Control of the motion of a boat

Lionel Rosier

Université Henri Poincaré Nancy 1

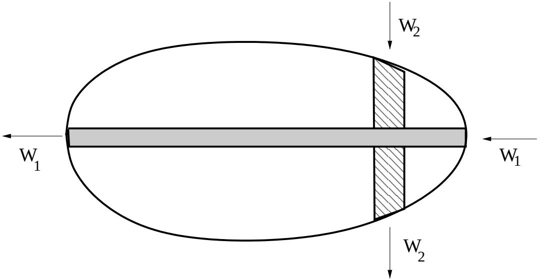
Poitiers 26/08/2010

Joint work with

Olivier Glass, Université Paris-Dauphine

Control of the motion of a boat

- We consider a rigid body $S \subset \mathbb{R}^2$ with one axis of symmetry, surrounded by a fluid, and which is controlled by two fluid flows, a longitudinal one and a transversal one.



Aims

- We aim to control the position and velocity of the rigid body by the control inputs. System of dimension 3+3 with a PDE in the dynamics. Control living in $\mathbb{R}^2$. No control objective for the fluid flow (exterior domain!!).
- Model for the motion of a boat with a longitudinal propeller, and a transversal one (thruster) in the framework of the theory of fluid-structure interaction problems. Rockets and planes also concerned.

Bowthruster

Longitudinal thruster

What is a fluid-structure interaction problem?

- Consider a rigid (or flexible) structure in touch with a fluid.
- The velocity of the fluid obeys **Navier-Stokes (or Euler)** equations in a **variable domain**
- The dynamics of the rigid structure is governed by **Newton** laws. Great role played by the pressure.
- Questions of interest: **existence** of (weak, strong, global) solutions of the system fluid+solid, **uniqueness**, **long-time behavior**, **control**, **inverse problems**, **optimal design**, ...

Main difficulties

1. The systems describing the motions of the fluid and the solid are nonlinear and **strongly coupled**; e.g., the **pressure of the fluid** gives rise to a force and a torque applied to the solid, and the fluid domain changes when the solid is moving.
2. The fluid domain $\mathbb{R}^N \setminus S(t)$ is an **unknown** function of time

Why to consider perfect fluids?

1. Euler equations provide a good model for the motion of boats or submarines in a reasonable time-scale.
2. **Explicit** computations may be performed with the aid of **Complex Analysis** when the flow is potential and 2D.
3. There is a **natural** choice for the boundary conditions $u_{rel} \cdot n = 0$ for Euler equations. For Navier-Stokes flows, one often takes $u_{rel} = 0$
4. The control theory of Euler flows is well understood (Coron, Glass).

System under investigation

$$\Omega(t) = \mathbb{R}^2 \setminus S(t)$$

Euler

$$\begin{aligned} & u_t + (u \cdot \nabla) u + \nabla p = 0, \;\; x \in \Omega(t) \\ & \mathrm{div}\; u = 0, \;\; x \in \Omega(t) \\ & u \cdot \vec{n} = (h' + r(x-h)^{\perp}) \cdot \vec{n} + w(x,t), \;\; x \in \partial\Omega(t) \\ & \lim_{|x| \to \infty} u(x,t) = 0 \end{aligned}$$

Newton

$$\begin{aligned} & m\, h''(t) = \int_{\partial\Omega(t)} p\, \vec{n}\, d\sigma \\ & J\, r' = \int_{\partial\Omega(t)} (x-h)^{\perp} \cdot p \vec{n}\, d\sigma \end{aligned}$$

System supplemented with Initial Conditions, and with the value of the vorticity at the incoming flow (in $\Omega(t)$) for the uniqueness

System in a frame linked to the solid

After a change of variables and unknown functions, we obtain in $\Omega := \mathbb{R}^2 \setminus S(0)$

$$
\begin{aligned}
& v_t + (v - l - ry^{\perp}) \cdot \nabla v + rv^{\perp} + \nabla q = 0, \ y \in \Omega \\
& \mathrm{div}\ v = 0, \ y \in \Omega \\
& v \cdot \vec{n} = (h' + ry^{\perp}) \cdot \vec{n} + \sum_{1 \leq j \leq 2} w_j(t) \chi_j(y), \ y \in \partial\Omega \\
& \lim_{|y| \to \infty} v(y, t) = 0 \\
& m\, h''(t) = \int_{\partial S(t)} q\, \vec{n}\, d\sigma - mrl^{\perp} \\
& J\, r' = \int_{\partial\Omega} qn \cdot y^{\perp}\, d\sigma
\end{aligned}
$$

