10-ème Colloque Franco - Roumain en Mathématiques Appliquées

A Non Equilibrium Steady State as an Adiabatic Limit

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Introduction

I shall present some results obtained these last three years in collaboration with Horia Cornean and Pierre Duclos,
results that allowed us to construct a Non-Equilibrium Steady State associated to an adiabatic modification of some thermodynamical parameter.

Reference

H. D. Cornean, P. Duclos, R. Purice:
Adiabatic non-equilibrium steady states in the partition free approach,
preprint arXiv:1006.4272

Plan of the talk

- The System
- The Main Result
- Main lines of the proofs

The System

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The configuration space:

A closed subset $\mathcal{L} \subset \mathbb{R}^{d+1}$ with $d \geq 0$ of the form

$$\mathcal{L} := \big[(-\infty, -a_0) \times \mathfrak{D}\big] \cup \mathcal{C}_0 \cup \big[(a_0, \infty) \times \mathfrak{D}\big], \qquad \text{for some } a_0 > 0.$$

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where:

1. $\mathfrak{D} \subset \mathbb{R}^d$ is a bounded simply connected open set awith regular boundary $\partial\mathfrak{D}$,
2. $\mathcal{C}_0 \subset \mathbb{R}^{d+1}$ is bounded and satisfies: $\left([-a_0, a_0] \times \mathbb{R}^d\right) \cap \mathcal{L} = \mathcal{C}_0$,
3. $\boldsymbol{\Sigma} := \partial\mathcal{L}$ is a regular surface in $\mathbb{R}^{d+1}$.

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so that

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$$\mathcal{L} = \mathcal{L}_- \cup \mathcal{C} \cup \mathcal{L}_+;$$
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We shall use the notations: $\mathcal{I}_- := (-\infty, -a)$ and $\mathcal{I}_+ := (a, \infty)$.

The One-body Dynamics

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We use the orthogonal decomposition:

- $\Pi_- : \mathcal{H} \to \mathcal{H}_- := L^2(\mathcal{I}_- \times \mathfrak{D})$,
- $\Pi_+ : \mathcal{H} \to \mathcal{H}_+ := L^2(\mathcal{I}_+ \times \mathfrak{D})$,
- $\Pi_0 : \mathcal{H} \to \mathcal{H}_0 := L^2(\mathcal{C})$,

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- The one-particle Hamiltonian is of the form:

$$H := -\Delta_D + \Pi_0 w \Pi_0$$

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- We denote by $R(z)$ the rezolvent of H.

The One-body Dynamics

Remark

Due to our hypothesis on w all the iterated commutators $\big[Q_1,\big[Q_1,\ldots\big[Q_1,\Pi_0 w\Pi_0\big]\ldots\big]\big] = 0$ and $\big[P_1,\big[P_1,\ldots\big[P_1,\Pi_0 w\Pi_0\big]\ldots\big]\big]$ are bounded operators in $\mathcal{H}$.

We denoted by Q_1 the operator of multiplication with the variable $x \in \mathbb{R}$ on $\mathcal{H}$ and by $P_1 := -i\partial_x$; we consider the factorization $\mathbb{R}^{d+1} \cong \mathbb{R} \times \mathbb{R}^d$.

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Hypothesis 1

We shall suppose that

$$\sigma_{pp}(H)=\sigma_{disc}(H),\quad \#\sigma_{pp}(H)<\infty.$$

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The time-dependent Hamiltonian

$$K_\eta(t) := H + V_\eta(t)$$

with domain

$$\mathbb{H}_D(\boldsymbol{\mathcal{L}}) := H_0^1(\boldsymbol{\mathcal{L}}) \cap H^2(\boldsymbol{\mathcal{L}})$$

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The Electric Bias

The non-homogenous evolution

For $-\infty < s \leq t \leq 0$, the unitary propagator $W_\eta(t,s)$ is the solution of the Cauchy problem:

$$i\partial_t W_\eta(t,s) = K_\eta(t) W_\eta(t,s)$$
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For any $\eta > 0$ the family $\{K_\eta(t)\}_{t\in\mathbb{R}}$ are self-adjoint operators in $\mathcal{H}$, having a common domain equal to $\mathbb{H}_D(\mathcal{L})$ and depending differentiable on $t \in \mathbb{R}$ with a bounded self-adjoint norm derivative

$$\partial_t K_\eta(t) = \eta\,\chi(\eta t) V.$$

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Thus it is described by a quasi-free state having as two-point function the usual Fermi-Dirac density at temperature T and chemical potential μ:

$$\rho_e(E) := \frac{1}{1 + e^{(E-\mu)/kT}}$$

applied to the total Hamiltonian $H = \big(-\Delta_D \big) \otimes 1 + \Pi_0 w \Pi_0$.

Initial state at $t = -\infty$: $\rho_e(H)$.

The State

The state at time $t \in \mathbb{R}_-$

$$\rho_\eta(t) := s-\lim_{s \searrow -\infty} W_\eta(t,s)\rho_e(H)W_\eta(t,s)^*.$$

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Proposition

The following limit exists

$$\Omega_\eta(t) := \underset{s \searrow -\infty}{n - \lim} \Omega_\eta(t,s).$$

but not uniformly with respect to η.

Proof of the Proposition:

Let us write the equation in integral form:

$$\Omega_\eta(t,s) = 1 + i\int_s^t \chi(\eta r)\Omega_\eta(t,r)e^{i(r-t)H}V(Q)e^{-i(r-t)H}\,dr$$

so that

$$\|\Omega_\eta(t,s_1) - \Omega_\eta(t,s_2)\| \le \int_{s_2}^{s_1} \chi(\eta r)\|V(Q)\|\,dr$$

verifying thus the Cauchy criterion for convergence with respect to the uniform topology on $\mathbb{B}[\mathcal{H}]$ due to the integrability of χ.

