

Non-convex variational inequalities with singular inputs and applications to stochastic inclusions

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Object

An existence and uniqueness result is given for the multivalued differential equation

$$\begin{cases} dx(t) + \partial^- \varphi(x(t))(dt) \ni dm(t) \\ x(0) = x_0, \end{cases}$$

where $m : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ is a continuous function and $\partial^- \varphi$ is the Fréchet subdifferential of a semiconvex function φ ; the domain of φ **can be non-convex**, but with some regularity of the boundary. The compactness of the map $m \mapsto x : C([0, T]; \mathbb{R}^d) \rightarrow C([0, T]; \mathbb{R}^d)$ and tightness criteria permit to pass from the deterministic case to a stochastic variational inequality

$$\begin{cases} X_t + K_t = \xi + \int_0^t F(s, X_s) ds + \int_0^t G(s, X_s) dB_s, & t \geq 0, \\ dK_t(\omega) \in \partial^- \varphi(X_t(\omega))(dt). \end{cases}$$

★ Variational Inequalities (VI)

(A) in convex domain:

(α) *with integrable inputs*: completely solved (J. Lions, H. Brezis, V. Barbu, etc. etc.)

(β) *with singular inputs*:

– *reflected problem*: Skorohod '61, '62, Tanaka '79

– *variational inequalities*: V.Barbu & Rascanu ('97)

(B) non-convex domain:

(α) *with integrable inputs* : Marino & Tosques ('90), Rossi & Savaré (2004)

(β) *with singular inputs*:

– *reflected problem*: P-L. Lions & Sznitman ('84), Saisho ('87)

– *variational inequalities*: **OPEN PROBLEM !!!**

★ Stochastic Variational Inequalities (SVI)

(A) **convex domain**: Rascanu ('81,'82, '96), Asiminoaei & Rascanu ('97), Bensoussan & Rascanu ('94, '97), Pardoux & Răşcanu ('98, '99), Es-Saky & Ouknine (2002), Ren & Hu (2007), Ren (2010), etc.

(B) non-convex domain:

(α) *reflected diffusions*: P-L.Lions & Sznitman ('84), Dupuis & Ishii ('93), Saisho ('87), etc.

(β) **SVI: OPEN PROBLEM !!!**

Outline

★ Preliminaries

★ Variational inequalities in non-convex domains

- ▲ *Classical Skorohod problem*
- ▲ *Generalized Skorohod problem*
- ▲ *Generalized Skorohod equations*

★ Stochastic variational inequalities in non-convex domains

- ▲ *Existence and uniqueness result*
- ▲ *Markov solutions of reflected SDEs.*

★ Annex

- ▲ *A forward stochastic inequality*
- ▲ *Tightness and some Itô's integral properties*

★ References

1 Preliminaries

Let E be a nonempty closed subset of \mathbb{R}^d and $N_E(x)$ be the closed external normal cone of E at $x \in E$ i.e.

$$N_E(x) \stackrel{\text{def}}{=} \left\{ u \in \mathbb{R}^d : \lim_{\delta \searrow 0} \frac{d_E(x + \delta u)}{\delta} = |u| \right\},$$

Definition 1 We say that E satisfies the **uniform exterior ball condition**, abbreviated $UEBC$, if

- $N_E(x) \neq \{0\}$ for all $x \in Bd(E)$,
- there exists $r_0 > 0$, such that $\forall x \in Bd(E)$ and $\forall u \in N_E(x)$, $|u| = r_0$ it holds:

$$d_E(x + u) = r_0 \quad (\text{or equivalent } B(x + u, r_0) \cap E = \emptyset)$$

(we say that E satisfies the $r_0 - UEBC$).

We have the following equivalence:

Lemma 1 The following assertions are equivalent:

- (i) E satisfies the uniform exterior ball condition;

(ii) $\exists \gamma \geq 0$ and for all $x \in Bd(E)$ there exists $\hat{x} \in \mathbb{R}^d \setminus \{0\}$ such that

$$\langle \hat{x}, y - x \rangle \leq \gamma |\hat{x}| |y - x|^2; \quad \forall y \in E.$$

Lemma 2 *If E satisfies the $r_0 - UEBC$, then*

$$N_E(x) = \left\{ \hat{x} : \langle \hat{x}, y - x \rangle \leq \frac{1}{2r_0} |\hat{x}| |y - x|^2; \quad \forall y \in E \right\},$$

Definition 2 *Let $x, v \in \mathbb{R}^d$, $r > 0$. We call the set*

$$\begin{aligned} D_x(v, r) &= \text{conv} \{x, \overline{B}(x + v, r)\} \\ &= \{x + t(u - x) : u \in \overline{B}(x + v, r), t \in [0, 1]\} \end{aligned}$$

a $(|v|, r)$ -drop (or a comet) of vertex x and running direction v .

Definition 3 The nonempty closed set $E \subset \mathbb{R}^d$ satisfies *UIDC (uniform interior (h_0, r_0) – drop condition)* if there exist $h_0, r_0 > 0$ and for all $x \in E$ there exists $v_x \in \mathbb{R}^d$, $|v_x| = h_0$, such that:

$$D_x(v_x, r_0) \subset E$$

Note that

- (UIDC) implies that $\text{int}(E) \neq \emptyset$;
- if E satisfies the uniform interior ball condition then E satisfies (UIDC).

Example 1 Let $E \subset \mathbb{R}^d$ defined by

$$\left\{ \begin{array}{l} (i) \quad E = \{x \in \mathbb{R}^d : \phi(x) \leq 0\}, \text{ where } \phi \in C_b^2(\mathbb{R}^d), \\ (ii) \quad \text{int}(E) = \{x \in \mathbb{R}^d : \phi(x) < 0\}, \\ (iii) \quad \text{Bd}(E) = \{x \in \mathbb{R}^d : \phi(x) = 0\} \text{ and } |\nabla\phi(x)| = 1 \forall x \in \text{Bd}(E). \end{array} \right.$$

The set E satisfies *UEBC (the uniform exterior ball condition)* and *UIDC (the uniform interior drop condition)*.

Proof. Note that at any boundary point $x \in \text{Bd}(E)$, $\nabla\phi(x)$ is a unit normal vector to the boundary, pointing towards the exterior of E . Hence for all $x \in \text{Bd}(E)$ we have $N_E(x) = \{\delta \nabla\phi(x) : \delta \geq 0\}$ and $N_{E^c}(x) = \{-\delta \nabla\phi(x) : \delta \geq 0\}$.

Since for all $y \in E$ and $x \in Bd(E)$,

$$\begin{aligned} \langle \nabla \phi(x), y - x \rangle &= \phi(y) - \phi(x) - \int_0^1 \langle \nabla \phi(x + \lambda(y - x)) - \nabla \phi(x), y - x \rangle d\lambda \\ &\leq M |y - x|^2 \end{aligned}$$

then by Lemma 1 E satisfies (UEBC).