Potential flows

Assuming that the initial vorticity and circulation are null

$$\omega_0 := \text{curl}\, u_0 \equiv 0, \qquad \Gamma_0 := \int_{\partial\Omega} u_0 \cdot n^{\perp} d\sigma = 0$$

and that the vorticity at the inflow part of $\partial\Omega$ is null

$$\omega(y,t) = 0 \quad \text{if } \ w_i(t)\chi_i(y) < 0 \qquad \text{for some } i = 1,2$$

then the flow remains potential, i.e. $v = \nabla\phi$ where ϕ solves

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega \times [0,T] \\ \dfrac{\partial\phi}{\partial n} = (l + ry^{\perp}) \cdot n + \displaystyle\sum_{i=1,2} w_i(t)\chi_i(y) & \text{on } \partial\Omega \times [0,T] \\ \lim_{|y|\to\infty} \nabla\phi(y) = 0 & \text{on } [0,T] \end{cases}$$

Potential flows (continued)

$v = \nabla \phi$ decomposed as

$$\nabla \phi = \sum_{i=1,2} l_i(t) \nabla \psi_i(y) + r(t) \nabla \varphi(y) + \sum_{i=1,2} w_i(t) \nabla \theta_i(y)$$

where the functions $\varphi,\ \psi_i$ and θ_i are harmonic on Ω and fulfill the following boundary conditions on $\partial\Omega$

$$\frac{\partial \varphi}{\partial n} = y^{\perp} \cdot n, \qquad \frac{\partial \psi_i}{\partial n} = n_i(y), \qquad \frac{\partial \theta_i}{\partial n} = \chi_i(y)$$

This gives the following expression for the pressure

$$q = -\{ \sum_{i=1,2} l_i' \psi_i + \dot{r} \varphi + \sum_{i=1,2} \dot{w}_i \theta_i + \frac{|v|^2}{2} - l \cdot v - r y^{\perp} \cdot v \}$$

Plugging this expression in Newton's law yields a

Finite dimension control system

$$
\begin{aligned}
h' &= l \\
\mathcal{I} l' &= \mathcal{C} w' + B(l, w)
\end{aligned}
$$

where $h = [h_1, h_2, \theta]^T$, $l = [l_1, l_2, r]^T$, $w = [w_1, w_2]^T$, and

$$
\mathcal{I} = \begin{bmatrix} m + \int \psi_1 n_1 & 0 & 0 \\ 0 & m + \int \psi_2 n_2 & \int \varphi n_2 \\ 0 & \int \psi_2 y^\perp \cdot n & J + \int \varphi y^\perp \cdot n \end{bmatrix}
$$

$$
\mathcal{C} = \begin{bmatrix} -\int \theta_1 n_1 & 0 \\ 0 & -\int \theta_2 n_2 \\ 0 & -\int \theta_2 y^\perp \cdot n \end{bmatrix}
$$

where $\int = \int_{\partial\Omega}$ and $B(l, w)$ is bilinear in (l, w)

Toy problem $w_2 = 0,\ h_2 = l_2 = 0$

$$(*) \begin{cases} h_1' = & l_1 \\ l_1' = & \alpha w_1' + \beta w_1 l_1 + \gamma w_1^2 \end{cases}$$

where

$$(\alpha, \beta, \gamma) := (m + \int_{\partial\Omega} \psi_1 n_1)^{-1} (\int_{\partial\Omega} \theta_1 n_1, \int_{\partial\Omega} \chi_1 \partial_1 \psi_1, \int_{\partial\Omega} \chi_1 \partial_1 \theta_1)$$

Claims

- If we add the equation ${w_1}' = v_1$ to $(*)$, the system with state (h_1, l_1, w_1) and input v_1 is NOT controllable!
- **In general** we cannot impose the condition $w_1(0) = w_1(T) = 0$ when $l_1(0) = l_1(T) = 0$ (i.e. fluid at rest at $t = 0, T$). Actually we can do that if and only if $\gamma + \alpha\beta = 0$.

Proof of the claims

Introduce $z_1 := l_1 - \alpha w_1$ From

$$l_1' = \alpha w_1' + \beta w_1 l_1 + \gamma w_1^2$$

we derive

$$z_1' = \beta w_1 z_1 + (\gamma + \alpha\beta) w_1^2$$

hence

$$z_1(t) = [z_1(0) + (\gamma + \alpha\beta) \int_0^t w_1^2(\tau) e^{-\int_0^\tau \beta w_1(s) ds} d\tau] e^{\int_0^t \beta w_1(s) ds}$$

Generic assumption

We shall assume that $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \neq 0$ where

$$
\begin{aligned}
a_{11} &= -\int_{\partial\Omega} \theta_2 n_2 \\
a_{12} &= -(m + \int_{\partial\Omega} \psi_1 n_1)(\int_{\partial\Omega} \chi_1 \partial_2 \theta_2 + \int_{\partial\Omega} \chi_2 \partial_2 \theta_1) \\
&\quad - \int_{\partial\Omega} \chi_2 \partial_2 \psi_1 \cdot \int_{\partial\Omega} \theta_1 n_1 \\
a_{21} &= -\int_{\partial\Omega} \theta_2 y^\perp \cdot n \\
a_{22} &= -(m + \int_{\partial\Omega} \psi_1 n_1)(\int_{\partial\Omega} \chi_1 \nabla \theta_2 \cdot y^\perp + \int_{\partial\Omega} \chi_2 \nabla \theta_1 \cdot y^\perp) \\
&\quad + (\int_{\partial\Omega} y_1 \nabla \theta_2 \cdot \tau - \int_{\partial\Omega} \chi_2 \nabla \psi_1 \cdot y^\perp) \int_{\partial\Omega} \theta_1 n_1
\end{aligned}
$$

Main result

Thm If

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \neq 0$$

then the system

$$\begin{aligned} h' &= l \\ \mathcal{I} l' &= \mathcal{C} w' + B(l, w) \end{aligned}$$

with state $(h, l) \in \mathbb{R}^6$ and control $w \in \mathbb{R}^2$ is locally controllable around 0.