The State (useful formula)

Using the previous Proposition let us denote by:

$$\Omega_\eta := s - \lim_{s \searrow -\infty} \Omega_\eta(0, s).$$

so that it is easy to verify that

$$\rho_\eta(t) \; = \; W_\eta(t, 0)\Omega_\eta \rho_e(H)\Omega_\eta^* W_\eta(t, 0)^*.$$

The Main Result

The Adiabatic Limit

In oredr to study the limit for $\eta \searrow 0$
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decoupling the system at $x = \pm a$ by imposing Dirichelt conditions on $\mathcal{D}_{\pm}$.

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This trick will allow us to compare in a more precise way
the asymptotic evolution $W_{\eta}(t, s)$
with the one associated to the Hamiltonian H.

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$$\mathbb{H}_D(\mathcal{L}_\pm) := H^1_0(\mathcal{L}_\pm) \cap H^2(\mathcal{L}_\pm);\ \mathbb{H}_D(\mathcal{C}) := H^1_0(\mathcal{C}) \cap H^2(\mathcal{C})$$

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- $\overset{\circ}{\Delta}_D : \overset{\circ}{\mathbb{H}}_D(\mathcal{L}) \to L^2(\mathcal{L})$ the self-adjoint Laplace operator with Dirichlet conditions on $\partial\mathcal{L} \cup \mathcal{D}_- \cup \mathcal{D}_+$;

  we have $\overset{\circ}{\Delta}_D = \overset{\circ}{\Delta}_{D,-} \oplus \overset{\circ}{\Delta}_{D,0} \oplus \overset{\circ}{\Delta}_{D,+}$.

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  we have $\qquad \overset{\circ}{\Delta}_D = \overset{\circ}{\Delta}_{D,-} \oplus \overset{\circ}{\Delta}_{D,0} \oplus \overset{\circ}{\Delta}_{D,+}.$

We can write $\qquad \overset{\circ}{\Delta}_{D,\pm} = \mathfrak{l}_\pm \otimes 1 + 1 \otimes \mathfrak{L}_{\mathcal{D}}$ with:

- $\mathfrak{L}_{\mathcal{D}}$ the Laplacean on the bounded domain $\mathcal{D} \subset \mathbb{R}^d$ with Dirichlet conditions on the boundary $\partial\mathcal{D}$
- $\mathfrak{l}_\pm$ is (-1) times the operator of second derivative on $\mathcal{I}_\pm$ with Dirichlet condition at $\pm a$ resp.

The Decoupled System

The decoupled Hamiltonian

$$\overset{\circ}{H} := -\overset{\circ}{\Delta}_D + \Pi_0 w \Pi_0 : \quad \overset{\circ}{\mathbb{H}}_D(\boldsymbol{\mathcal{L}}) \longrightarrow \mathcal{H}.$$

(having Dirichlet conditions on $\partial\boldsymbol{\mathcal{L}} \cup \boldsymbol{\mathcal{D}}_- \cup \boldsymbol{\mathcal{D}}_+$).

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The decoupled Hamiltonian with bias

$$\overset{\circ}{K}_\eta(t) := \overset{\circ}{H} + V_\eta(t) = \overset{\circ}{H} + \chi_\eta(t) V : \quad \overset{\circ}{\mathbb{H}}_D(\boldsymbol{\mathcal{L}}) \longrightarrow \mathcal{H}.$$

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The decoupled non-homogeneous evolution

$\overset{\circ}{W}_\eta(t,s)$ defined as the solution of the following Cauchy problem:

$$\begin{cases} -i\partial_t \overset{\circ}{W}_\eta(t,s) = -\overset{\circ}{K}_\eta(t) \overset{\circ}{W}_\eta(t,s) \\ \overset{\circ}{W}_\eta(s,s) = 1 \end{cases} .$$

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- All the above operators commute with $\Pi_\pm$ and thus with V.
- We have the formula $\overset{\circ}{W}_\eta(t,s) =$

$$= e^{-i(t-s)\overset{\circ}{H}}\left[1 + \Pi_-\left(e^{iv_- \int_s^t \chi(\eta u)du}\right) + \Pi_+\left(e^{iv_+ \int_s^t \chi(\eta u)du}\right)\right]$$

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Hypothesis 2

a) $\sigma_{pp}(K_\kappa) \cap \sigma_c(K_\kappa) = \emptyset$,

b) $\#\sigma_{pp}(K_\kappa) = N < \infty \; \forall \kappa \in [0,1]$, $\sigma_{pp}(K_\kappa) = \{\varepsilon_j(\kappa)\}_{j=1}^N$.

c) $\min\limits_{\kappa \in [0,1]} \{\text{dist}\,(\sigma_{pp}(K(\kappa)), \sigma_{ac}(K(\kappa)))\} \geq d > 0$.

The Main Result

Theorem

1. The limit $\rho_\eta(t) := \underset{s \searrow -\infty}{s-\lim} \rho_\eta(t,s)$ exists for any $t \leq 0$, uniformly with respect to $\eta > 0$.
2. The wave operator Ξ_0 for $\{\overset{\circ}{K}_1, K_1\}$ exists and is complete.
3. The following limit exists, is independent of t and is given by

$$\rho_{ne} := \underset{\eta \searrow 0}{s-\lim} \rho_\eta(t) = \Xi_0 \rho(\overset{\circ}{H}) \Xi_0^* + \sum_{j=1}^{N} \rho(\varepsilon_j(0)) E_j(K_1),$$

where $\{\varepsilon_j(0)\}_{j=1}^N$ are the eigenvalues of $H = K_0$ in ascending order and $\{E_j(K_1)\}$ the eigenprojections for the Hamiltonian K_1 obtained by analytically continuing $\{E_j(K_\kappa)\}_{j=1}^N$ from $\kappa = 0$ to $\kappa = 1$.

