A uniform Berry-Esseen theorem on $M-$estimators for geometrically ergodic Markov chains

Valentin Patilea

IRMAR-INSA & CREST-ENSAI, Rennes, France

en collaboration avec
Loïc Hervé & James Ledoux, IRMAR-INSA, Rennes

Colloque Franco-Roumain, Poitiers, Août 2010

The problem (1/2)

- Let $\{X_n\}_{n\geq 0}$ be a V-geometrically ergodic Markov chain with $V(\cdot) \geq 1$ some fixed *unbounded* real-valued function.

The problem (1/2)

- Let $\{X_n\}_{n\geq 0}$ be a V-geometrically ergodic Markov chain with $V(\cdot) \geq 1$ some fixed *unbounded* real-valued function.

- Consider
$$M_n(\alpha) = n^{-1} \sum_{k=1}^{n} F(\alpha, X_{k-1}, X_k),$$
$\alpha \in \mathcal{A} \subset \mathbb{R}$ with $F(\cdot,\cdot,\cdot)$ real-valued functional and $\mathcal{A}$ some open interval.

The problem (1/2)

- Let $\{X_n\}_{n\geq 0}$ be a V-geometrically ergodic Markov chain with $V(\cdot) \geq 1$ some fixed *unbounded* real-valued function.

- Consider

$$M_n(\alpha) = n^{-1} \sum_{k=1}^{n} F(\alpha, X_{k-1}, X_k),$$

  $\alpha \in \mathcal{A} \subset \mathbb{R}$ with $F(\cdot, \cdot, \cdot)$ real-valued functional and $\mathcal{A}$ some open interval.

- Define the $M-$estimator $\widehat{\alpha}_n$ such that

$$M_n(\widehat{\alpha}_n) \leq \inf_{\alpha \in \mathcal{A}} M_n(\alpha) + c_n$$

  with c_n, $n \geq 1$ some sequence of real numbers decreasing to zero.

The problem (1/2)

- Let $\{X_n\}_{n\geq 0}$ be a V-geometrically ergodic Markov chain with $V(\cdot) \geq 1$ some fixed *unbounded* real-valued function.

- Consider
$$M_n(\alpha) = n^{-1} \sum_{k=1}^{n} F(\alpha, X_{k-1}, X_k),$$
$\alpha \in \mathcal{A} \subset \mathbb{R}$ with $F(\cdot,\cdot,\cdot)$ real-valued functional and $\mathcal{A}$ some open interval.

- Define the $M-$estimator $\widehat{\alpha}_n$ such that
$$M_n(\widehat{\alpha}_n) \leq \inf_{\alpha\in\mathcal{A}} M_n(\alpha) + c_n$$
with c_n, $n \geq 1$ some sequence of real numbers decreasing to zero.

- Let $\alpha_0 \in \mathcal{A}$ be the “true” value of the parameter of interest.

The problem (2/2)

- Suppose that

$$\left(\left|\frac{\partial F}{\partial \alpha}(\alpha, x, y)\right| + \left|\frac{\partial^2 F}{\partial \alpha^2}(\alpha, x, y)\right|\right)^{3+\varepsilon} \leq C\,(V(x) + V(y))$$

for some constants $\varepsilon > 0$ and $C > 0$.

The problem (2/2)

- Suppose that

$$\left(\left|\frac{\partial F}{\partial \alpha}(\alpha, x, y)\right| + \left|\frac{\partial^2 F}{\partial \alpha^2}(\alpha, x, y)\right|\right)^{3+\varepsilon} \leq C\,(V(x) + V(y))$$

for some constants $\varepsilon > 0$ and $C > 0$.

- Aim: show that the estimator $\widehat{\alpha}_n$ satisfies a Berry-Esseen theorem uniformly with respect to the underlying probability distribution of the Markov chain.