If $y \in E^c$ and $x \in Bd(E)$, then $\phi(y) > 0$, $\phi(x) = 0$ and

$$\begin{aligned} \langle -\nabla \phi(x), y - x \rangle &= -\phi(y) + \int_0^1 \langle \nabla \phi(x + \lambda(y - x)) - \nabla \phi(x), y - x \rangle d\lambda \\ &\leq M |y - x|^2 \end{aligned}$$

that is E^c satisfies (UEBC) and consequently E satisfies (UIDC). ■

A function $\varphi : \mathbb{R}^d \rightarrow]-\infty, +\infty]$ is proper if

$$Dom(\varphi) \stackrel{def}{=} \{v \in \mathbb{R}^d : \varphi(v) < +\infty\} \neq \emptyset$$

and $Dom(\varphi)$ has no isolated point.

Definition 4 The Fréchet subdifferential of φ at $x \in \mathbb{R}^d$ is defined by $\partial^- \varphi(x) = \emptyset$, if $x \notin Dom(\varphi)$ and

for $x \in \text{Dom}(\varphi)$,

$$\partial^- \varphi(x) = \{\hat{x} \in \mathbb{R}^d : \liminf_{y \rightarrow x} \frac{\varphi(y) - \varphi(x) - \langle \hat{x}, y - x \rangle}{|y - x|} \geq 0\};$$

the domain of the multivalued operator $\partial^- \varphi : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is defined by

$$\text{Dom}(\partial^- \varphi) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d : \partial^- \varphi(x) \neq \emptyset\}$$

If E is a nonempty closed subset of \mathbb{R}^d and

$$\varphi(x) = I_E(x) = \begin{cases} 0, & \text{if } x \in E, \\ +\infty, & \text{if } x \notin E, \end{cases}$$

then φ is l.s.c. and

$$\partial^- I_E(x) = \{\hat{x} \in \mathbb{R}^d : \limsup_{y \rightarrow x, y \in E} \frac{\langle \hat{x}, y - x \rangle}{|y - x|} \leq 0\}$$

is the *Fréchet normal cone* at E in x . By a result of Colombo&Goncharov (2001) we have for any closed subset E of a Hilbert space

$$\partial^- I_E(x) = N_E(x)$$

Definition 5 $\varphi : \mathbb{R}^d \rightarrow]-\infty, +\infty]$ is a semiconvex function if

- (i) $\overline{\text{Dom}(\varphi)}$ satisfies the *UEBC* ;
- (ii) $\text{Dom}(\partial^- \varphi) \neq \emptyset$
- (iii) there exist $\gamma_1, \gamma_2 \geq 0$ such that for all $(x, \hat{x}) \in \partial^- \varphi, y \in \mathbb{R}^d$:

$$\langle \hat{x}, y - x \rangle + \varphi(x) \leq \varphi(y) + (\gamma_1 + \gamma_2 |\hat{x}|) |y - x|^2.$$

We also say that φ is a (γ_1, γ_2) –semiconvex function.

Remark 1

- If E satisfies r_0 – *UEBC* then $\varphi = I_E$ is (γ_1, γ_2) –semiconvex l.s.c. function for all $\gamma_1 \geq 0$ and $\gamma_2 \geq \frac{1}{2r_0}$.
- A convex function φ is (γ_1, γ_2) –semiconvex function for all $\gamma_1, \gamma_2 \geq 0$.
- If E is a closed bounded subset of \mathbb{R}^d satisfying the *UEBC* and $g \in C^2(\mathbb{R}^d)$ (or $g \in C(\mathbb{R}^d)$ is a convex function), then $f : \mathbb{R}^d \rightarrow]-\infty, +\infty]$, $f(x) = I_E(x) + g(x)$ is a l.s.c. semiconvex function.
- If $\varphi : \mathbb{R}^d \rightarrow]-\infty, +\infty]$ is a semiconvex function, then there exists $a \in \mathbb{R}$ such that

$$\varphi(y) + a|y|^2 + a \geq 0 \quad \text{for all } y \in \mathbb{R}^d.$$

2 Variational inequalities in non-convex domains

2.1 Classical Skorohod problem

Let $E \subset \mathbb{R}^d$ be a *non empty closed subset* of \mathbb{R}^d . We assume

(UEBC) E satisfies the uniform exterior ball condition

and

(UIDC) E satisfies the uniform interior drop condition

The *classical Skorohod problem* is to find a solution of the multivalued differential equation

$$\begin{cases} dx(t) + \partial^- I_E(x(t))(dt) \ni dm(t), & t \geq 0, \\ x(0) = x_0, \end{cases}$$

or equivalent

$$\begin{cases} x(t) + k(t) = x_0 + m(t), & t \geq 0, \\ dk(t) \in \partial^- I_E(x(t))(dt), \end{cases}$$

where

$$(H_0) \begin{cases} (i) & x_0 \in E \\ (ii) & m \in C(\mathbb{R}_+; \mathbb{R}^d), \quad m(0) = 0 \end{cases}$$

Definition 6 Let E satisfy the $r_0 - UEB$ condition. A pair (x, k) is a solution of the Skorohod problem (and we write $(x, k) = \mathcal{SP}(E; x_0, m)$) if

- ◆ $x, k : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ are continuous functions and
- ◆ for all $0 \leq s \leq t$:

$$\left\{ \begin{array}{l} j) \quad x(t) \in E, \\ jj) \quad k \in BV_{loc}([0, \infty[; \mathbb{R}^d), \quad k(0) = 0, \\ jjj) \quad x(t) + k(t) = x_0 + m(t), \\ jv) \quad \forall y \in C(\mathbb{R}_+; E) : \\ \qquad \int_s^t \langle y(r) - x(r), dk(r) \rangle \leq \frac{1}{2r_0} \int_s^t |y(r) - x(r)|^2 d\uparrow k\downarrow_r . \end{array} \right. \quad (1)$$

Theorem 1 Let $x_0 \in E$ and $m : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ a continuous function such that $m(0) = 0$. Under the assumptions (UEBC) and (UIDC) the Skorohod problem (1) has a unique solution. Moreover $(1 - jv) \Leftrightarrow (2) \Leftrightarrow (3)$, where

$$\left\{ \begin{array}{l} \exists \gamma > 0 \text{ such that } \forall y : [0, \infty[\rightarrow E \text{ continuous :} \\ \int_s^t \langle y(r) - x(r), dk(r) \rangle \leq \gamma \int_s^t |y(r) - x(r)|^2 d\uparrow k\downarrow_r . \end{array} \right. \quad (2)$$

and

$$\left\{ \begin{array}{l} (a) \quad \Downarrow k \Downarrow_t = \int_0^t \mathbf{1}_{x(s) \in Bd(E)} d \Downarrow k \Downarrow_s \quad , \\ (b) \quad k(t) = \int_0^t n_{x(s)} d \Downarrow k \Downarrow_s, \quad \text{where } n_{x(s)} \in N_E(x(s)) \\ \quad \quad \quad \text{and } |n_{x(s)}| = 1, \quad d \Downarrow k \Downarrow_s \text{ -a.e..} \end{array} \right. \quad (3)$$

