A new result of exact controllability for the magnetohydrodynamic equations

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1. The control action is distributed in the entire domain Ω

Let Ω be a bounded connected open set in $\mathbb{R}^n$ and let $T > 0$. Consider

(1)
$$\begin{array}{ll} \dfrac{\partial y}{\partial t} - \Delta y = u & \text{in } \Omega \times (0,T), \\ y = 0 & \text{on } \partial\Omega \times (0,T), \\ y(\cdot,0) = y_0 & \text{in } \Omega. \end{array}$$

Problem 1

Let y_0 and y_T be given states. Find u such that the corresponding solution y of (1) also satisfies

$$y(\cdot,T) = y_T.$$

Assume that $y_0, y_T \in H^2(\Omega) \cap H_0^1(\Omega)$.
Solution to Problem 1:

(2)
$$y = \frac{T-t}{T} y_0 + \frac{t}{T} y_T \quad (y(\cdot,T) = y_T),$$

(3)
$$u = \frac{1}{T}(y_T - y_0) - \frac{T-t}{T}\Delta y_0 + \frac{t}{T}\Delta y_T.$$

2. The control action is distributed in an arbitrary small subregion ω of Ω

Let ω be an open subset of Ω. Consider

(4)
$$\begin{array}{ll} \dfrac{\partial y}{\partial t} - \Delta y = \chi_\omega u & \text{in } \Omega \times (0,T), \\ y = 0 & \text{on } \partial\Omega \times (0,T), \\ y(\cdot,0) = y_0 & \text{in } \Omega, \end{array}$$

where

$$\chi_\omega(x) = \begin{cases} 1 \text{ for } x \in \omega, \\ 0 \text{ for } x \in \Omega \setminus \omega. \end{cases}$$

Let $\widetilde{y}$ satisfy

(5)
$$\begin{array}{ll} \dfrac{\partial \widetilde{y}}{\partial t} - \Delta \widetilde{y} = 0 & \text{in } \Omega \times (0,T), \\ \widetilde{y} = 0 & \text{on } \partial\Omega \times (0,T). \end{array}$$

Problem 2

Find $u \in L^2(\Omega \times (0,T))$ such that the corresponding weak solution y of (4) also satisfies

(6)
$$y(\cdot,T) = \widetilde{y}(\cdot,T).$$

II. THE RELATIONSHIP BETWEEN CONTROLLABILITY AND OBSERVABILITY

Consider the (homogeneous) adjoint of equation (4):

$$\begin{aligned} &\frac{\partial z}{\partial t} + \Delta z = 0 && \text{in } \Omega \times (0,T), \\ &z = 0 && \text{on } \partial\Omega \times (0,T). \end{aligned} \tag{7}$$

Theorem 1

If the controlled heat equation (4) is globally exactly controllable, then there exists a constant $c > 0$ such that

$$\int_\Omega z^2(x,0)dx \le c \int_0^T \int_\omega z^2 dxdt \tag{8}$$

for all the solutions z of (7).

Theorem 2

If there exists a constant $c > 0$ such that inequality (8) holds for all the solutions z of (7), then the controlled heat equation (4) is globally exactly controllable.

III. THE OBSERVABILITY INEQUALITY AND CARLEMAN ESTIMATES

Let z be an arbitrary solution of the adjoint heat equation (7).
The energy estimate for z:

$$\begin{aligned}&\int_\Omega z^2(x,0)dx + \int_{\Omega\times(0,T)} |\nabla z(x,t)|^2 dxdt \\ &\leq c\int_\Omega z^2(x,T)dx.\end{aligned} \tag{9}$$

Weighted variant for (9).
Let $\rho \in C([0,T]) \cap C^1([0,T))$ with $\rho'(0) \neq 0$ and $\rho(T) = 0$. Set

$$w = \rho z.$$

We have

$$\begin{aligned}&\frac{\partial w}{\partial t} + \Delta w = \rho' z && \text{in } Q = \Omega \times (0,T),\\ &w = 0 && \text{on } \Sigma = \partial\Omega \times (0,T),\\ &w(\cdot,T) = 0 && \text{in } \Omega.\end{aligned}$$