Example: AR(1) with ARCH(1) errors

- Consider the the observations are generated by the process

$$X_n = \rho_0 X_{n-1} + \sigma(X_{n-1}; \zeta_0, \gamma_0)\varepsilon_n, \quad n = 1, 2, \ldots$$

Example: AR(1) with ARCH(1) errors

- Consider the the observations are generated by the process

$$X_n = \rho_0 X_{n-1} + \sigma(X_{n-1}; \zeta_0, \gamma_0)\varepsilon_n, \quad n = 1, 2, \ldots$$

  - $X_0 = 0$, $\sigma^2(x; \zeta, \gamma) = \zeta + \gamma x^2$;
  - $\{\varepsilon_n\}_{n\geq 1}$ are i.i.d. zero mean of variance equal to 1, with finite pth order moment, $p > 0$ and (unknown) continuous density $f_\varepsilon > 0$;
  - $|\rho_0| < 1$ and $\zeta_0, \gamma_0 > 0$ are the true values of the parameters.

Example: AR(1) with ARCH(1) errors

- Consider the the observations are generated by the process

$$X_n = \rho_0 X_{n-1} + \sigma(X_{n-1}; \zeta_0, \gamma_0)\varepsilon_n, \quad n = 1, 2, \ldots$$

  - $X_0 = 0$, $\sigma^2(x; \zeta, \gamma) = \zeta + \gamma x^2$;
  - $\{\varepsilon_n\}_{n\geq 1}$ are i.i.d. zero mean of variance equal to 1, with finite pth order moment, $p > 0$ and (unknown) continuous density $f_\varepsilon > 0$;
  - $|\rho_0| < 1$ and $\zeta_0, \gamma_0 > 0$ are the true values of the parameters.

- The "true" parameter is $\theta = (\rho_0, \zeta_0, \gamma_0) \in \Theta \subset [-\overline{\rho}, \overline{\rho}] \times [m_\zeta, M_\zeta] \times [m_\gamma, M_\gamma] \subset \mathbb{R}^3$, where

Example: AR(1) with ARCH(1) errors

- Consider the the observations are generated by the process

$$X_n = \rho_0 X_{n-1} + \sigma(X_{n-1}; \zeta_0, \gamma_0)\varepsilon_n, \quad n = 1, 2, \dots$$

  - $X_0 = 0$, $\sigma^2(x; \zeta, \gamma) = \zeta + \gamma x^2$;
  - $\{\varepsilon_n\}_{n\geq 1}$ are i.i.d. zero mean of variance equal to 1, with finite pth order moment, $p > 0$ and (unknown) continuous density $f_\varepsilon > 0$;
  - $|\rho_0| < 1$ and $\zeta_0, \gamma_0 > 0$ are the true values of the parameters.

- The "true" parameter is $\theta = (\rho_0, \zeta_0, \gamma_0) \in \Theta \subset [-\overline{\rho}, \overline{\rho}] \times [m_\zeta, M_\zeta] \times [m_\gamma, M_\gamma] \subset \mathbb{R}^3$, where
  - $\overline{\rho} \in (0, 1)$ is given;
  - $0 < m_\zeta < M_\zeta < \infty$ and $0 < m_\gamma < M_\gamma < 1$ are given such that $\overline{\rho} + \sqrt{M_\gamma} < 1$.

Example: AR(1) with ARCH(1) errors

- Consider the the observations are generated by the process

$$X_n = \rho_0 X_{n-1} + \sigma(X_{n-1}; \zeta_0, \gamma_0)\varepsilon_n, \quad n = 1, 2, \ldots$$

  - $X_0 = 0$, $\sigma^2(x; \zeta, \gamma) = \zeta + \gamma x^2$;
  - $\{\varepsilon_n\}_{n \geq 1}$ are i.i.d. zero mean of variance equal to 1, with finite pth order moment, $p > 0$ and (unknown) continuous density $f_\varepsilon > 0$;
  - $|\rho_0| < 1$ and $\zeta_0, \gamma_0 > 0$ are the true values of the parameters.
- The "true" parameter is $\theta = (\rho_0, \zeta_0, \gamma_0) \in \Theta \subset [-\overline{\rho}, \overline{\rho}] \times [m_\zeta, M_\zeta] \times [m_\gamma, M_\gamma] \subset \mathbb{R}^3$, where
  - $\overline{\rho} \in (0, 1)$ is given;
  - $0 < m_\zeta < M_\zeta < \infty$ and $0 < m_\gamma < M_\gamma < 1$ are given such that $\overline{\rho} + \sqrt{M_\gamma} < 1$.
- We are interested to estimate ρ_0 and γ_0 and to derive BE bounds.