2.2 Generalized Skorohod Problem

PROBLEM : $\exists?$ a pair (x, k) of continuous functions $x, k : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ solution of the **generalized Skorohod problem**

$$\left\{ \begin{array}{l} x(t) + k(t) = x_0 + m(t), \quad t \geq 0, \\ dk(t) \in \partial^- \varphi(x(t))(dt), \end{array} \right. \quad (4)$$

By $dk(t) \in \partial^- \varphi(x(t)) (dt)$ we understand

$$\begin{aligned}
 a) \quad & x \in C\left(\mathbb{R}_+; \overline{Dom(\varphi)}\right), \quad \varphi(x) \in L^1_{loc}(\mathbb{R}_+), \\
 b) \quad & k \in C\left(\mathbb{R}_+; \mathbb{R}^d\right) \cap BV_{loc}\left(\mathbb{R}_+; \mathbb{R}^d\right), \quad k(0) = 0, \\
 c) \quad & \forall 0 \leq s \leq t, \quad \forall y : [0, \infty[\rightarrow \mathbb{R}^d \text{ continuous :} \\
 & \int_s^t \langle y(r) - x(r), dk(r) \rangle + \int_s^t \varphi(x(r)) dr \\
 & \leq \int_s^t \varphi(y(r)) dr + \int_s^t |y(r) - x(r)|^2 (\gamma_1 dr + \gamma_2 d\uparrow k\downarrow_r)
 \end{aligned} \tag{5}$$

Assumptions:

(A1)

$$\begin{cases} (i) & x_0 \in \overline{Dom(\varphi)} \\ (ii) & m \in C([0, T]; \mathbb{R}^d), \quad m(0) = 0; \end{cases} \tag{6}$$

(A2)

$$\begin{aligned}
 (i) \quad & \varphi : \mathbb{R}^d \rightarrow]-\infty, +\infty] \text{ is proper l.s.c. } (\gamma_1, \gamma_2) - \text{semiconvex function;} \\
 (ii) \quad & |\varphi(x) - \varphi(y)| \leq L + L|x - y|, \quad \forall x, y \in Dom(\varphi) \\
 (iii) \quad & Dom(\varphi) \text{ satisfies uniform interior drop condition (UIDC);}
 \end{aligned} \tag{7}$$

Theorem 2 *Let $T > 0$ and the assumptions (A1) and (A2) be satisfied.*

I. $\exists C_T : C([0, T] : \mathbb{R}^d) \rightarrow \mathbb{R}_+$ which is bounded on compact subsets of $C([0, T] : \mathbb{R}^d)$ such that for any solution (x, k) is solution of

$$\begin{cases} x(t) + k(t) = x_0 + m(t), & t \geq 0, \\ dk(t) \in \partial^- \varphi(x(t))(dt), \end{cases} \quad (8)$$

(we write $(x, k) = \mathcal{SP}(\partial^- \varphi; x_0, m)$), then

$$\begin{aligned} (a) \quad & \uparrow k \downarrow_T = \|k\|_{BV([0, T]; \mathbb{R}^d)} \leq C_T(m), \\ (b) \quad & \|x\|_T = \|x\|_{C([0, T]; \mathbb{R}^d)} \leq |x_0| + C_T(m), \\ (c) \quad & |x(t) - x(s)| + \uparrow k \downarrow_t - \uparrow k \downarrow_s \leq C_T(m) \times \sqrt{\mu_m(t-s)}, \\ & \forall 0 \leq s \leq t \leq T. \end{aligned} \quad (9)$$

If moreover $\hat{m} \in C([0, T] : \mathbb{R}^d)$, $\hat{x}_0 \in \text{Dom}(\varphi)$ and $(\hat{x}, \hat{k}) = \mathcal{SP}(\partial^- \varphi; \hat{x}_0, \hat{m})$, then

$$\|x - \hat{x}\|_T + \|k - \hat{k}\|_T \leq \kappa_T(m, \hat{m}) \times \left[|x_0 - \hat{x}_0| + \sqrt{\|m - \hat{m}\|_T} \right], \quad (10)$$

II. *The problem (8) has a unique solution (x, k) .*

Proof. steps:

- we prove the estimates (9) and (10);

- the uniqueness follows from (10);
- if $m \in C^1(\mathbb{R}_+; \mathbb{R}^d)$ the existence follows from **Marino, A. & Tosques, M. : Some variational problems with lack of convexity**, Methods of Nonconvex Analysis, Proceedings from CIME, Varenna 1989 , Springer, Berlin (1990)
- if $m \in C(\mathbb{R}_+; \mathbb{R}^d)$ the existence follows from (9) by Arzela-Ascoli Theorem:
Let $m_n \in C([0, T]; \mathbb{R}^d)$, s.t. $m_n \rightarrow m$ in $C([0, T]; \mathbb{R}^d)$. Let $(x_n, k_n) = \mathcal{SP}(\partial^- \varphi; x_0, m_n)$

$$\begin{cases} x_n(t) + k_n(t) = x_0 + m_n(t), & t \geq 0, \\ dk_n(t) \in \partial^- \varphi(x_n(t))(dt), \end{cases}$$

– By (9)

(x_n, k_n) is bounded and uniformly equicontinuous sequence from $C([0, T]; \mathbb{R}^d) \times C([0, T]; \mathbb{R}^d)$.

– By Arzela-Ascoli Theorem

$$(x_n, k_n) \rightarrow (x, k) \quad \text{in } C([0, T]; \mathbb{R}^{2d}) \quad (\text{on a subsequence})$$

– By l.s.c. property of $\uparrow \cdot \downarrow : C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}$:

$$\uparrow k \downarrow_T \leq \liminf_{n \rightarrow +\infty} \uparrow k_n \downarrow_T \leq C < \infty.$$

– By Helly-Bray Theorem and the l.s.c. property of φ we can pass to $\liminf_{n \rightarrow +\infty}$ in

$$\begin{aligned} & \int_s^t \langle y(r) - x_n(r), dk_n(r) \rangle + \int_s^t \varphi(x_n(r)) dr \\ & \leq \int_s^t \varphi(y(r)) dr + \int_s^t |y(r) - x_n(r)|^2 (\gamma_1 dr + \gamma_2 d\uparrow k_n \downarrow_r). \end{aligned}$$

■

2.3 Generalized Skorohod equations

PROBLEM : $\exists?$ a pair (x, k) of continuous functions $x, k : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ solution of the **generalized Skorohod differential equation**

$$\begin{cases} x(t) + k(t) = x_0 + \int_0^t f(s, x(s)) ds + m(t), & t \geq 0, \\ dk(t) \in \partial^- \varphi(x(t))(dt), \end{cases} \quad (11)$$

where

(A3)