The energy estimate for w:

$$\begin{aligned}&\int_\Omega \rho^2(0) z^2(x,0)dx + \int_Q \rho^2(t) z^2(x,t)dxdt \\ &\leq c\int_Q (\rho'(t))^2 z^2(x,t)dxdt.\end{aligned} \tag{10}$$

Take

$$\rho(t) = e^{\alpha(t)}, \qquad \alpha(t) = -\frac{1}{T-t}. \tag{11}$$

Inequality (10) becomes

$$\begin{aligned} &\int_\Omega e^{2\alpha(0)} z^2(x,0)dx + \int_Q e^{2\alpha} z^2 dxdt \\ &\leq c \int_Q e^{2\alpha}(\alpha')^2 z^2 dxdt = c \int_Q e^{2\alpha} \alpha^4 z^2 dxdt. \end{aligned} \tag{12}$$

Take

$$\begin{aligned} \rho(x,t) &= e^{s\alpha(x,t)}, \\ \alpha(x,t) &= -\frac{\beta(x)}{t(T-t)}, \\ \beta(x) &= \gamma - e^{\lambda\psi(x)} \end{aligned} \tag{13}$$

with $\psi = 0$ on $\partial\Omega$ and $\nabla\psi \neq 0$ in $\overline{\Omega} \setminus \omega$. We can obtain

$$\begin{aligned} &\int_Q e^{2s\alpha} \varphi |\nabla z|^2 dxdt + s^2 \int_Q e^{2s\alpha} \varphi^3 z^2 dxdt \\ &\leq cs^2 \int_{Q_\omega} e^{2s\alpha} \varphi^3 z^2 dxdt \end{aligned} \tag{14}$$

for s and λ large enough, where $Q = \Omega \times (0,T)$, $Q_\omega = \omega \times (0,T)$, and

$$\varphi = \frac{e^{\lambda\psi}}{t(T-t)}, \text{ with } \psi \text{ in (13).} \tag{15}$$

IV. THE CONTROLLABILITY OF THE NAVIER–STOKES EQUATIONS

Consider

$$(16)\qquad \begin{cases} \dfrac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y + \nabla p = f + \chi_\omega u & \text{in } Q = \Omega\times(0,T), \\ \text{div } y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma = \partial\Omega\times(0,T), \\ y(\cdot,0) = y_0 & \text{in } \Omega. \end{cases}$$

Let $\widetilde{y}$ satisfy

$$(17)\qquad \begin{cases} \dfrac{\partial \widetilde{y}}{\partial t} - \nu\Delta \widetilde{y} + (\widetilde{y}\cdot\nabla)\widetilde{y} + \nabla \widetilde{p} = f & \text{in } Q, \\ \text{div } \widetilde{y} = 0 & \text{in } Q, \\ \widetilde{y} = 0 & \text{on } \Sigma. \end{cases}$$

Problem 3.

Find $u \in (L^2(Q))^n$ and a corresponding weak solution y of system (16) which also satisfies

$$(18)\qquad y(\cdot,T) = \widetilde{y}(\cdot,T) \text{ a.e. in } \Omega.$$

Set

$$H = \{y \in (L^2(\Omega))^n : \text{div } y = 0 \text{ in } \Omega,\ y\cdot N = 0 \text{ on } \partial\Omega\}.$$

Theorem 3 (E. Fernandez–Cara, S. Guerrero, O. Imanuvilov, J.P. Puel)

Let $n = 2$ *or* 3 *and* $f \in (L^2(Q))$*. If* $\widetilde{y}$ *is a weak solution of* (17) *which satisfies*

$$\widetilde{y} \in (L^\infty(Q))^n \text{ and } \frac{\partial \widetilde{y}}{\partial t} \in L^2(0,T;\ (L^\infty(\Omega))^n), \tag{19}$$

then there exists $r > 0$ *such that, for any* $y_0 \in H \cap (L^{2n-2}(\Omega))^n$ *satisfying*

$$|y_0 - \widetilde{y}(\cdot, 0)|_{(L^{2n-2}(\Omega))^n} \leq r,$$

Problem 3 *is solvable.*

The second condition in (19) can be replaced by

$$\frac{\partial \widetilde{y}}{\partial t} \in L^2(0,T;(L^\sigma(\Omega))^n) \text{ for } \sigma > \begin{cases} 1 \text{ when } n = 2, \\ \frac{6}{5} \text{ when } n = 3. \end{cases} \tag{20}$$