Proof of the main result

Reduction to $t = 0$.

It is enough to prove our main formula giving the *non-equilibrium* state ρ_{ne} for $t = 0$.

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- Indeed, once this formula is proved for $t = 0$ it shows that the strong limit of $\rho_\eta(0)$ when $\eta \searrow 0$ is commuting with $K_1 = H + V$.
- It is elementary to check that $W_\eta(t,0)$ and $W_\eta(t,0)^*$ converge in norm to e^{-itK_1} and respectively e^{itK_1} when $\eta \searrow 0$ (with t fixed).

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- It is elementary to check that $W_\eta(t,0)$ and $W_\eta(t,0)^*$ converge in norm to e^{-itK_1} and respectively e^{itK_1} when $\eta \searrow 0$ (with t fixed).
- Since $e^{\pm itK_1}$ commute with $s - \lim_{\eta \searrow 0} \rho_\eta(0)$ it follows that the adiabatic strong limit of $\rho_\eta(t)$ must also exist satisfy the equation in our main Theorem.

Preliminary result

Proposition 0

Let $\kappa \in [0,1]$. There exists a discrete set $\mathfrak{N} \subset \mathbb{R}$ such that for any closed interval $I \subset \mathbb{R}_+ \setminus \mathfrak{N}$ we have the estimate (here $\langle x \rangle := \sqrt{x^2+1}$):

$$\sup_{z \in \{x+iy \mid x \in I, 0<y<\delta\}} \left\| e^{-\langle Q_1 \rangle} R_\kappa(z) e^{-\langle Q_1 \rangle} \right\| \leq C(I,\delta,\kappa) < \infty.$$

In particular, K_κ has no singular continuous spectrum.

Road Map of the Proof

- We have to study the existence of the following double limit:

$$\rho_{ne} = s - \lim_{\eta \searrow 0} \left\{ s - \lim_{s \to -\infty} W_\eta(0,s) \rho_e(H) W_\eta(0,s)^* \right\}.$$

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- Proposition 0 implies $\sigma_{sc}(K_\kappa) = \emptyset$ so that

$$\mathcal{H} \;=\; E_\infty(\kappa)\mathcal{H} \oplus \left\{ \bigoplus_{1 \le j \le N} E_j(\kappa)\mathcal{H} \right\},$$

where $E_\infty(\kappa) := E_{ac}(K_\kappa)$ and $E_j(\kappa) := E_j(K_\kappa)$.

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  where $E_\infty(\kappa) := E_{ac}(K_\kappa)$ and $E_j(\kappa) := E_j(K_\kappa)$.
- Then we can write

$$\rho_\eta(t, s) = W_\eta(t, s)\rho(H)E_\infty(0)W_\eta(t, s)^* +$$

$$+ \left\{ \sum_{1 \leq j \leq N} \rho(\varepsilon_j)W_\eta(t, s)E_j(0)W_\eta(t, s)^* \right\}.$$

Road Map of the Proof. *The discrete spectrum*

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Let us compute (for some $j \in \{1, \ldots, N\}$)

$$s - \lim_{\eta \searrow 0} \left[s - \lim_{s \searrow -\infty} W_\eta(s)^* E_j(0) W_\eta(s) \right].$$

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- As V is a bounded analytic perturbation of H, the map $[0,1] \ni \kappa \mapsto E_j(\kappa) \in \mathbb{B}(\mathcal{H})$ is Lipschitz continuous in norm.
- Thus there exists a constant $C > 0$ such that:

$$\left\| W_\eta^*(s) E_j(0) W_\eta(s) - W_\eta^*(s) E_j(\chi(\eta s)) W_\eta(s) \right\| \leq C \chi(\eta s), \quad s \leq 0.$$

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- Thus we can replace $E_j(0)$ with $E_j(\chi(\eta s))$ and the limit does not change.

Road Map of the Proof. *The discrete spectrum*

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Proposition D-1 (a weak version of the gap-less adiabatic theorem)

The following limit exists in the norm topology and we have the equality:

$$n-\lim_{\eta\searrow 0}\left[n-\lim_{s\searrow-\infty} W_\eta^*(s)E_j(\chi(\eta s))W_\eta(s)\right]=E_j(1).$$

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Corollary

$$n-\lim_{\eta\searrow 0}\left[n-\lim_{s\searrow -\infty} W_\eta^*(s)E_j(0)W_\eta(s)\right]=E_j(1)'$$

$$n-\lim_{\eta\searrow 0}\left[n-\lim_{s\searrow -\infty} e^{isH}E_{ac}(H)W_\eta(s)E_{pp}(K(1))\right]=0.$$

Road Map of the Proof. *The discrete spectrum*

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The first equality concludes the proof of the adiabatic limit for the discrete part of the spectrum (in the norm topology !).
The second one will play a role further.

Road Map of the Proof. *The continuous spectrum*

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We consider the term coming from the absolutely continuous spectrum:

$$s-\lim_{\eta\searrow 0}\left[s-\lim_{s\searrow -\infty} W_\eta^*(s)\rho_{eq}(H)E_{ac}(H)W_\eta(s)\right].$$

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The second equality in the previous Corollary implies:

$$s-\lim_{\eta\searrow 0}\left[s-\lim_{s\searrow-\infty} W_\eta^*(s)\rho_{eq}(H)E_{ac}(H)W_\eta(s)\right]$$
$$= s-\lim_{\eta\searrow 0}\left[s-\lim_{s\searrow-\infty} W_\eta^*(s)E_{ac}(H)e^{-isH}\rho_{eq}(H)e^{isH}E_{ac}(H)W_\eta(s)\right]$$
$$= s-\lim_{\eta\searrow 0}\left[s-\lim_{s\searrow-\infty} E_{\mathrm{ac}}(K(1))W_\eta^*(s)\rho_{eq}(H)E_{ac}(H)W_\eta(s)E_{ac}(K(1))\right],$$

provided that the last double strong limit exists.
Note that all errors go to zero in norm.