- ◆ $f(\cdot, \cdot) : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a *Carathéodory function*
- ◆ there exist $\mu \in L^1_{loc}(0, \infty)$, such that the *monotonicity condition*

$$\langle x - y, f(t, x) - f(t, y) \rangle \leq \mu(t) |x - y|^2, \quad \forall x, y \in \mathbb{R}^d$$

is satisfied *a.e.* $t \geq 0$

- ◆ *Boundedness* : for all $T \geq 0$,

$$\int_0^T f^\#(s) ds < \infty, \quad \text{where} \quad f^\#(t) \stackrel{\text{def}}{=} \sup \left\{ |f(t, u)| : u \in \overline{\text{Dom}(\varphi)} \right\}$$

Theorem 3 (Generalized Skorohod Equation) Under the assumptions (A1), (A2), (A3) the Eq. (11)

has a unique solution (x, k) .

Proof. The uniqueness is obtained in a standard manner. The existence is obtained passing to the limit (via Arzela-Ascoli Theorem) in a the approximating equation

$$\begin{cases} x_n(t) + k_n(t) = x_0 + \int_{0 \vee (t-1/n)}^{t-1/n} f(s, x_n(s)) ds + m(t), & t \geq 0, \\ dk_n(t) \in \partial^- \varphi(x_n(t))(dt). \end{cases}$$

■

3 Stochastic variational inequalities in non-convex domains

3.1 Existence and uniqueness result

Consider the following SVI (also called *generalized stochastic Skorohod equation*)

$$\begin{cases} X_t + K_t = \xi + \int_0^t F(s, X_s) ds + \int_0^t G(s, X_s) dB_s, & t \geq 0, \\ dK_t(\omega) \in \partial^- \varphi(X_t(\omega))(dt), \end{cases} \quad (12)$$

where we have

- ▲ $\{B_t : t \geq 0\}$ is a \mathbb{R}^k -valued Brownian motion with respect to the given stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$
- ▲ **Carathéodory conditions:** $F(\cdot, \cdot, \cdot) : \Omega \times [0, +\infty[\times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $G(\cdot, \cdot, \cdot) : \Omega \times [0, +\infty[\times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times k}$ are $(\mathcal{P}, \mathbb{R}^d)$ -Carathéodory functions that is

$$\begin{cases} (a) & F(\cdot, \cdot, x) \text{ and } G(\cdot, \cdot, x) \text{ are p.m.s.p., } \forall x \in \mathbb{R}^d; \\ (b) & F(\omega, t, \cdot) \text{ and } G(\omega, t, \cdot) \text{ are continuous function } d\mathbb{P} \otimes dt - a.e. \end{cases} \quad (13)$$

Denoting

$$\begin{aligned} F^\#(s) &= \sup \left\{ |F(t, u)| : u \in \overline{Dom(\varphi)} \right\}, \\ G^\#(s) &= \sup \left\{ |G(t, u)| : u \in \overline{Dom(\varphi)} \right\} \end{aligned}$$

we assume that the following are satisfied

- ▲ **Boundedness conditions :** for all $T \geq 0$:

$$\begin{cases} (a) & \int_0^T F^\#(s) ds < \infty, \mathbb{P} - \text{a.s.} \\ (b) & \int_0^T |G^\#(s)|^2 ds < \infty, \mathbb{P} - \text{a.s.} \end{cases} \quad (14)$$

- ▲ **Monotonicity and Lipschitz conditions :** there exist $\mu \in L^1_{loc}(0, \infty)$ and $\ell \in L^2_{loc}(0, \infty; \mathbb{R}_+)$,

such that $d\mathbb{P} \otimes dt - a.e. :$

$$\left\{ \begin{array}{l} (\mathbf{M}_F) \quad \text{Monotonicity condition :} \\ \quad \langle x - y, F(t, x) - F(t, y) \rangle \leq \mu(t) |x - y|^2, \quad \forall x, y \in \mathbb{R}^d, \\ (\mathbf{L}_G) \quad \text{Lipschitz condition :} \\ \quad |G(t, x) - G(t, y)| \leq \ell(t) |x - y|, \quad \forall x, y \in \mathbb{R}^d. \end{array} \right. \quad (15)$$

▲ $\partial^- \varphi$ is the Fréchet subdifferential of a *proper l.s.c. semiconvex function* $\varphi : \mathbb{R}^d \rightarrow]-\infty, +\infty]$ and $Dom(\varphi) \stackrel{def}{=} \{x \in \mathbb{R}^d : \varphi(x) < +\infty\}$ (in general a non-convex domain) has the property (UIDC) : *uniform interior drop condition* . Moreover of satisfies

$$|\varphi(x) - \varphi(y)| \leq L + L|x - y|, \quad \forall x, y \in Dom(\varphi).$$

Definition 7 (I) Given a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)_{t \geq 0}$ and a \mathbb{R}^k -valued \mathcal{F}_t -Brownian motion $\{B_t : t \geq 0\}$, a pair $(X, K) : \Omega \times [0, \infty[\rightarrow \mathbb{R}^d \times \mathbb{R}^d$ of continuous \mathcal{F}_t -progressively measurable stochastic

processes is a **strong solution** of the SDE (12) if $\mathbb{P} - a.s. \omega \in \Omega$:

$$\left\{ \begin{array}{l} j) \quad X_t \in \overline{Dom(\varphi)}, \quad \forall t \geq 0, \quad \varphi(X_\cdot) \in L_{loc}^1(0, \infty) \\ jj) \quad K_\cdot \in BV_{loc}(\mathbb{R}_+; \mathbb{R}^d), \quad K_0 = 0, \\ jjj) \quad X_t + K_t = \xi + \int_0^t F(s, X_s) ds + \int_0^t G(s, X_s) dB_s, \quad \forall t \geq 0, \\ jv) \quad \forall 0 \leq s \leq t, \quad \forall y : [0, \infty[\rightarrow \mathbb{R}^d \text{ continuous} : \\ \qquad \int_s^t \langle y(r) - X_r, dK_r \rangle + \int_s^t \varphi(X_r) dr \\ \qquad \leq \int_s^t \varphi(y(r)) dr + \int_s^t |y(r) - X_r|^2 (\gamma_1 dr + \gamma_2 d\uparrow K \downarrow_r) \end{array} \right. \quad (16)$$

that is

$$(X_\cdot(\omega), K_\cdot(\omega)) = \mathcal{SP}(\partial^- \varphi; \xi(\omega), M_\cdot(\omega)), \quad \mathbb{P} - a.s. \omega \in \Omega,$$

with

$$M_t = \int_0^t F(s, X_s) ds + \int_0^t G(s, X_s) dB_s.$$

(II) Let $F(\omega, t, x) = f(t, x)$, $G(\omega, t, x) = g(t, x)$ and $\xi(\omega) = x_0$ be independent of ω . If there exists a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)_{t \geq 0}$, a \mathbb{R}^k -valued \mathcal{F}_t -Brownian motion $\{B_t : t \geq 0\}$ and a pair

$(X., K.) : \Omega \times [0, \infty[\rightarrow \mathbb{R}^d \times \mathbb{R}^d$ of \mathcal{F}_t -p.m.c.s.p. such that

$$(X.(\omega), K.(\omega)) = \mathcal{SP}(\partial^- \varphi; x_0, M.(\omega)), \quad \mathbb{P} - a.s. \omega \in \Omega,$$

with

$$M_t = \int_0^t f(s, X_s) ds + \int_0^t g(s, X_s) dB_s,$$

then the collection $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t, X_t, K_t)_{t \geq 0}$ is called a **weak solution** of the SDE (12).

