not a Hilbert space.
- $W(\cdot)$ is not a a $C([-T,0])$-valued semimartingale.

Metivier-Pellaumail and Dinculeanu

- B is a general Banach space.
- $\mathbb{X}$ essentially semimartingale.
- The natural generalization concept of quadratic variation for Banach valued processes $\mathbb{X}$ is a $(B\hat{\otimes}_\pi B)$-valued process denoted by $[\mathbb{X}]^{\otimes}$ and called tensor quadratic variation.

but... $W(\cdot)$ does not admit a tensor quadratic variation. In fact the limit for ϵ going to zero of

$$\frac{1}{\epsilon}\int_0^t \|W_{s+\epsilon}(\cdot) - W_s(\cdot)\|^2_{C([-T,0])}\, ds.$$

4 Stochastic calculus via regularization in Banach spaces

- A stochastic integral for B^*-valued integrand with respect to B-valued integrators, which are not necessarily semimartingales.
- χ-quadratic variation of $\mathbb{X}$
  A new concept of quadratic variation which generalizes the tensor quadratic variation and which involves a Banach subspace χ of $(B\hat{\otimes}_\pi B)^*$.

Definition 5 *Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be a B-valued (resp. a B^*-valued) continuous stochastic process.*
Suppose that the random function defined for every fixed $t \in [0,T]$ by

$$\int_0^t {}_{B^*}\langle \mathbb{Y}_s, d^-\mathbb{X}_s \rangle_B := \lim_{\epsilon \to 0} \int_0^t {}_{B^*}\langle \mathbb{Y}(s), \frac{\mathbb{X}_{s+\epsilon} - \mathbb{X}_s}{\epsilon} \rangle_B ds$$

in probability exists and admits a continuous version. Then, the corresponding process will be called **forward stochastic integral of $\mathbb{Y}$ with respect to $\mathbb{X}$**.

Connection with Da Prato-Zabczyk integral

Let $B = H$ is separable Hilbert space.

Theorem 6 *Let* $\mathbb{W}$ *be a* H*-valued* Q*-Brownian motion with* $Q \in L^1(H)$ *and* $\mathbb{Y}$ *be* H^**-valued process such that* $\int_0^t \|\mathbb{Y}_s\|_{H^*}^2 ds < \infty$ *a.s. Then, for every* $t \in [0, T]$,

$$\int_0^t {}_{H^*}\langle \mathbb{Y}_s, d^- \mathbb{W}_s \rangle_H \quad = \quad \int_0^t \mathbb{Y}_s \cdot d\mathbb{W}_s^{dz}$$

$($*Da Prato-Zabczyk integral*$)$

Notion of Chi-subspace

Definition 7 *A Banach subspace χ continuously injected into $(B\hat{\otimes}_\pi B)^*$ will be called* **Chi-subspace** *(of $(B\hat{\otimes}_\pi B)^*$). In particular it holds*

$$\| \cdot \|_\chi \geq \| \cdot \|_{(B\hat{\otimes}_\pi B)^*}.$$

Notion of Chi-quadratic variation

Let

- $\mathbb{X}$ be a B-valued continuous process,
- χ a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$,
- $\mathcal{C}([0,T])$ space of real continuous processes equipped with the ucp topology.

Let $[\mathbb{X}]^\epsilon$ be the application

$$[\mathbb{X}]^\epsilon : \chi \longrightarrow \mathcal{C}([0,T])$$

defined by

$$\phi \mapsto \left(\int_0^t {}_{\chi}\langle \phi, \frac{J\left(\left(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s \right) \otimes^2 \right)}{\epsilon} \rangle_{\chi^*} \, ds \right)_{t \in [0,T]},$$

where $J : B\hat{\otimes}_\pi B \rightarrow (B\hat{\otimes}_\pi B)^{**}$ is the canonical injection between a Banach space and its bidual (omitted in the sequel).
Let E be a Banach space and $J : E \rightarrow E^{**}$:

- J is an isometry with respect to the strong topology,
- $J(E)$ is weak star dense in E^{**},
- If E is not reflexive, $J(E) \subsetneq E^{**}$.