Road Map of the Proof. *The continuous spectrum*

Let us replace $\rho_{eq}(H)$ with $\rho_{eq}(\overset{\circ}{H})E_{\mathrm{ac}}(\overset{\circ}{H})$ in the previous formula.

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- We have the identity:

$$\{\rho_{eq}(\overset{\circ}{H})E_{\mathrm{ac}}(\overset{\circ}{H}) - \rho_{eq}(H)\}E_{\mathrm{ac}}(H)W_\eta(s)$$
$$= -\rho_{eq}(\overset{\circ}{H})E_{\mathrm{pp}}(\overset{\circ}{H})e^{-isH}E_{\mathrm{ac}}(H)\left\{e^{isH}W_\eta(s)\right\} +$$
$$+\{\rho_{eq}(\overset{\circ}{H}) - \rho_{eq}(H)\}e^{-isH}E_{\mathrm{ac}}(H)\left\{e^{isH}W_\eta(s)\right\}.$$

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- Both terms on the right hand side are of the form $Ce^{-isH}E_{\mathrm{ac}}(H)T$ with C compact (use C-1) and $T \in \mathbb{B}(\mathcal{H})$.

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- Both terms on the right hand side are of the form $Ce^{-isH}E_{\mathrm{ac}}(H)T$ with C compact (use C-1) and $T \in \mathbb{B}(\mathcal{H})$.
- RAGE Theorem implies the vanishing of the limit for $s \searrow -\infty$.

Road Map of the Proof. *The continuous spectrum*

Up to now we have shown that

$$s-\lim_{\eta\searrow 0}\left[s-\lim_{s\searrow -\infty} W_\eta^*(s)\rho_{eq}(H)E_{ac}(H)W_\eta(s)\right] =$$

$$= s-\lim_{\eta\searrow 0}\left\{s-\lim_{s\searrow -\infty} E_{\mathrm{ac}}(K_1)W_\eta^*(s)E_{\mathrm{ac}}(H)\rho_{eq}(\mathring{H})E_{\mathrm{ac}}(\mathring{H})E_{\mathrm{ac}}(H)W_\eta(s)E_{\mathrm{ac}}(K_1)\right\}.$$

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- Let us use formula: $\overset{\circ}{W}_\eta(t,s) =$

$$= e^{-i(t-s)\overset{\circ}{H}}\left[1+\Pi_-\left(e^{iv_-\int_s^t\chi(\eta u)du}\right)+\Pi_+\left(e^{iv_+\int_s^t\chi(\eta u)du}\right)\right]$$

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- in order to obtain

$$s-\lim_{\eta\searrow 0}\ s-\lim_{s\searrow-\infty}\Big[\ W_\eta^*(s)\rho_{eq}(H)E_{ac}(H)W_\eta(s)\ -$$
$$-\ E_{\mathrm{ac}}(K_1)W_\eta^*(s)E_{\mathrm{ac}}(H)\mathring{W}_\eta(s)\rho_{eq}(\mathring{H})E_{\mathrm{ac}}(\mathring{H})\mathring{W}_\eta^*(s)E_{\mathrm{ac}}(H)W_\eta(s)E_{\mathrm{ac}}(K_1)\Big]=0.$$

Road Map of the Proof. *The continuous spectrum*

Proposition C-2

For any $\eta > 0$ the following limits exist in the strong operator topology:

$$\Xi_\eta := \lim_{s \searrow -\infty} E_{\mathrm{ac}}(K_1) W_\eta^*(s) E_{\mathrm{ac}}(H) \overset{\circ}{W}_\eta(s) E_{\mathrm{ac}}(\overset{\circ}{H}).$$

Road Map of the Proof. *The continuous spectrum*

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- It follows that $\Xi_\eta^* = w - \lim_{s\searrow -\infty} E_{\mathrm{ac}}(\mathring{H}) \mathring{W}_\eta^*(s) E_{\mathrm{ac}}(H) W_\eta(s) E_{\mathrm{ac}}(K_1)$.

Road Map of the Proof. *The continuous spectrum*

Proposition C-2

For any $\eta > 0$ the following limits exist in the strong operator topology:

$$\Xi_\eta := \lim_{s \searrow -\infty} E_{\rm ac}(K_1) W_\eta^*(s) E_{\rm ac}(H) \mathring{W}_\eta(s) E_{\rm ac}(\mathring{H}).$$

- It follows that $\Xi_\eta^* = w - \lim_{s \searrow -\infty} E_{\rm ac}(\mathring{H}) \mathring{W}_\eta^*(s) E_{\rm ac}(H) W_\eta(s) E_{\rm ac}(K_1)$.
- Let us notice that

$$\begin{aligned}
&E_{\rm ac}(\mathring{H}) \mathring{W}_\eta^*(s) E_{\rm ac}(H) W_\eta(s) E_{\rm ac}(K_1) \\
&= E_{\rm ac}(\mathring{H}) \mathring{W}_\eta^*(s) e^{-isH} E_{\rm ac}(H) \left\{ e^{isH} W_\eta(s) \right\} E_{\rm ac}(K_1) \\
&= E_{\rm ac}(\mathring{H}) \left\{ \mathring{W}_\eta^*(s) e^{-is\mathring{H}} \right\} \left\{ e^{is\mathring{H}} e^{-isH} E_{\rm ac}(H) \right\} \left\{ e^{isH} W_\eta(s) \right\} E_{\rm ac}(K_1).
\end{aligned}$$

Road Map of the Proof. *The continuous spectrum*

- The factor $e^{isH}W_\eta(s)$ converges in norm to ω_η^* when $s \to -\infty$.