We first give a uniqueness result for strong solutions.

Proposition 1 (Pathwise uniqueness). Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t)_{t \geq 0}$ be given and the assumption (7) be satisfied. Assume that

$$F(\cdot, \cdot, \cdot) : \Omega \times [0, +\infty[\times \mathbb{R}^d \rightarrow \mathbb{R}^d \quad \text{and} \quad G(\cdot, \cdot, \cdot) : \Omega \times [0, +\infty[\times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times k}$$

satisfy (13), (14) and (15). Then the SDE (12) has at most one strong solution.

Proof. Let (X, K) and (\hat{X}, \hat{K}) be two solutions corresponding to ξ and respectively $\hat{\xi}$. Since

$$dK_t \in \partial^- \varphi(X_t)(dt) \quad \text{and} \quad d\hat{K}_t \in \partial^- \varphi(\hat{X}_t)(dt)$$

then for $p \geq 1$ and $\lambda > 0$

$$\begin{aligned} & \left\langle X_t - \hat{X}_t, (F(t, X_t) dt - dK_t) - (F(t, \hat{X}_t) dt - d\hat{K}_t) \right\rangle \\ & + \left(\frac{1}{2} m_p + 9p\lambda \right) \left| G(t, X_t) - G(t, \hat{X}_t) \right|^2 dt \leq |X_t - \hat{X}_t|^2 dV_t \end{aligned}$$

with

$$V_t = \int_0^t \left[\mu(s) ds + \left(\frac{1}{2} m_p + 9p\lambda \right) \ell^2(s) ds + 2\gamma_1 ds + \gamma_2 d\downarrow K\downarrow_s + \gamma_2 d\downarrow \hat{K}\downarrow_s \right].$$

Therefore, by Proposition 5 (Annex), we get

$$\mathbb{E} \left[1 \wedge \left\| e^{-V} (X - \hat{X}) \right\|_T^p \right] \leq C_{p,\lambda} \mathbb{E} \left[1 \wedge |\xi - \hat{\xi}|^p \right].$$

Hence the uniqueness follows. ■

Lemma 3 (The case of additive noise) *If $\xi \in L^0 \left(\Omega, \mathcal{F}_0, \mathbb{P}; \overline{Dom(\varphi)} \right)$ and $M \in S_d^0$, $M_0 = 0$, then there*

exists a unique pair $(X, K) \in S_d^0 \times S_d^0$ **strong solution** of the problem

$$\begin{cases} X_t(\omega) + K_t(\omega) = \xi(\omega) + \int_0^t F(\omega, s, X_s(\omega)) ds + M_t(\omega), & t \geq 0, \\ dK_t(\omega) \in \partial^- \varphi(X_t(\omega))(dt), \end{cases}$$

\mathbb{P} – a.s. $\omega \in \Omega$, *that is*

$$(X.(\omega), K.(\omega)) = \mathcal{SP}(\partial^- \varphi; \xi(\omega), M.(\omega)), \quad \mathbb{P} - a.s. \omega \in \Omega.$$

To study the general SDE (12) we consider only the case when F, G and ξ are independent of ω . Hence we have

$$\begin{cases} X_t + K_t = x_0 + \int_0^t f(s, X_s) ds + \int_0^t g(s, X_s) dB_s, & t \geq 0, \\ dK_t(\omega) \in \partial^- \varphi(X_t(\omega))(dt), \end{cases} \quad (17)$$

where $f(\cdot, \cdot) : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $g(\cdot, \cdot) : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times k}$.

As above, denote

$$\begin{aligned} f^\#(s) &\stackrel{def}{=} \sup \left\{ |f(t, u)| : u \in \overline{Dom(\varphi)} \right\}, \\ g^\#(s) &\stackrel{def}{=} \sup \left\{ |g(t, u)| : u \in \overline{Dom(\varphi)} \right\}. \end{aligned}$$

Proposition 2 *If $x_0 \in \overline{Dom(\varphi)}$ and*

$$\int_0^T \left[f^\#(s)^2 + g^\#(s)^4 \right] ds < \infty, \quad \forall T \geq 0.$$

then the problem (17) has a weak solution $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, X_t, K_t, B_t)_{t \geq 0}$.

Proof.

- Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t^B, B_t)_{t \geq 0}$ be a Brownian motion.
- (X^n, K^n) solution of the approximating problem

$$\begin{cases} X_t^n + K_t^n = x_0 + \int_0^t f\left(s, X_{s-\frac{1}{n}}^n\right) ds + \int_0^t g\left(s, X_{s-\frac{1}{n}}^n\right) dB_s, & t \geq 0, \\ dK_t^n(\omega) \in \partial^- \varphi(X_t^n(\omega))(dt). \end{cases} \quad (18)$$

- Let

$$M_t^n = \int_0^t f\left(s, X_{s-\frac{1}{n}}^n\right) ds + \int_0^t g\left(s, X_{s-\frac{1}{n}}^n\right) dB_s$$

Since

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \sup_{n \geq 1} \mathbb{E} \left[\sup_{0 \leq \theta \leq \varepsilon} |M_{t+\theta}^n - M_t^n|^4 \right] = 0$$

$\{M^n : n \geq 1\}$ is **tight** on $C(\mathbb{R}_+; \mathbb{R}^d)$.