Example cont'd

- To estimate ρ_0 one can use the least squares estimator

$$\widehat{\rho}_n = \frac{\sum_{k=1}^n X_k X_{k-1}}{\sum_{k=1}^n X_{k-1}^2} = \arg\min_{\rho} \frac{1}{n} \sum_{k=1}^{n} F(\rho, X_{k-1}, X_k),$$

where $F(\rho, X_{k-1}, X_k) = (X_k - \rho X_{k-1})^2$.

Example cont'd

- To estimate ρ_0 one can use the least squares estimator

$$\widehat{\rho}_n = \frac{\sum_{k=1}^n X_k X_{k-1}}{\sum_{k=1}^n X_{k-1}^2} = \arg\min_{\rho} \frac{1}{n} \sum_{k=1}^n F(\rho, X_{k-1}, X_k),$$

where $F(\rho, X_{k-1}, X_k) = (X_k - \rho X_{k-1})^2$.

- The parameter γ_0 can also be estimated by a suitable least squares approach.

Agenda of the talk

1. Introduction

Agenda of the talk

1. Introduction
2. A general method for deriving Berry-Esseen bounds for $M-$estimators

Agenda of the talk

1. Introduction
2. A general method for deriving Berry-Esseen bounds for $M-$estimators
3. A uniform Berry-Esseen statement with geometrically ergodic Markov chains

Agenda of the talk

1. Introduction
2. A general method for deriving Berry-Esseen bounds for $M-$estimators
3. A uniform Berry-Esseen statement with geometrically ergodic Markov chains
4. Application: Berry-Esseen bound for $M-$estimators with geometrically Markov chains

Agenda of the talk

1. Introduction
2. A general method for deriving Berry-Esseen bounds for $M-$estimators
3. A uniform Berry-Esseen statement with geometrically ergodic Markov chains
4. Application: Berry-Esseen bound for $M-$estimators with geometrically Markov chains
5. Example of application: AR(1) with ARCH(1) errors

Definitions, notation (1/2)

- Let E be a measurable space equipped with a countably generated σ-field $\mathcal{E}$

Definitions, notation (1/2)

- Let E be a measurable space equipped with a countably generated σ-field $\mathcal{E}$
- Let $\{X_n\}_{n\geq 0}$ be a Markov chain with state space E and transition kernels $\{Q_\theta(x,\cdot) : x \in E\}$ where θ is a parameter in some general set Θ.

Definitions, notation (1/2)

- Let E be a measurable space equipped with a countably generated σ-field $\mathcal{E}$
- Let $\{X_n\}_{n\geq 0}$ be a Markov chain with state space E and transition kernels $\{Q_\theta(x,\cdot) : x \in E\}$ where θ is a parameter in some general set Θ.
- Let μ be the initial distribution of the chain. It can be fixed or dependent on θ.

Definitions, notation (1/2)

- Let E be a measurable space equipped with a countably generated σ-field $\mathcal{E}$
- Let $\{X_n\}_{n\geq 0}$ be a Markov chain with state space E and transition kernels $\{Q_\theta(x,\cdot) : x \in E\}$ where θ is a parameter in some general set Θ.
- Let μ be the initial distribution of the chain. It can be fixed or dependent on θ.
- Let $\mathbb{P}_{\theta,\mu}$ be the probability distribution of $\{X_n\}_{n\geq 0}$ and $\mathbb{E}_{\theta,\mu}$ be the expectation w.r.t. $\mathbb{P}_{\theta,\mu}$.