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Proposition C-3

The limit $\omega_- := \underset{s\searrow-\infty}{s-\lim} e^{is\overset{\circ}{H}}e^{-isH}E_{ac}(H)$ exists,

defines a unitary map $E_{ac}(H)\mathcal{H} \to E_{ac}(\overset{\circ}{H})\mathcal{H}$

and one has: $\underset{s\searrow-\infty}{s-\lim} e^{isH}e^{-is\overset{\circ}{H}}E_{ac}(\overset{\circ}{H}) = \omega_-^* = E_{ac}(H)\omega_-^*$.

Road Map of the Proof. *The continuous spectrum*

- The factor $e^{isH}W_\eta(s)$ converges in norm to ω_η^* when $s \to -\infty$.
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and one has: $\underset{s\searrow-\infty}{s-\lim} e^{isH}e^{-is\overset{\circ}{H}}E_{ac}(\overset{\circ}{H}) = \omega_-^* = E_{ac}(H)\omega_-^*$.

Conclusion

$$\Xi_\eta^* = \underset{s\searrow-\infty}{s-\lim}\, E_{\mathrm{ac}}(\overset{\circ}{H})\overset{\circ}{W}{}_\eta^*(s)E_{\mathrm{ac}}(H)W_\eta(s)E_{\mathrm{ac}}(K(1)),$$

Road Map of the Proof. *The continuous spectrum*

We have thus proved that the following limit exists and is given by

$$s-\lim_{s\searrow-\infty} W_\eta^*(s)\rho_{eq}(H)E_{ac}(H)W_\eta(s) \;=\; \Xi_\eta\rho(\mathring{H})\Xi_\eta^*.$$

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We have thus proved that the following limit exists and is given by

$$\underset{s\searrow -\infty}{s-\lim}\, W_\eta^*(s)\rho_{eq}(H)E_{ac}(H)W_\eta(s) \;=\; \Xi_\eta \rho(\overset{\circ}{H})\Xi_\eta^*.$$

We have to study now the strong limits of Ξ_η and Ξ_η^* when $\eta \searrow 0$.
In fact we shall prove that they are equal to the wave operator associated to the pair $(\overset{\circ}{K}_1, K_1)$ and to its adjoint (resp.).

Road Map of the Proof. *The continuous spectrum*

Proposition C-4

1. For any $\kappa \in [0,1]$ we have $E_{ac}(\overset{\circ}{K}_\kappa) = E_{ac}(\overset{\circ}{H})$.
2. The following limits exist:

$$\Xi_0 := \underset{s\searrow-\infty}{s-\lim} e^{isK(1)} e^{-is\overset{\circ}{K}_1} E_{ac}(\overset{\circ}{H}) = E_{ac}(K_1)\Xi_0 E_{ac}(\overset{\circ}{H});$$

$$\Xi_0^* = \underset{s\searrow-\infty}{s-\lim} e^{is\overset{\circ}{K}_1} e^{-isK_1} E_{ac}(K_1) = E_{ac}(\overset{\circ}{H})\Xi_0^* E_{ac}(K_1).$$

Thus the wave operators associated to the pair $(\overset{\circ}{K}_1, K_1)$ exist and are complete.

Road Map of the Proof. *The continuous spectrum*

Proposition C-5

Ξ_η has a strong limit when $\eta \searrow 0$ and moreover

$$s - \lim_{\eta \searrow 0} \Xi_\eta = \Xi_0.$$

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The completeness of Ξ_0 implies that $\Xi_0^* : E_{ac}(K(1))\mathcal{H} \to E_{ac}(\overset{\circ}{H})\mathcal{H}$ is unitary.

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Ξ_η has a strong limit when $\eta \searrow 0$ and moreover

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Thus, $\forall f \in E_{ac}(K(1))\mathcal{H}, \quad \left\| \left[\Xi_0^* - \Xi_\eta^*\right] f \right\|_{\mathcal{H}}^2 \leq$

$$\leq 2\|f\|_{\mathcal{H}}^2 - 2\Re\big(\,\langle \Xi_\eta \Xi_0^* f, f\rangle\,\big) \underset{\eta \searrow 0}{\rightarrow} 2\|f\|_{\mathcal{H}}^2 - 2\Re\big(\,\langle \Xi_0 \Xi_0^* f, f\rangle\,\big) = 0,$$

Road Map of the Proof. *The continuous spectrum*

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and thus for $\eta \searrow 0$, Ξ_η^* converges strongly to Ξ_0^* on $E_{ac}(K_1)\mathcal{H}$.

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Ξ_η has a strong limit when $\eta \searrow 0$ and moreover

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Thus, $\forall f \in E_{ac}(K(1))\mathcal{H}, \quad \big\| \left[\Xi_0^* - \Xi_\eta^*\right] f \big\|_{\mathcal{H}}^2 \leq$

$$\leq 2\|f\|_{\mathcal{H}}^2 - 2\Re\big(\langle \Xi_\eta \Xi_0^* f, f\rangle \big) \underset{\eta\searrow 0}{\to} 2\|f\|_{\mathcal{H}}^2 - 2\Re\big(\langle \Xi_0 \Xi_0^* f, f\rangle \big) = 0,$$

and thus for $\eta \searrow 0$, Ξ_η^* converges strongly to Ξ_0^* on $E_{ac}(K_1)\mathcal{H}$.