The local controllability also holds true in the presence of vorticity and circulation.

Step 1. Loop-shaped trajectory

We consider a special trajectory of the toy problem ($w_2 \equiv 0$) constructed as in the **flatness approach** due to M. Fliess, J. Levine, P. Martin, P. Rouchon

- We **first define** the trajectory

$$
\begin{aligned}
\overline{h_1}(t) &= \lambda(1 - \cos(2\pi t/T)) \\
\overline{l_1}(t) &= \lambda(2\pi/T)) \sin(2\pi t/T)
\end{aligned}
$$

- We **next solve** the Cauchy problem

$$
\begin{cases}
\overline{w_1}' &= \alpha^{-1}\{\overline{l_1}' - \gamma \overline{w_1}^2 - \beta \overline{w_1}\,\overline{l_1}\} \\
\overline{w_1}(0) &= 0
\end{cases}
$$

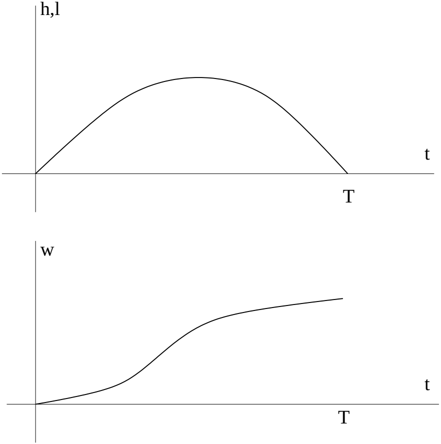
  to design the control input.

- Then $\overline{w_1}$ exists on $[0, T]$ for $0 < \lambda << 1$. $(\overline{h_1}, \overline{l_1}) = 0$ at $t = 0, T$. **Nothing** can be said about $\overline{w_1}(T)$.

Step 2. Return Method

We linearize along the above (non trivial) reference trajectory to use the nonlinear terms. We obtain a system of the form

$$x' = A(t)x + B(t)u + Cu'$$



Linearization along the reference trajectory

Fact. The reachable set from the origin for the system

$$x' = A(t)x + B(t)u + Cu', \; x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$

is

$$\mathcal{R} = \mathcal{R}_T(A, B + AC) + C\mathbb{R}^m + \Phi(T, 0)C\mathbb{R}^m$$

where $\Phi(t, t_0)$ is the resolvent matrix associated with the system $x' = A(t)x$, and $\mathcal{R}_T(A, B)$ denotes the reachable set in time T from 0 for $x' = A(t)x + B(t)u$, i.e.

$$\mathcal{R}_T(A, B) = \{x(T); \; x' = A(t)x + B(t)u, \; x(0) = 0, \; u \in L^2(0, T, \mathbb{R}^m)\}$$

Silverman-Meadows test of controllability

Consider a (smooth) time-varying control system

$$x = A(t)x + B(t)u, \qquad x \in \mathbb{R}^n,\ t \in [0, T],\ u \in \mathbb{R}^m.$$

Define a sequence $(M_i(\cdot))_{i\geq 0}$ by

$$M_0(t) = B(t), \qquad M_i(t) = \frac{dM_{i-1}}{dt} - A(t)M_i(t) \qquad i \geq 1,\ t \in [0, T]$$

Then for any $t_0 \in [0, T]$ and any $i \geq 0$

$$\Phi(T, t_0)M_i(t_0)\mathbb{R}^m \subset \mathcal{R}_T(A, B)$$

Proof of the main result (continued)

To complete the proof of the theorem in the case of potential flows, we use

- the generic assumption to prove that the linearized system is controllable. (We use the term $w_1 w_2$ to control r)
- the Implicit Function Theorem to conclude.

Proof of the main result (continued)

In the general case (vorticity $+$ circulation), we prove/use

- a Global Well-Posedness result using an extension argument (which enables to define the vorticity at the incoming part of the flow), and Schauder Theorem in Kikuchi's spaces;
- Linear estimates for the difference of the velocities corresponding to potential (resp. general) flows in terms of the vorticity and circulation at time 0;
- a topological argument to conclude when the vorticity and circulation are small;
- a scaling argument due to J.-M. Coron

Conclusion

- Local exact controllability result for a boat with a general shape
- Two linearization arguments: in $\mathbb{R}^6$ (for potential flows) and next to deal with general flows
- Prospects:
  - Motion planning
  - 3D (submarine)
  - Numerics??