Definitions, notation (2/2)

- For all θ, let

$$M_\theta(\alpha) = \lim_{n\to\infty} \mathbb{E}_{\theta,\mu}[M_n(\alpha)].$$

Assume that there exists a unique "true" value $\alpha_0 = \alpha_0(\theta)$ of the parameter of interest, that is

$$M_\theta(\alpha_0) < M_\theta(\alpha), \quad \forall \alpha \neq \alpha_0.$$

Definitions, notation (2/2)

- For all θ, let
$$M_\theta(\alpha) = \lim_{n\to\infty} \mathbb{E}_{\theta,\mu}[M_n(\alpha)].$$
Assume that there exists a unique “true” value $\alpha_0 = \alpha_0(\theta)$ of the parameter of interest, that is
$$M_\theta(\alpha_0) < M_\theta(\alpha), \quad \forall \alpha \neq \alpha_0.$$

- We want to prove
$$\sup_{\theta\in\Theta} \sup_{u\in\mathbb{R}} \left| \mathbb{P}_{\theta,\mu}\left\{ \frac{\sqrt{n}}{\tau(\theta)} (\widehat{\alpha}_n - \alpha_0) \leq u \right\} - \Gamma(u) \right| = O(n^{-1/2})$$
where

Definitions, notation (2/2)

- For all θ, let
$$M_\theta(\alpha) = \lim_{n\to\infty} \mathbb{E}_{\theta,\mu}[M_n(\alpha)].$$
Assume that there exists a unique "true" value $\alpha_0 = \alpha_0(\theta)$ of the parameter of interest, that is
$$M_\theta(\alpha_0) < M_\theta(\alpha), \quad \forall \alpha \neq \alpha_0.$$

- We want to prove
$$\sup_{\theta\in\Theta} \sup_{u\in\mathbb{R}} \left| \mathbb{P}_{\theta,\mu} \left\{ \frac{\sqrt{n}}{\tau(\theta)} (\widehat{\alpha}_n - \alpha_0) \leq u \right\} - \Gamma(u) \right| = O(n^{-1/2})$$

 where
  - $\Gamma(\cdot)$ denotes the $N(0,1)$ d.f.
  - $\tau(\theta)$ is some suitable positive real number.

Related work

- IID framework and functional $F(\alpha, X_k)$:
  - Pfanzagl (1971): $\{X_n\}_{n\geq 0}$ a sequence of i.i.d. random variables and with functionals of the form $F(\alpha, X_k)$.
  - Bentkus, Bloznelis & Götze (1997): obtain the result of Pfanzagl (with constants) using convexity arguments that allow for non differentiable functionals $\alpha \mapsto F(\alpha, X_k)$

Related work

- IID framework and functional $F(\alpha, X_k)$:
  - Pfanzagl (1971): $\{X_n\}_{n\geq 0}$ a sequence of i.i.d. random variables and with functionals of the form $F(\alpha, X_k)$.
  - Bentkus, Bloznelis & Götze (1997): obtain the result of Pfanzagl (with constants) using convexity arguments that allow for non differentiable functionals $\alpha \mapsto F(\alpha, X_k)$
- Markov framework and functional $F(\alpha, X_k, X_{k-1})$:
  - Rao (1973): extends the framework of Pfanzagl to $\{X_n\}_{n\geq 0}$ a uniformly ergodic Markov chain
  - Milhaud & Raugi (1989): ML in linear autoregressive models under strong conditions on the error term law. Moreover, they obtain a suboptimal rate $n^{-1/2} \ln^{1/2} n$.

Pfanzagl's method revisited

- Consider a statistical model $(\Omega, \mathcal{F}, \{\mathbb{P}_\theta,\ \theta \in \Theta\})$, where Θ denotes some parameter space

Pfanzagl's method revisited

- Consider a statistical model $(\Omega, \mathcal{F}, \{\mathbb{P}_\theta,\ \theta \in \Theta\})$, where Θ denotes some parameter space
- Let $\{X_n\}_{n\geq 0}$ be any sequence of observations (not necessarily markovian). Let $\mathbb{E}_\theta$ denote the expectation with respect to $\mathbb{P}_\theta$.