The linearized Navier–Stokes system around $\widetilde{y}$:

$$\begin{array}{ll} \dfrac{\partial y}{\partial t} - \nu \Delta y + (\widetilde{y} \cdot \nabla) y + (y \cdot \nabla)\widetilde{y} + \nabla p = f + \chi_\omega u & \text{in } Q, \\ \text{div } y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(\cdot, 0) = y_0 & \text{in } \Omega. \end{array} \tag{21}$$

The adjoint of system (21):

$$\begin{array}{ll} \dfrac{\partial z}{\partial t} + \nu \Delta z + (\nabla z +^t \nabla z)\widetilde{y} + \nabla q = h & \text{in } Q, \\ \text{div } z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma. \end{array} \tag{22}$$

The equation of q:

$$\Delta q = \text{div } h - \text{div } ((\nabla z +^t \nabla z)\widetilde{y}) \text{ in } Q. \tag{23}$$

Let $n = 3$. Consider

$$
(24)\quad
\begin{cases}
\dfrac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y - (B\cdot\nabla)B + \nabla p + \nabla\left(\dfrac{1}{2}B^2\right) = f + \chi_\omega u & \text{in } Q,\\
\dfrac{\partial B}{\partial t} + \eta\ \mathrm{curl}(\mathrm{curl}B) + (y\cdot\nabla)B - (B\cdot\nabla)y = P(\chi_\omega v) & \text{in } Q,\\
\mathrm{div}\ y = 0, \mathrm{div}\ B = 0 & \text{in } Q,\\
y = 0, B\cdot N = 0, (\mathrm{curl}B)\times N = 0 & \text{on } \Sigma,\\
y(\cdot,0) = y_0, B(\cdot,0) = B_0 & \text{in } \Omega.
\end{cases}
$$

Let $(\widetilde{y}, \widetilde{B})$ satisfy

$$
(25)\quad
\begin{cases}
\dfrac{\partial \widetilde{y}}{\partial t} - \nu\Delta \widetilde{y} + (\widetilde{y}\cdot\nabla)\widetilde{y} - (\widetilde{B}\cdot\nabla)\widetilde{B} + \nabla \widetilde{p} + \nabla\left(\dfrac{1}{2}\widetilde{B}^2\right) = f & \text{in } Q,\\
\dfrac{\partial \widetilde{B}}{\partial t} + \eta\ \mathrm{curl}(\mathrm{curl}\widetilde{B}) + (\widetilde{y}\cdot\nabla)\widetilde{B} - (\widetilde{B}\cdot\nabla)\widetilde{y} = 0 & \text{in } Q,\\
\mathrm{div}\ \widetilde{y} = 0, \mathrm{div}\ \widetilde{B} = 0 & \text{in } Q,\\
\widetilde{y} = 0, \widetilde{B}\cdot N = 0, (\mathrm{curl}\widetilde{B})\times N = 0 & \text{on } \Sigma.
\end{cases}
$$

Problem 4.

Find $(u, v) \in (L^2(Q))^6$ and a corresponding weak solution (y, B) of system (24) which also satisfy

$$(26) \qquad y(\cdot, T) = \widetilde{y}(\cdot, T) \text{ and } B(\cdot, T) = \widetilde{B}(\cdot, T) \text{ a.e. in } \Omega.$$

Theorem 4 (T. Havârneanu, C. Popa, S.S. Sritharan)

Let $f \in (L^2(Q))^3$. *If* $(\widetilde{y}, \widetilde{B})$ *is a weak solution of* (25) *which satisfies*

$$(27) \qquad (\widetilde{y}, \widetilde{B}) \in (L^\infty(Q))^6 \text{ and } \left(\frac{\partial \widetilde{y}}{\partial t}, \frac{\partial \widetilde{B}}{\partial t}\right) \in L^2(0, T; (L^\infty(\Omega))^6),$$

then there exists $r > 0$ *such that, for any* $(y_0, B_0) \in (H \cap (L^4(\Omega))^3)^2$ *satisfying*

$$|y_0 - \widetilde{y}(\cdot, 0)|_{(L^4(\Omega))^3} + |B_0 - \widetilde{B}(\cdot, 0)|_{(L^4(\Omega))^3} \leq r,$$

Problem 4 *is solvable.*