- If $U^n = (X^n, K^n, \updownarrow K^n \updownarrow)$ then

$$\mathbf{m}_{U^n}(\varepsilon; [0, T]) \leq C \sqrt{\varepsilon + \mathbf{m}_{M^n}(\varepsilon)}, \quad \text{a.s.}$$

- $U^n = (X^n, K^n, \updownarrow K^n \updownarrow)$ is tight on $\mathbb{X} = C([0, T]; \mathbb{R}^{2d+1})$.
- By the Prohorov theorem

$$(X^n, K^n, \updownarrow K^n \updownarrow, B) \rightarrow (X, K, V, B) \quad \text{in law}$$

on $C([0, T]; \mathbb{R}^{2d+1+k})$

- By the Skorohod theorem, $\exists (\Omega, \mathcal{F}, \mathbb{P}), \exists (\bar{X}^n, \bar{K}^n, \bar{V}^n, \bar{B}^n), (\bar{X}, \bar{K}, \bar{V}, \bar{B})$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$

$$\mathcal{L}(\bar{X}^n, \bar{K}^n, \bar{V}^n, \bar{B}^n) = \mathcal{L}(X^n, K^n, \updownarrow K^n \updownarrow, B) \quad \text{and} \quad \mathcal{L}(\bar{X}, \bar{K}, \bar{V}, \bar{B}) = \mathcal{L}(X, K, V, B)$$

such that, in $C([0, T]; \mathbb{R}^{2d+1+k})$, as $n \rightarrow \infty$,

$$(\bar{X}^n, \bar{K}^n, \bar{V}^n, \bar{B}^n) \xrightarrow{\mathbb{P}\text{-a.s.}} (\bar{X}, \bar{K}, \bar{V}, \bar{B}).$$

- By Proposition 9, $\left(\bar{B}^n, \{\mathcal{F}_t^{\bar{X}^n, \bar{K}^n, \bar{V}^n, \bar{B}^n}\}\right)$, $n \geq 1$, and $\left(\bar{B}, \{\mathcal{F}_t^{\bar{X}, \bar{K}, \bar{V}, \bar{B}}\}\right)$ are \mathbb{R}^k –Brownian motion.
- For all $0 \leq s \leq t$, $\mathbb{P} - a.s.$

$$\begin{aligned} \bar{X}_0 = x_0, \quad \bar{K}_0 = 0, \quad \bar{X}_t \in E, \\ \updownarrow \bar{K} \updownarrow_t - \updownarrow \bar{K} \updownarrow_s \leq \bar{V}_t - \bar{V}_s \quad \text{and} \quad 0 = \bar{V}_0 \leq \bar{V}_s \leq \bar{V}_t \end{aligned} \quad (19)$$

- Since for all $0 \leq s < t$, $n \in \mathbb{N}^*$

$$\begin{aligned} \int_s^t \varphi(X_r^n) dr &\leq \int_s^t \varphi(y(r)) dr - \int_s^t \langle y(r) - X_r^n, dK_r^n \rangle \\ &\quad + \int_s^t |y(r) - X_r^n|^2 (\rho dr + \gamma d\updownarrow K^n \updownarrow_r), \quad a.s. \end{aligned}$$

then by Proposition 7 we infer

$$\begin{aligned} \int_s^t \varphi(\bar{X}_r) dr &\leq \int_s^t \varphi(y(r)) dr - \int_s^t \langle y(r) - \bar{X}_r, d\bar{K}_r \rangle \\ &\quad + \int_s^t |y(r) - \bar{X}_r|^2 (\rho dr + \gamma d\bar{V}_r). \end{aligned} \quad (20)$$

- Based on (19), (20) we have

$$d\bar{K}_r \in \partial^- \varphi(\bar{X}_r)(dr).$$

- $\mathbb{P} - a.s.$

$$\bar{X}_t + \bar{K}_t = x_0 + \int_0^t f(s, \bar{X}_s) ds + \int_0^t g(s, \bar{X}_s) d\bar{B}_s, \quad \forall t \in [0, T].$$

- Consequently $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \mathcal{F}_t^{\bar{B}, \bar{X}}, \bar{X}_t, \bar{K}_t, \bar{B}_t)_{t \geq 0}$ is a weak solution of the SDE (17). ■

Finally with similar arguments as used by Ykeda-Watanabe *weak existence + pathwise uniqueness* implies strong existence. Hence

Theorem 4 Under the assumptions of Proposition 2 the problem (17) has a unique strong solution $(X_t, K_t)_{t \geq 0}$.

3.2 Markov solutions of reflected SDEs.

In this subsection we study the Markov property of the solutions of reflected SDEs. when the set E has a particular form. Assume that

$$\left\{ \begin{array}{l} (i) \quad E = \{x \in \mathbb{R}^d : \phi(x) \leq 0\}, \text{ where } \phi \in C_b^2(\mathbb{R}^d), \\ (ii) \quad \text{int}(E) = \{x \in \mathbb{R}^d : \phi(x) < 0\}, \\ (iii) \quad \text{Bd}(E) = \{x \in \mathbb{R}^d : \phi(x) = 0\} \text{ and } |\nabla\phi(x)| = 1 \forall x \in \text{Bd}(E). \end{array} \right. \quad (21)$$

The set E satisfies (*UEBC: the uniform exterior ball condition*) and (*UIDC: the uniform interior drop condition*)

Let $t \in [0, \infty[$ and $\xi \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}; E)$ be fixed. Consider the equation

$$\begin{cases} X_s^{t,\xi} + K_s^{t,\xi} = \xi + \int_t^s f(r, X_r^{t,\xi}) dr + \int_t^s g(r, X_r^{t,\xi}) dB_r, & s \geq t, \\ dK_r(\omega) \in \partial^- I_E(X_r^{t,\xi}(\omega))(dr), \end{cases} \quad (22)$$

where $f(\cdot, \cdot) : [0, +\infty[\times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $g(\cdot, \cdot) : [0, +\infty[\times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times k}$ are continuous functions and satisfy: there exist $\mu \in \mathbb{R}$ and $\ell > 0$ such that for all $u, v \in \mathbb{R}^d$

$$\begin{aligned} (i) \quad & \langle u - v, f(t, u) - f(t, v) \rangle \leq \mu |u - v|^2 \\ (ii) \quad & |g(t, u) - g(t, v)| \leq \ell |u - v|. \end{aligned} \quad (23)$$

It follows from Theorem 4 and Theorem 1 that there exists a unique pair $(X^{t,\xi}, K^{t,\xi}) : \Omega \times [0, \infty[\rightarrow \mathbb{R}^d \times \mathbb{R}^d$ of continuous progressively measurable stochastic processes is a strong solution of the SDE

(22) that is $\mathbb{P} - a.s. \omega \in \Omega$:

$$\left\{ \begin{array}{l} (j) \quad X_s^{t,\xi} \in E \text{ and } X_{s \wedge t}^{t,\xi} = \xi \text{ for all } s \geq 0 \\ (jj) \quad K_s^{t,\xi} \in BV_{loc}([0, \infty[; \mathbb{R}^d), \quad K_s^{t,\xi} = 0 \text{ for all } 0 \leq s \leq t, \\ (jjj) \quad X_s^{t,\xi} + K_s^{t,\xi} = \xi + \int_t^s f(r, X_r^{t,\xi}) dr + \int_t^s g(r, X_r^{t,\xi}) dB_r, \quad \forall s \geq t, \\ (jv) \quad \uparrow K^{t,\xi} \downarrow_s = \int_t^s \mathbf{1}_{Bd(E)}(X_r^{t,\xi}) d \uparrow K^{t,\xi} \downarrow_r, \quad \forall s \geq t, \\ (v) \quad K_s^{t,\xi} = \int_t^s \nabla \phi(X_r^{t,\xi}) d \uparrow K^{t,\xi} \downarrow_r, \quad \forall s \geq t. \end{array} \right. \quad (24)$$