Pfanzagl's method revisited

- Consider a statistical model $(\Omega, \mathcal{F}, \{\mathbb{P}_\theta,\ \theta \in \Theta\})$, where Θ denotes some parameter space
- Let $\{X_n\}_{n\geq 0}$ be any sequence of observations (not necessarily markovian). Let $\mathbb{E}_\theta$ denote the expectation with respect to $\mathbb{P}_\theta$.
- For each n, let $M_n(\alpha)$ be a measurable function depending on $X_0, X_1, ..., X_n$ and the parameter of interest $\alpha \in \mathcal{A} \subset \mathbb{R}$ with $\mathcal{A}$ some open interval.

Pfanzagl's method revisited

- Consider a statistical model $(\Omega, \mathcal{F}, \{\mathbb{P}_\theta,\ \theta \in \Theta\})$, where Θ denotes some parameter space
- Let $\{X_n\}_{n\geq 0}$ be any sequence of observations (not necessarily markovian). Let $\mathbb{E}_\theta$ denote the expectation with respect to $\mathbb{P}_\theta$.
- For each n, let $M_n(\alpha)$ be a measurable function depending on $X_0, X_1, ..., X_n$ and the parameter of interest $\alpha \in \mathcal{A} \subset \mathbb{R}$ with $\mathcal{A}$ some open interval.
- Let $c_n \downarrow 0$

Pfanzagl's method revisited

- Consider a statistical model $(\Omega, \mathcal{F}, \{\mathbb{P}_\theta,\ \theta \in \Theta\})$, where Θ denotes some parameter space
- Let $\{X_n\}_{n\geq 0}$ be any sequence of observations (not necessarily markovian). Let $\mathbb{E}_\theta$ denote the expectation with respect to $\mathbb{P}_\theta$.
- For each n, let $M_n(\alpha)$ be a measurable function depending on $X_0, X_1, ..., X_n$ and the parameter of interest $\alpha \in \mathcal{A} \subset \mathbb{R}$ with $\mathcal{A}$ some open interval.
- Let $c_n \downarrow 0$
- *$M-$estimator*: a measurable function $\widehat{\alpha}_n$ of the observations $(X_0, \ldots, X_n)$ such that

$$M_n(\widehat{\alpha}_n) \leq \min_{\alpha\in\mathcal{A}} M_n(\alpha) + c_n.$$

Pfanzagl's method revisited: Assumptions (1/3)

$\forall\ n \geq 1$ and $\alpha, \alpha' \in \mathcal{A}$, there exist $M_n'(\alpha)$, $M_n''(\alpha)$ measurable functions depending on $X_0, X_1, ..., X_n$ and on α such that:

Pfanzagl's method revisited: Assumptions (1/3)

$\forall\ n \geq 1$ and $\alpha, \alpha' \in \mathcal{A}$, there exist $M_n'(\alpha)$, $M_n''(\alpha)$ measurable functions depending on $X_0, X_1, ..., X_n$ and on α such that:

(A1) $\forall \theta \in \Theta$, there exists a unique $\alpha_0 = \alpha_0(\theta) \in \mathcal{A}$ such that $M_\theta'(\alpha_0) = 0$ where $M_\theta'(\alpha) = \lim_{n\to\infty} \mathbb{E}_\theta[M_n'(\alpha)]$.

Pfanzagl's method revisited: Assumptions (1/3)

$\forall\ n \geq 1$ and $\alpha, \alpha' \in \mathcal{A}$, there exist $M_n'(\alpha)$, $M_n''(\alpha)$ measurable functions depending on $X_0, X_1, ..., X_n$ and on α such that:

(A1) $\forall \theta \in \Theta$, there exists a unique $\alpha_0 = \alpha_0(\theta) \in \mathcal{A}$ such that $M_\theta'(\alpha_0) = 0$ where $M_\theta'(\alpha) = \lim_{n\to\infty} \mathbb{E}_\theta[M_n'(\alpha)]$.

(A2) $0 < \inf_\theta m(\theta) \leq \sup_\theta m(\theta) < \infty$ where $m(\theta) = \lim_{n\to\infty} \mathbb{E}_\theta[M_n''(\alpha_0)]$.