With this, the proof of the Theorem is concluded.

Proof of Proposition 0

- **A quadratic partition of unity:** $\chi_-^2 + \chi_0^2 + \chi_+^2 = 1$

$$\chi_\pm \in C^\infty(\mathbb{R}), \quad \chi_\pm(x) = 1 \text{ for } \pm x > 2a, \quad \chi_\pm(x) = 0 \text{ for } |x| < a,$$
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- **A cut-off Hamiltonian:** $K_{\kappa,L}$
  For $L > 2a$, obtained from K_κ on the region $\mathcal{L} \cap (-L, L)$ by imposing Dirichlet boundary conditions at $x = \pm L$.
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- **It is easy to verify that**

$$(K_\kappa - z)\widetilde{R}_\kappa(z) = \mathrm{Id} + X(z), \qquad \text{with } e^{\langle Q_1 \rangle} X(z) \in \mathbb{B}(\mathcal{H}).$$

Proof of Proposition 0

Since for large values of $\Im(z)$ the norm of $e^{\langle Q_1\rangle}X(z)$ tends to 0, we can write at least for those values of z that:

$$e^{-\langle Q_1\rangle}R_\kappa(z)e^{-\langle Q_1\rangle} = e^{-\langle Q_1\rangle}\widetilde{R}_\kappa(z)e^{-\langle Q_1\rangle}\left[1+e^{\langle Q_1\rangle}X(z)e^{-\langle Q_1\rangle}\right]^{-1}.$$

Now $e^{\langle Q_1\rangle}X(z)e^{-\langle Q_1\rangle}$ is compact and analytic in the upper complex plane, and has a bounded limit from above on any interval I which avoids the discrete set of thresholds in the leads and the discrete spectrum of $K_{\kappa,L}$.
Due to the exponential decaying on the right and the compactly supported cut-offs on the left, $e^{\langle Q_1\rangle}X(z)e^{-\langle Q_1\rangle}$ can be analytically continued to the set $\{x+iy|x\in I, -\delta<y<\delta\}$ for δ small enough.
Thus we can apply the analytic Fredholm alternative on this set and conclude that $\left[1+e^{\langle Q_1\rangle}X(z)e^{-\langle Q_1\rangle}\right]^{-1}$ exists on I outside a discrete set of points.

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$$|\varepsilon_1(\kappa) - \varepsilon_2(\kappa)| \geq C|\kappa - \kappa_0|^M, \quad \kappa \in [0, 1].$$

- their corresponding orthogonal projections $E_j(\kappa)$ can also be chosen to be real analytic on $[0, 1]$.

Proof of Proposition D-1

The Proposition follows easily from

Lemma

If $B_\eta(s) := W_\eta(s)^* E_1(\chi(\eta s)) W_\eta(s)$,

Then $B_\eta(0) = E_1(\chi(0)) = E_1(1)$

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- For the limit let us remember that there exists a unique critical time $t_0 < 0$ when $\chi(t_0) = \kappa_0$ (where two eigenvalues intersect).
  Fix some $0 < \delta < 1$ (to be chosen later in a more precise way).
  We split the negative semi-axis $\mathbb{R}_-$ in three parts:

$$\mathbb{R}_- = \left(-\infty, \frac{t_0 - \eta^\delta}{\eta}\right] \cup \left[\frac{t_0 - \eta^\delta}{\eta}, \frac{t_0 + \eta^\delta}{\eta}\right] \cup \left[\frac{t_0 + \eta^\delta}{\eta}, 0\right]. \qquad (1)$$

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- We write:

$$B_\eta\left(\frac{t_0+\eta^\delta}{\eta}\right) - B_\eta\left(\frac{t_0-\eta^\delta}{\eta}\right) = \int_{\frac{t_0-\eta^\delta}{\eta}}^{\frac{t_0+\eta^\delta}{\eta}} \partial_s B_\eta(s) ds.$$

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- This implies:

$$\left\| B_\eta\left(\frac{t_0+\eta^\delta}{\eta}\right) - B_\eta\left(\frac{t_0-\eta^\delta}{\eta}\right) \right\| \leq C\eta^\delta.$$

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- a positively oriented simple closed contour $\Gamma_\eta \subset \mathbb{C}$ (with interior domain U_η) such that:
    - $U_\eta \cap \sigma(K_{\chi(\eta s)}) = \varepsilon_1(\chi(\eta s))$,
    - $D_\eta := \sup_{s\in\mathbb{R}_-\setminus\left[\frac{t_0-\eta^\delta}{\eta},\frac{t_0+\eta^\delta}{\eta}\right]} \sup_{z\in\Gamma_\eta} \|(K_{\chi(\eta s)} - z)^{-1}\| \leq C\eta^{-M\delta}$,
    - the length of the contour Γ_η is of order $1/D_\eta$.

Proof of Proposition D-1 - *Far from the crossing*

Use the second order of the adiabatic expansion.

$$F_\eta(s) := B_\eta(s) + \eta\chi'(\eta s) W_\eta^*(s) Y(\chi(\eta s)) W_\eta(s),$$

$$Y(\chi(\eta s)) := -\frac{1}{2\pi}\oint_{\Gamma_\eta} dz \left(K(\chi(\eta s)) - z\right)^{-1} X(\chi(\eta s)) \left(K(\chi(\eta s)) - z\right)^{-1},$$

$$X(\kappa) := \left[E_1^\perp(\kappa)E_1'(\kappa)E_1(\kappa) - E_1(\kappa)E_1'(\kappa)E_1^\perp(\kappa)\right],$$

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- and the commutator equation $i[K(\kappa), Y(\kappa)] = -E_1'(\kappa)$, so that:

$$\partial_s F_\eta(s) =$$
$$-\eta^2 W_\eta^*(s)\left\{\partial_x \frac{\chi'(x)}{2\pi}\oint_{\Gamma_\eta} dz\,\big(K_{\chi(x)} - z\big)^{-1}X(\chi(x))\big(K_{\chi(x)} - z\big)^{-1}\right\}_{x=\eta s} W_\eta(s).$$

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Far from the crossing

Thus:

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Conclusion

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Conclusion

$$||B_\eta(s) - B_\eta(0)|| \leq C(\eta^\delta + \eta^{1-2M\delta}), \quad \forall s \leq 0.$$

Choose now any $\delta \in (0, 1/(2M))$ to conclude the proof.