Definition 8 $\mathbb{X}$ **admits a** χ***-quadratic variation*** *if*

H1 *For all* $(\epsilon_n) \downarrow 0$ *it exists a subsequence* (ϵ_{n_k}) *such that*

$$\sup_k \int_0^T \frac{\left\|(\mathbb{X}_{s+\epsilon_{n_k}} - \mathbb{X}_s)^{\otimes 2}\right\|_{\chi^*}}{\epsilon_{n_k}} ds \quad < \infty \quad a.s.$$

H2 *There exists* $[\mathbb{X}] : \chi \longrightarrow \mathcal{C}([0,T])$ *such that*

$$[\mathbb{X}]^\epsilon(\phi) \xrightarrow[\epsilon \longrightarrow 0]{ucp} [\mathbb{X}](\phi) \quad \forall\, \phi \in \chi$$

H3 There is a χ^*-valued bounded variation process $\widetilde{[\mathbb{X}]}$, such that $\widetilde{[\mathbb{X}]}_t(\phi) = [\mathbb{X}](\phi)_t$ a.s. for all $\phi \in \chi$.
For every fixed $\phi \in \chi$, processes $\widetilde{[\mathbb{X}]}_t(\phi)$ and $[\mathbb{X}](\phi)_t$ are indistinguishable.

Definition 9 *When* $\mathbb{X}$ *admits a* χ*-quadratic variation, the* χ^**-valued process* $\widetilde{[\mathbb{X}]}$ *(and even the application* $[\mathbb{X}]$*) will be called* χ**-quadratic variation** *of* $\mathbb{X}$

Remark 10 1. $\widetilde{[\mathbb{X}]}$ *is the quadratic variation intervening in the second order derivative term of Itô's formula.*

2. *For every fixed* $\phi \in \chi$, *processes* $\widetilde{[X]}_t(\phi)$ *and* $[X](\phi)_t$ *are indistinguishable.*

3. *The* χ^**-valued process* $\widetilde{[X]}$ *is weakly star continuous, i.e.* $\widetilde{[X]}(\phi)$ *is continuous for every fixed* $\phi \in \chi$.

Definition 11 *We say that $\mathbb{X}$ admits a* ***global quadratic variation (g.q.v.)*** *if it admits a χ-quadratic variation with $\chi = (B\hat{\otimes}_\pi B)^*$.*

When $\chi = (B\hat{\otimes}_\pi B)^*$

- H2 requires a weak* convergence in $(B\hat{\otimes}_\pi B)^{**}$, i.e.

$$\widetilde{[\mathbb{X}]}^\epsilon \xrightarrow[\epsilon\to 0]{w^*} \widetilde{[\mathbb{X}]},$$

  where $\widetilde{[\mathbb{X}]}^\epsilon$ is a χ^*-valued bounded variation process associated to $[\mathbb{X}]^\epsilon$, defined by
  $\widetilde{[\mathbb{X}]}^\epsilon_t(\phi) := [\mathbb{X}]^\epsilon(\phi)_t \qquad a.s.$

- The g.q.v. $\widetilde{[\mathbb{X}]}$ is $(B\hat{\otimes}_\pi B)^{**}$-valued.

Connection with other concepts of quadratic variation

Global quadratic variation permit us to recover quadratic variation concepts in literature.

1. $B = \mathbb{R}^n$. $\mathbb{X} = (X^1, \ldots, X^n)$ admits a covariations matrix $[\mathbb{X}^*, \mathbb{X}] = ([X^i, X^j])_{1 \le i,j \le n}$ if and only if $\mathbb{X}$ admits g.q.v. $\widetilde{[\mathbb{X}]} = [\mathbb{X}^*, \mathbb{X}]$.
2. $B = H$ Hilbert separable. If $\mathbb{X}$ has $L^1(H)$-valued quadratic variation $[\mathbb{X}]^{dz}$ then $\mathbb{X}$ admits g.q.v. $\widetilde{[\mathbb{X}]} = [\mathbb{X}]^{dz}$.
3. B general. If $\mathbb{X}$ admits tensor quadratic variation $[\mathbb{X}]^{\otimes}$ then $\mathbb{X}$ admits g.q.v. $\widetilde{[\mathbb{X}]} = [\mathbb{X}]^{\otimes}$.