Remark that by Theorem 1 the conditions (jv) & (v) are equivalent to

$$\left\{ \begin{array}{l} \exists \gamma > 0 \text{ such that } \forall y : [0, \infty[\rightarrow E \text{ continuous :} \\ \langle y(r) - X_r^{t,\xi}, dK_r^{t,\xi} \rangle \leq \gamma |y(r) - X_r^{t,\xi}|^2 d \uparrow K^{t,\xi} \downarrow_r. \end{array} \right.$$

If $(X^{t,\xi}, K^{t,\xi})$ and $(X^{t,\eta}, K^{t,\eta})$ are two solutions, then from this last inequality it follows

$$\langle X_r^{t,\xi} - X_r^{t,\eta}, dK_r^{t,\xi} - dK_r^{t,\eta} \rangle + \gamma |X_r^{t,\xi} - X_r^{t,\eta}|^2 (d \uparrow K^{t,\xi} \downarrow_r + d \uparrow K^{t,\eta} \downarrow_r) \geq 0, \quad a.s. \quad (25)$$

Proposition 3 *Let the assumptions (21), (23) be satisfied and*

$$\sup_{(t,x) \in [0,T] \times E} [|f(t,x)| + |g(t,x)|] < \infty.$$

Then for all $p \geq 1$, $\lambda > 0$ and $s \geq t$,

$$\begin{aligned} (j) \quad & \mathbb{E}^{\mathcal{F}_t} \sup_{r \in [t,s]} |X_r^{t,\xi} - X_r^{t,\eta}|^p \leq C |\xi - \eta|^p \exp [C (s - t)], \text{ a.s.}, \\ (jj) \quad & \mathbb{E}^{\mathcal{F}_t} \sup_{r \in [t,s]} |K_r^{t,\xi}|^p \leq \mathbb{E}^{\mathcal{F}_t} \uparrow K^{t,\xi} \downarrow_s^p \leq C (1 + (s - t)^p), \text{ a.s.} \\ (jjj) \quad & \mathbb{E}^{\mathcal{F}_t} \sup_{r \in [t,s]} |X_r^{t,\xi}|^p \leq C (1 + (s - t)^p + |\xi|^p), \\ (jv) \quad & \mathbb{E}^{\mathcal{F}_t} e^{\lambda |K_s^{t,\xi}|} \leq \mathbb{E}^{\mathcal{F}_t} e^{\lambda \uparrow K^{t,\xi} \downarrow_s} \leq \exp \left\{ C\lambda + \left(C\lambda + \frac{C^2 \lambda^2}{2} \right) t \right\}, \text{ a.s..} \end{aligned} \tag{26}$$

where C is a constant independent of ξ, η, s, t and λ .

If the monotonicity condition (23-i) is replaced by

$$|f(t,u) - f(t,v)| \leq \mu |u - v|,$$

and $\phi \in C_b^3(\mathbb{R}^d)$, then we moreover have

$$\mathbb{E}^{\mathcal{F}_t} \sup_{r \in [t,s]} |K_r^{t,\xi} - K_r^{t,\eta}|^p + \mathbb{E}^{\mathcal{F}_t} \sup_{r \in [t,s]} |\uparrow K^{t,\xi} \downarrow_r - \uparrow K^{t,\eta} \downarrow_r|^p \leq C e^{C(s-t)} |\xi - \eta|^p, \tag{27}$$

where C is a constant independent of ξ, η, s, t .

Corollary 1 *Let the assumptions of Proposition (3) be satisfied and E a bounded set. Then for every $T > 0$ and $p \geq 1$ the mapping*

$$(t, x) \mapsto (X_s^{t,x}, K_s^{t,x}, \updownarrow K_s^{t,x} \updownarrow) : [0, T] \times E \rightarrow S_d^p[0, T] \times S_d^p[0, T] \times S_1^p[0, T]$$

is continuous. Moreover if $h_1, h_2 : [0, T] \times E \rightarrow \mathbb{R}$ are bounded continuous functions, then

$$(t, x) \mapsto \mathbb{E} \int_t^T h_1(s, X_s^{t,x}) ds + \mathbb{E} \int_t^T h_2(s, X_s^{t,x}) d \updownarrow K_s^{t,x} \updownarrow : [0, T] \times E \rightarrow \mathbb{R} \quad (28)$$

is continuous.

Moreover

Proposition 4 *The solution $\{X_s^{0,x} : s \geq 0\}$ of the SDE (22) is a strong Markov process with*

(i) the transition probability

$$P(t, x; s, G) = \mathbb{P}(X_s^{t,x} \in G)$$

for $t, s \geq 0$ and $G \in \mathcal{B}_d$;

(ii) the evolution operator $P_{t,s} : B_b(\mathbb{R}^d) \rightarrow B_b(\mathbb{R}^d)$, $0 \leq t \leq s$,

$$(P_{t,s}\psi)(x) = \mathbb{E}\psi(X_s^{t,x});$$

(iii) the *infinitesimal generator* \mathcal{A}_t satisfying

$$\mathcal{D} \stackrel{\text{def}}{=} \{\psi \in C_b^2(\mathbb{R}^d) : \nabla\psi(x) = 0 \text{ if } x \in Bd(E)\} \subset \text{Dom}(\mathcal{A}_t), \text{ for all } t \geq 0$$

and for $\psi \in \mathcal{D}$

$$\begin{aligned} \mathcal{A}_t(\psi)(x) &= \frac{1}{2} \mathbf{Tr} [g(t,x)g^*(t,x)\psi''_{xx}(x)] + \langle f(t,x), \psi'_x(x) \rangle \\ &= \frac{1}{2} \sum_{i,j=1}^d (gg^*)_{ij}(t,x) \frac{\partial^2 \psi(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d f_i(t,x) \frac{\partial \psi(x)}{\partial x_i}. \end{aligned}$$

4 Annex

4.1 A forward stochastic inequality

Let $X \in S_d^0$ be a continuous local semimartingale of the form

$$X_t = X_0 + K_t + \int_0^t G_s dB_s, \quad t \geq 0, \quad \mathbb{P} - a.s., \quad (29)$$

and for some $p \geq 1$ and $\lambda > 1$ as signed measures on $[0, \infty[$

$$dD_t + \langle X_t, dK_t \rangle + \left(\frac{1}{2}m_p + 9p\lambda \right) |G_t|^2 dt \leq \mathbf{1}_{p \geq 2} dR_t + |X_t| dN_t + |X_t|^2 dV_t,$$

where $m_p = 1 \vee (p - 1)$ and

- ◇ $K \in S_d^0$; $K. \in BV_{loc}([0, \infty[; \mathbb{R}^d)$, $K_0 = 0$, $\mathbb{P} - a.s.$;
- ◇ $G \in \Lambda_{d \times k}^0$;
- ◇ D, R, N are \mathcal{P} -m.i.c.s.p. $D_0 = R_0 = N_0 = V_0 = 0$;
- ◇ V is a \mathcal{P} -m.local b-v.c.s.p., $V_0 = 0$.