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- By some classical arguments it is enough to prove that $R(z) - \mathring{R}(z)$ is compact for some $z \in \mathbb{C}$ with $|Im(z) \neq 0$.

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- Looking at H and $\overset{\circ}{H}$ as self-adjoint extensions of a given symetric operator, one proves that any $v \in (R(z) - \overset{\circ}{R}(z))\mathcal{H}$ satisfies the following equation on $\mathcal{L}_- \cup \mathcal{L}_+$: $-\overset{\circ}{\Delta}_{D,\pm} u_\pm = z u_\pm$.

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Lemma *(Exponential Decay)*

Let $z \in \mathbb{C} \setminus [0, \infty)$. There exists $\gamma_0(z) > 0$ such that for $\gamma_\pm \in (0, \gamma_0(z))$, we have: $$\left\| e^{\pm\gamma_\pm Q_1} \Pi_\pm (R(z) - \mathring{R}(z)) e^{\pm\gamma_\pm Q_1} \Pi_\pm \right\| \leq c,$$
and for $\Psi_\alpha(x) := e^{\alpha\sqrt{x^2+1}}$ $(\alpha \in (0, \gamma_0(z)))$ we have:

$$\left\| \Psi_\alpha(Q_1) \big(R(z) - \mathring{R}(z)\big) \Psi_\alpha(Q_1) \right\| \leq c.$$

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Thus $\big(R(z) - \overset{\circ}{R}(z)\big)\mathcal{H} \subset H^1_0(\mathcal{L})$.

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Thus $\big(R(z) - \overset{\circ}{R}(z)\big)\mathcal{H} \subset H_0^1(\boldsymbol{\mathcal{L}})$.

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Thus, the compactness of Sobolev embeddings for compact domains implies that

$$R(z) - \overset{\circ}{R}(z) \text{ is a compact operator}$$

for any $z \in \mathbb{C} \setminus \mathbb{R}$.

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$$R_\kappa(z)^p - \mathring{R}_\kappa(z)^p = \sum_{0\leq j\leq p-1} R_\kappa(z)^j \big(R_\kappa(z) - \mathring{R}_\kappa(z)\big) \mathring{R}_\kappa(z)^{p-1-j} =$$

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Moreover:

$$R_\kappa(z)^{k+l}\Psi_\alpha(Q_1)^{-1} =$$

$$\Psi_{\alpha/2}(Q_1)^{-1}\big[\Psi_{\alpha/2}(Q_1)R_\kappa(z)^k\Psi_{\alpha/2}(Q_1)^{-1}\big]\Psi_{\alpha/2}(Q_1)^{-1}\big[\Psi_\alpha(Q_1)R_\kappa(z)^l\Psi_\alpha(Q_1)^{-1}\big].$$

Proof of Propositions C-2, C-3

Lemma

Fix $\kappa \in [0,1]$ and $z \in \mathbb{C} \setminus [0,\infty)$.
Then there exist $k_d \in \mathbb{N}$ large enough and $\alpha(z) > 0$ small enough, such that for any $k \geq k_d$ we have that $F(Q_1)\mathring{R}(z)^k$, $F(Q_1)R_\kappa(z)^k$ and $F(Q_1)\Psi_\alpha(Q_1)\mathring{R}(z)^k\Psi_\alpha(Q_1)^{-1}$, $F(Q_1)\Psi_\alpha(Q_1)R_\kappa(z)^k\Psi_\alpha(Q_1)^{-1}$ with $|\alpha| < \alpha(z)$ are Hilbert-Schmidt operators on $\mathcal{H}$ for any measurable function $F \in L^2(\mathbb{R})$.

Proof of Propositions C-2, C-3

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$$\overset{\circ}{R}_\kappa(z)^k\mathcal{H} \subset \left(H^{2k}(\mathcal{L}_-)\bigcap H^1_0(\mathcal{L}_-)\right)\oplus\left(H^{2k}(\mathcal{C})\bigcap H^1_0(\mathcal{C})\right)\oplus\left(H^{2k}(\mathcal{L}_+)\bigcap H^1_0(\mathcal{L}_+)\right)$$

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- and taking k sufficiently large the Sobolev embeding Theorem implies $\overset{\circ}{R}_\kappa(z)^k\mathcal{H} \subset BC(\mathcal{L})$.

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Remark

If $T\, L^2(\mathcal{L}) \subset BC(\mathcal{L})$ then:

- T has an integral kernel K_T such that $\sup_{x\in\mathcal{L}} \int_{\mathcal{L}} |K_T(x,y)|^2 dy < \infty$
- $F(Q)T$ is Hilbert-Schmidt for any $F \in L^2(\mathcal{L})$.

Proof of Propositions C-2, C-3

- We notice that $\Psi_\alpha(Q_1)\overset{\circ}{H}\Psi_\alpha(Q_1)^{-1} = \overset{\circ}{H} + T_\alpha$, where T_α is a first order differential operator which has the following mapping property:

$$T_\alpha : H^k(\mathcal{L}_-) \oplus H^k(\mathcal{C}) \oplus H^k(\mathcal{L}_+) \longrightarrow H^{k-1}(\mathcal{L}_-) \oplus H^{k-1}(\mathcal{C}) \oplus H^{k-1}(\mathcal{L}_+).$$

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- A similar result is obtained for H and R by working with a slight modification of the weight Ψ that is constant on a neighbourhood of $\{\pm a\}$.