In all the recovered cases we have a strong convergence, so the g.q.v. $\widetilde{[\mathbb{X}]}$ is always $B\hat{\otimes}_\pi B$-valued.

We recall $B\hat{\otimes}_\pi B \subseteq (B\hat{\otimes}_\pi B)^{**}$.

Infinite dimensional Itô's formula

Let B a separable Banach space

Theorem 12 *Let $\mathbb{X}$ a B-valued continuous process admitting a χ-quadratic variation.*
Let $F : [0,T] \times B \longrightarrow \mathbb{R}$ be $C^{1,2}$ Fréchet such that

$$D^2F : [0,T] \times B \longrightarrow \chi \subset (B\hat{\otimes}_\pi B)^* \quad \textit{continuously}$$

Then for every $t \in [0,T]$ the forward integral

$$\int_0^t {}_{B^*}\langle DF(s,\mathbb{X}_s), d^-\mathbb{X}_s\rangle_B$$

exists and the following formula holds.

$$F(t,\mathbb{X}_t) = F(0,\mathbb{X}_0) + \int_0^t \partial_s F(s,\mathbb{X}_s)ds +$$

$$+ \int_0^t {}_{B^*}\langle DF(s,\mathbb{X}_s), d^-\mathbb{X}_s\rangle_B +$$

$$+ \frac{1}{2}\int_0^t {}_{\chi}\langle D^2F(s,\mathbb{X}_s), d\widetilde{[\mathbb{X}]}_s\rangle_{\chi^*}$$

5 Window processes

- We fix attention now on $B = C([-T, 0])$-valued window processes.
- X continuous real valued process and $X(\cdot)$ its window process.
- $\mathbb{X} = X(\cdot)$

- If X has Hölder continuous paths of parameter $\gamma > 1/2$, then $X(\cdot)$ has a zero g.q.v.
  For instance:
  - $X = B^H$ fractional Brownian motion with parameter $H > 1/2$.
  - $X = B^{H,K}$ bifractional Brownian motion with parameters $H \in]0,1[$, $K \in]0,1]$ s.t. $HK > 1/2$.
- $W(\cdot)$ does not admit a g.q.v.

Some examples of Chi-subspaces

- χ Chi-subspace of $(B\hat{\otimes}_\pi B)^*$. For instance:
  - $\mathcal{M}([-T,0]^2)$ equipped with the total variation norm.
  - $L^2([-T,0]^2)$.
  - $\mathcal{D}_{0,0} = \{\mu(dx,dy) = \lambda\,\delta_0(dx)\otimes\delta_0(dy)\}$.
  - $(\mathcal{D}_0 \oplus L^2)\,\hat{\otimes}_h^2 = \mathcal{D}_{0,0} \oplus L^2([-T,0])\hat{\otimes}_h D_0 \oplus D_0\hat{\otimes}_h L^2([-T,0]) \oplus L^2([-T,0]^2)$.
  - $Diag :=$ $\{\mu(dx,dy) = g(x)\delta_y(dx)dy; g \in L^\infty([-T,0])\}$.

Evaluations of χ-quadratic variation for window processes

- $W(\cdot)$ does not admit a $\mathcal{M}([-T,0]^2)$-quadratic variation.
- If X is a real finite quadratic variation process, then
  - $X(\cdot)$ has zero $L^2([-T,0]^2)$-quadratic variation.
  - $X(\cdot)$ has $\mathcal{D}_{0,0}$-quadratic variation

$$[X(\cdot)] : \mathcal{D}_{0,0} \longrightarrow \mathcal{C}[0,T] , \qquad [X(\cdot)]_t(\mu) = \mu(\{0,0\})[X$$

  - $X(\cdot)$ has $(\mathcal{D}_0 \oplus L^2)\,\hat{\otimes}_h^2$-quadratic variation

$$[X(\cdot)] : \left(\mathcal{D}_0 \oplus L^2\right)\hat{\otimes}_h^2 \longrightarrow \mathcal{C}[0,T] , \qquad [X(\cdot)]_t(\mu) = \mu($$

  - $X(\cdot)$ has $Diag$-quadratic variation

6 About robustness of Black-Scholes formula

Let (S_t) be the price of a financial asset of the type

$$S_t = \exp(\sigma W_t - \frac{\sigma^2}{2} t)\,, \quad \sigma > 0\,.$$

Let $h = \tilde{f}(S_T) = f(W_T)$ where $f(y) = \tilde{f}\left(\exp(\sigma y - \frac{\sigma^2}{2} T)\right)$.