Proposition 5 *There exists a constant $C_{p,\lambda}$ such that: $\mathbb{P} - a.s.$, for all $0 \leq t \leq s$ and $\delta \geq 0$*

$$\begin{aligned} & \mathbb{E}^{\mathcal{F}_t} \frac{\|e^{-V} X\|_{[t,s]}^p}{(1+\delta\|e^{-V} X\|_{[t,s]}^2)^{p/2}} + \mathbb{E}^{\mathcal{F}_t} \int_t^s \frac{e^{-pV_r} |X_r|^{p-2}}{(1+\delta|e^{-V_r} X_r|^2)^{(p+2)/2}} dD_r + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s \frac{e^{-2V_r}}{(1+\delta|e^{-V_r} X_r|^2)^2} (dD_r + |G_r|^2 dr) \right)^{p/2} \\ & \leq C_{p,\lambda} \left[\frac{e^{-pV_t} |X_t|^p}{(1+\delta e^{-2V_t} |X_t|^2)^{p/2}} + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-2V_r} \mathbf{1}_{p \geq 2} dR_r \right)^{p/2} + \mathbb{E}^{\mathcal{F}_t} \left(\int_t^s e^{-V_r} dN_r \right)^p \right]. \end{aligned} \quad (30)$$

In particular if $R = N = 0$ then $\mathbb{P} - a.s.$

$$\mathbb{E}^{\mathcal{F}_t} \left[1 \wedge \|e^{-V} X\|_{[t,s]}^p \right] \leq C_{p,\lambda} \left[1 \wedge |e^{-V_t} X_t|^p \right],$$

4.2 Tightness and some Itô's integral properties

Proposition 6 Let $\{X_t^n : t \geq 0\}$, $n \in \mathbb{N}^*$, be a family of \mathbb{R}^d -valued continuous stochastic processes defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that for every $T \geq 0$, there exist $\alpha = \alpha_T > 0$ and $b = b_T \in C(\mathbb{R}_+)$ with $b(0) = 0$, (both independent of n) such that

$$(j) \quad \lim_{N \rightarrow \infty} \left[\sup_{n \in \mathbb{N}^*} \mathbb{P}(\{|X_0^n| \geq N\}) \right] = 0,$$

$$(jj) \quad \mathbb{E} \left[1 \wedge \sup_{0 \leq s \leq \varepsilon} |X_{t+s}^n - X_t^n|^\alpha \right] \leq \varepsilon \cdot b(\varepsilon), \quad \forall \varepsilon > 0, n \geq 1, t \in [0, T],$$

Then $\{X^n : n \in \mathbb{N}^*\}$ is tight in $C(\mathbb{R}_+; \mathbb{R}^d)$.

Proposition 7 $\varphi : \mathbb{R}^d \rightarrow]-\infty, +\infty]$ is a l.s.c. function. Let (X, K, V) , (X^n, K^n, V^n) , $n \in \mathbb{N}$, be $C([0, T]; \mathbb{R}^d)^2 \times C([0, T]; \mathbb{R})$ -valued random variables, such that

$$(X^n, K^n, V^n) \xrightarrow[n \rightarrow \infty]{law} (X, K, V)$$

and for all $0 \leq s < t$, and $n \in \mathbb{N}^*$,

$$\begin{aligned} \uparrow K^n \downarrow_t - \uparrow K^n \downarrow_s &\leq V_t^n - V_s^n \quad a.s. \\ \int_s^t \varphi(X_r^n) dr &\leq \int_s^t \langle X_r^n, dK_r^n \rangle, \quad a.s., \end{aligned}$$

then $\uparrow K \downarrow_t - \uparrow K \downarrow_s \leq V_t - V_s$ a.s. and

$$\int_s^t \varphi(X_r) dr \leq \int_s^t \langle X_r, dK_r \rangle, \quad a.s..$$

Proposition 8 Let $X, \hat{X} \in S_d^0[0, T]$ and B, \hat{B} be two \mathbb{R}^k -Brownian motions and $g : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times k}$ be a function satisfying

$$\begin{aligned} g(\cdot, y) &\text{ is measurable } \forall y \in \mathbb{R}^d, \quad \text{and} \\ y &\mapsto g(t, y) \text{ is continuous } dt - a.e.. \end{aligned}$$

If

$$\mathcal{L}(X, B) = \mathcal{L}(\hat{X}, \hat{B}), \quad \text{on } C(\mathbb{R}_+, \mathbb{R}^{d+k})$$

then

$$\mathcal{L} \left(X, B, \int_0^\cdot g(s, X_s) dB_s \right) = \mathcal{L} \left(\hat{X}, \hat{B}, \int_0^\cdot g(s, \hat{X}_s) d\hat{B}_s \right), \text{ on } C(\mathbb{R}_+, \mathbb{R}^{d+k+d}).$$

Proposition 9 Let $B, B^n, \bar{B}^n : \Omega \times [0, \infty[\rightarrow \mathbb{R}^k$ and $X, X^n, \bar{X}^n : \Omega \times [0, \infty[\rightarrow \mathbb{R}^{d \times k}$, be c.s.p. such that

- (i) B^n is $\mathcal{F}_t^{B^n, X^n}$ -Brownian motion $\forall n \geq 1$;
- (ii) $\mathcal{L}(X^n, B^n) = \mathcal{L}(\bar{X}^n, \bar{B}^n)$ on $C(\mathbb{R}_+, \mathbb{R}^{d \times k} \times \mathbb{R}^k)$ for all $n \geq 1$;
- (iii) $\int_0^T |\bar{X}_s^n - \bar{X}_s|^2 ds + \sup_{t \in [0, T]} |\bar{B}_t^n - \bar{B}_t|$ in probability, as $n \rightarrow \infty$, for all $T > 0$.

Then $(\bar{B}^n, \{\mathcal{F}_t^{\bar{B}^n, \bar{X}^n}\})$, $n \geq 1$, and $(\bar{B}, \{\mathcal{F}_t^{\bar{B}, \bar{X}}\})$ are Brownian motions and as $n \rightarrow \infty$

$$\sup_{t \in [0, T]} \left| \int_0^t \bar{X}_s^n d\bar{B}_s^n \longrightarrow \int_0^t \bar{X}_s d\bar{B}_s \right| \longrightarrow 0 \text{ in probability.}$$

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Thank you for your attention !