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Proof of Proposition C-4

- First let us notice that using RAGE Theorem:

$$\lim_{s \searrow -\infty} \left[E_{\mathrm{ac}}(K_1) W_\eta^*(s) E_{\mathrm{ac}}(H) \mathring{W}_\eta(s) E_{\mathrm{ac}}(\mathring{H}) - E_{\mathrm{ac}}(K_1) W_\eta^*(s) \mathring{W}_\eta(s) E_{\mathrm{ac}}(\mathring{H}) \right] = 0.$$

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- For $\delta > 0$ let $\mathcal{V}_\delta$ be the set of vectors $f \in \mathcal{H}_{ac}(\overset{\circ}{H})$ with compact spectral support with respect to $\overset{\circ}{H}$ at distance larger than δ from all thresholds.
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- $f \in \mathcal{H}_{ac}(\mathring{H})$ is of the form $(f_-, f_+) \in \mathcal{H}_- \oplus \mathcal{H}_+$.
- Suppose $f \in \mathcal{H}_+ \cap \mathcal{V}_\delta$. Then:

$$f(x, \mathbf{x}_\perp) = \sum_{n=1}^{N} w_n(\mathbf{x}_\perp)\int_{\mathbb{R}} \sin[k(x-a)]f_n(k)dk$$

with $w_n \in L^2(\mathcal{D})$ and f_n with compact support at distanc at least δ from the thresholds.

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- Let us write: $\Xi_\eta(s)f := \Phi_\eta(s) - \Psi_\eta(s)$

$$\Phi_\eta(s) := E_{\mathrm{ac}}(K_1)W_\eta^*(s)\big(K_{\chi(\eta s)}+1\big)^{-1}\big(\mathring{K}_{\chi(\eta s)}+1\big)^{-1}\mathring{W}_\eta(s)\big(\mathring{K}_{\chi(\eta s)}+1\big)^2 f$$

$$\Psi_\eta(s) := E_{\mathrm{ac}}(K_1)W_\eta^*(s)\left[\big(K_{\chi(\eta s)}+1\big)^{-1} - \big(\mathring{K}_{\chi(\eta s)}+1\big)^{-1}\right]\mathring{W}_\eta(s)\big(\mathring{K}_{\chi(\eta s)}+1\big)f.$$

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$$\begin{aligned}\Phi_\eta(s) &:= E_{\mathrm{ac}}(K_1)W^*_\eta(s)\big(K_{\chi(\eta s)}+1\big)^{-1}\big(\mathring{K}_{\chi(\eta s)}+1\big)^{-1}\mathring{W}_\eta(s)\big(\mathring{K}_{\chi(\eta s)}+1\big)^2 f\\ \Psi_\eta(s) &:= E_{\mathrm{ac}}(K_1)W^*_\eta(s)\left[\big(K_{\chi(\eta s)}+1\big)^{-1} - \big(\mathring{K}_{\chi(\eta s)}+1\big)^{-1}\right]\mathring{W}_\eta(s)\big(\mathring{K}_{\chi(\eta s)}+1\big)f.\end{aligned}$$

- $\lim_{s\searrow-\infty}\Psi_\eta(s)=0$ because the difference of resolvents provides the exponential localization near the sample and the adiabatic decoupled free evolution decays with s (see the formula above).

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- A direct computation using estimations similar to those above gives:

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- Thus $\lim_{s\to-\infty} \Phi_\eta(s)$ exists and equals:

$$\Xi_\eta f = \Phi_\eta(0)$$
$$-i\int_{-\infty}^{0} E_{\mathrm{ac}}(K_1)W_\eta^*(s)\left[\left(K(\chi(\eta s))+1\right)^{-1} - \left(\mathring{K}(\chi(\eta s))+1\right)^{-1}\right]\mathring{W}_\eta(s)\left(\mathring{K}(\chi(\eta s))+1\right)^2 f ds$$
$$-\int_{-\infty}^{0} \eta\chi'(\eta s)E_{\mathrm{ac}}(K_1)W_\eta^*(s)\left(K(\chi(\eta s))+1\right)^{-1}$$
$$\cdot V\left[\left(K(\chi(\eta s))+1\right)^{-1} - \left(\mathring{K}(\chi(\eta s))+1\right)^{-1}\right]\mathring{W}_\eta(s)\left(\mathring{K}(\chi(\eta s))+1\right) f ds.$$

Proof of Proposition C-5

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- First, let us compute the limit $\eta \searrow 0$ in the above formula. Use the Lebesgue dominated convergence theorem to obtain:

$$\lim_{\eta \searrow 0} \Xi_\eta f = E_{\mathrm{ac}}(K_1)\big(K(1)+1\big)^{-1}\big(\overset{\circ}{K}(1)+1\big)f$$
$$- i \int_{-\infty}^{0} E_{\mathrm{ac}}(K_1)e^{isK(1)}\left[\big(K(1)+1\big)^{-1} - \big(\overset{\circ}{K}(1)+1\big)^{-1}\right] e^{-is\overset{\circ}{K}(1)}\big(\overset{\circ}{K}(1)+1\big)^2 f ds.$$

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  - Using the previous propagation estimates which were shown to be uniform in η, we can repeat the same argument as in Proposition C-4 but with $\eta = 0$ from the beginning.
    This will give a formula for $\Xi_0 f$ which will coincide with the one above.

Thank you for your attention !