Let $\tilde{u} : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ solving

$$\begin{cases} \partial_t \tilde{u}(t,x) + \frac{1}{2}\partial_{xx}\tilde{u}(t,x) = 0 & \\ \tilde{u}(T,x) = \tilde{f}(x) & x \in \mathbb{R}. \end{cases}$$

Applying classical Itô formula we obtain

$$\begin{aligned} h &= \tilde{u}(0,S_0) + \int_0^T \partial_x \tilde{u}(s,S_s)dS_s \\ &= u(0,W_0) + \int_0^T \partial_x u(s,W_s)dW_s \end{aligned}$$

for a suitable $u : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$.

Does one have a similar formula if W is replaced by a finite quadratic variation X such that $[X]_t = t$? **The answer is YES!**

Let X such that $[X]_t = t$

A1 $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous and polynomial growth

A2 $v \in C^{1,2}([0,T[\times\mathbb{R}) \cap C^0([0,T] \times \mathbb{R})$ such that

$$\begin{cases} \partial_t v(t,x) + \frac{1}{2}\partial_{xx} v(t,x) = 0 \\ v(T,x) = f(x) \end{cases}$$

Then

$$h := f(X_T) = v(0, X_0) + \underbrace{\int_0^T \partial_x v(s, X_s) d^- X_s}_{\text{improper forward integral}}$$

Schoenmakers-Kloeden (1999) Coviello-Russo (2006) Bender-Sottinen-Valkeila (2008)

Natural question

Generalization to the case of path dependent option?

As first step we revisit the toy model.

The toy model revisited

Proposition 13 *We set* $B = C([-T, 0])$ *and* $\eta \in B$ *and we define*

- $H : B \longrightarrow \mathbb{R}$, *by* $H(\eta) := f(\eta(0))$
- $u : [0, T] \times B \longrightarrow \mathbb{R}$, *by* $u(t, \eta) := v(t, \eta(0))$

Then

$$u \in C^{1,2}([0, T[\times B; \mathbb{R}) \cap C^0([0, T] \times B; \mathbb{R})$$

and solves

$$\left\{ \begin{array}{rcl} \partial_t u(t,\eta) & + & \frac{1}{2}\langle D^2 u\,(t,\eta)\ ,\ 1_D\rangle = 0 \\ u(T,\eta) & = & H(\eta) \end{array} \right.$$

Proof.

- $u(T,\eta) = v(T,\eta(0)) = f(\eta(0)) = H(\eta)$
- $\partial_t u\,(t,\eta) = \partial_t v\,(t,\eta(0))$
- $Du\,(t,\eta) = \partial_x v\,(t,\eta(0))\;\delta_0$
- $D^2 u\,(t,\eta) = \partial^2_{x\,x} v\,(t,\eta(0))\;\delta_0 \otimes \delta_0$
- $\partial_t u\,(t,\eta) + \frac{1}{2} D^2 u\,(t,\eta)(\{0,0\}) = 0$.

7 A generalized Clark-Ocone type formula

We set $B = C([-T, 0])$.

- X real continuous stochastic process with values in B.
- $X_0 = 0$,
- $[X]_t = t$.

Main task: to look for classes of functionals

$$H : B \longrightarrow \mathbb{R}$$

such that the r.v.

$$h := H(X_T(\cdot))$$

admits representation

$$h = H_0 + \int_0^T \xi_s d^- X_s$$

- Moreover we look for an explicit expression for
  - $H_0 \in \mathbb{R}$
  - ξ (adapted) process with respect to the canonical filtration of X

Idea

Obtain the representation formula by expressing $h = H(X_T(\cdot))$ as

$$h = H(X_T(\cdot)) = \lim_{t \uparrow T} u(t, X_t(\cdot))$$

where $u \in C^{1,2}\left([0, T[\times B\right)$ solves an infinite dimensional PDE, if previous limit exists.