Pfanzagl's method revisited: Assumptions (1/3)

$\forall\ n \geq 1$ and $\alpha, \alpha' \in \mathcal{A}$, there exist $M_n'(\alpha)$, $M_n''(\alpha)$ measurable functions depending on $X_0, X_1, ..., X_n$ and on α such that:

(A1) $\forall \theta \in \Theta$, there exists a unique $\alpha_0 = \alpha_0(\theta) \in \mathcal{A}$ such that $M_\theta'(\alpha_0) = 0$ where $M_\theta'(\alpha) = \lim_{n\to\infty} \mathbb{E}_\theta[M_n'(\alpha)]$.

(A2) $0 < \inf_\theta m(\theta) \leq \sup_\theta m(\theta) < \infty$ where $m(\theta) = \lim_{n\to\infty} \mathbb{E}_\theta[M_n''(\alpha_0)]$.

(A3) $\forall n \geq 1$, $\exists\ r_n > 0$ independent of θ such that $\sqrt{n} r_n \to 0$ and

$$\sup_{\theta\in\Theta} \mathbb{P}_\theta\left(|M_n'(\widehat{\alpha}_n)| \geq r_n\right) = O(n^{-1/2}).$$

Pfanzagl's method revisited: Assumptions (1/3)

$\forall\ n \geq 1$ and $\alpha, \alpha' \in \mathcal{A}$, there exist $M_n'(\alpha)$, $M_n''(\alpha)$ measurable functions depending on $X_0, X_1, ..., X_n$ and on α such that:

(A1) $\forall \theta \in \Theta$, there exists a unique $\alpha_0 = \alpha_0(\theta) \in \mathcal{A}$ such that $M_\theta'(\alpha_0) = 0$ where $M_\theta'(\alpha) = \lim_{n\to\infty} \mathbb{E}_\theta[M_n'(\alpha)]$.

(A2) $0 < \inf_\theta m(\theta) \leq \sup_\theta m(\theta) < \infty$ where $m(\theta) = \lim_{n\to\infty} \mathbb{E}_\theta[M_n''(\alpha_0)]$.

(A3) $\forall n \geq 1$, $\exists\ r_n > 0$ independent of θ such that $\sqrt{n} r_n \to 0$ and

$$\sup_{\theta\in\Theta} \mathbb{P}_\theta \left(|M_n'(\widehat{\alpha}_n)| \geq r_n \right) = O(n^{-1/2}).$$

(A4) for $j = 1, 2$, $\exists\ \sigma_j(\cdot)$ such that $0 < \inf_\theta \sigma_j(\theta) \leq \sup_\theta \sigma_j(\theta) < \infty$, and some constant $B > 0$ and integer P_0 such that for all $n \geq P_0$

$$\sup_{\theta\in\Theta} \sup_{u\in\mathbb{R}} \left| \mathbb{P}_\theta \left\{ \frac{\sqrt{n}}{\sigma_1(\theta)} M_n'(\alpha_0) \leq u \right\} - \Gamma(u) \right| \leq \frac{B}{\sqrt{n}}$$

$$\sup_{\theta\in\Theta} \sup_{u\in\mathbb{R}} \left| \mathbb{P}_\theta \left\{ \frac{\sqrt{n}}{\sigma_2(\theta)} (M_n''(\alpha_0) - m(\theta)) \leq u \right\} - \Gamma(u) \right| \leq \frac{B}{\sqrt{n}}.$$

Pfanzagl's method revisited: Assumptions (2/3)

(A4') for $n \geq P_0$, $|u| \leq 2\sqrt{\ln n}$, and $\theta \in \Theta$, $\exists\ \sigma^2_{n,u}(\theta) > 0$ such that, with some constants $A', B' > 0$ which do not depend on such n, u, θ,

$$|\sigma_{n,u}(\theta) - \sigma_1(\theta)| \leq A' \, |u| n^{-1/2},$$

$$\left| \mathbb{P}_\theta \left\{ \frac{\sqrt{n}}{\sigma_{n,u}(\theta)} \left(M_n'(\alpha_0) + \frac{u\,\sigma_1(\theta)}{\sqrt{n}\, m(\theta)} \left(M_n''(\alpha_0) - m(\theta)\right) \right) \leq u \right\} - \Gamma(u) \right| \leq \frac{B'}{\sqrt{n}}.$$