8 An infinite dimensional PDE

Let $H : B \longrightarrow \mathbb{R}$, we will show the existence of a function $u : [0,T] \times B \longrightarrow \mathbb{R}$ of class $C^{1,2}\left([0,T[\times B\right) \cap C^{0}\left([0,T] \times B\right)$ solving *Infinite dimensional PDE*

$$\begin{cases} \partial_t u(t,\eta) + '' \int_{-t}^{0} D^{ac} u\left(t,\eta\right) d\eta '' + \frac{1}{2}\langle D^2 u\left(t,\eta\right) ,\ 1_D\rangle = 0 \\ \\ u(T,\eta) = H(\eta) \end{cases} \tag{3}$$

where

- $1_D(x,y) := \begin{cases} 1 & \text{if } x = y,\ x,y \in [-T,0] \\ 0 & \text{otherwise} \end{cases}$
- $D^{ac}u\,(t,\eta)$ absolute continuous part of measure $Du\,(t,\eta)$
- If $x \mapsto D^{ac}_x u\,(t,\eta)$ has bounded variation, previous integral is defined by an integration by parts.

Then

$$h = H_0 + \int_0^T \xi_s d^- X_s \qquad (4)$$

with

- $H_0 = u(0, X_0(\cdot))$
- $\xi_s = D^{\delta_0} u(s, X_s(\cdot))$

Methodology: two steps

- We will choose a functional $u : [0, T] \times B \longrightarrow \mathbb{R}$ which solves the infinite dimensional PDE (3) with final condition H.
- Using Itô formula we establish a representation form (4).

Particular cases

1. $H(\eta) = f(\eta(0))$ where $f : \mathbb{R} \to \mathbb{R}$ continuous and polynomial growth $\Rightarrow u$ such that $D^2u(t,\eta) \in \mathcal{D}_{0,0}$
2. $H(\eta) = \left(\int_{-T}^{0} \eta(s)ds\right)^2 \Rightarrow u$ such that $D^2u(t,\eta) \in (\mathcal{D}_0 \oplus L^2)\hat{\otimes}_h^2$
3. $H(\eta) = \int_{-T}^{0} \eta(s)^2 ds \Rightarrow u$ such that $D^2u(t,\eta) \in (Diag \oplus \mathcal{D}_{0,0})$

A general representation theorem

Theorem 14

- $H : B \longrightarrow \mathbb{R}$
- $u \in C^{1,2}([0,T[\times B) \cap C^0([0,T] \times B)$
- $x \mapsto D_x^{ac} u(t,\eta)$ *has bounded variation*
- $D^2 u(t,\eta) \in (\mathcal{D}_0 \oplus L^2)\hat{\otimes}_h^2$

- u solves

$$\begin{cases} \partial_t u(t,\eta) & + & \int_{]-t,0]} D^{ac}u(t,\eta)\, d\eta + \frac{1}{2} D^2 u\,(t,\eta)(\{0,0\}) = \\ u(T,\eta) & = & H(\eta), \quad \eta \in B. \end{cases}$$

(5)

Then h has representation (4).

Proof. Application of Itô's formula. ■

Sufficient conditions to solve (5)

1. When X general process such that $[X]_t = t$.
   - H has a smooth Fréchet dependence on $L^2([-T, 0])$.
   - $h := H(X_T(\cdot)) = f\left(\int_0^T \varphi_1(s) d^- X_s, \dots, \int_0^T \varphi_n(s) d^- X_s\right)$,
     - $f : \mathbb{R}^n \to \mathbb{R}$ measurable and with linear growth
     - $(\varphi_i) \in C^2([0, T]; \mathbb{R})$
2. When $X = W$ if Clark-Ocone formula does not apply. For instance when $h \notin \mathbb{D}^{1,2}$, or $h \notin L^2(\Omega)$ (even not in $L^1(\Omega)$).