Pfanzagl's method revisited: Assumptions (2/3)

(A4') for $n \geq P_0$, $|u| \leq 2\sqrt{\ln n}$, and $\theta \in \Theta$, $\exists\ \sigma^2_{n,u}(\theta) > 0$ such that, with some constants $A', B' > 0$ which do not depend on such n, u, θ,

$$|\sigma_{n,u}(\theta) - \sigma_1(\theta)| \leq A'\,|u| n^{-1/2},$$

$$\left| \mathbb{P}_\theta \left\{ \frac{\sqrt{n}}{\sigma_{n,u}(\theta)} \left(M'_n(\alpha_0) + \frac{u\,\sigma_1(\theta)}{\sqrt{n}\,m(\theta)} \left(M''_n(\alpha_0) - m(\theta) \right) \right) \leq u \right\} - \Gamma(u) \right| \leq \frac{B'}{\sqrt{n}}.$$

(A5) For any $\alpha, \alpha' \in \mathcal{A}$, let $R_n(\alpha, \alpha')$ be defined by the equation

$$M'_n(\alpha') = M'_n(\alpha) + [M''_n(\alpha) + R_n(\alpha, \alpha')](\alpha' - \alpha).$$

For each n there exist $\omega_n \geq 0$ independent of θ and a real-valued measurable function W_n depending on $X_0, X_1, ..., X_n$ but independent of θ such that $\omega_n \to 0$ and $\forall (\alpha, \alpha') \in \mathcal{A}^2$,

$$\left| R_n(\alpha, \alpha') \right| \leq \{|\alpha - \alpha'| + \omega_n\}\, W_n,$$

and there exists a constant $c_W > 0$ such that

$$\sup_{\theta \in \Theta} \mathbb{P}_\theta\{c_W \leq W_n\} = O(n^{-1/2}).$$

Pfanzagl's method revisited: Assumptions (3/3)

(A6) Assume that $\widehat{\alpha}_n$ is uniformly consistent at some rate, that is there exists a sequence $\gamma_n \to 0$ such that

$$\sup_{\theta \in \Theta} \mathbb{P}_\theta \left\{ \, |\widehat{\alpha}_n - \alpha_0| \geq d \, \right\} \leq \gamma_n,$$

where $d = \inf_{\theta \in \Theta} m(\theta)/8c_W$ with c_W and $m(\theta)$ defined in (A5) and (A2) respectively.

Pfanzagl's method revisited: Assumptions (3/3)

(A6) Assume that $\widehat{\alpha}_n$ is uniformly consistent at some rate, that is there exists a sequence $\gamma_n \to 0$ such that

$$\sup_{\theta\in\Theta} \mathbb{P}_\theta\left\{ |\widehat{\alpha}_n - \alpha_0| \geq d \right\} \leq \gamma_n,$$

where $d = \inf_{\theta\in\Theta} m(\theta)/8c_W$ with c_W and $m(\theta)$ defined in (A5) and (A2) respectively.

- Comments:

Pfanzagl's method revisited: Assumptions (3/3)

(A6) Assume that $\widehat{\alpha}_n$ is uniformly consistent at some rate, that is there exists a sequence $\gamma_n \to 0$ such that

$$\sup_{\theta\in\Theta} \mathbb{P}_\theta\left\{ |\widehat{\alpha}_n - \alpha_0| \geq d \right\} \leq \gamma_n,$$

where $d = \inf_{\theta\in\Theta} m(\theta)/8c_W$ with c_W and $m(\theta)$ defined in (A5) and (A2) respectively.

- Comments:
  - The expectations $\mathbb{E}_\theta[M_n'(\alpha)]$ and $\mathbb{E}_\theta[M_n''(\alpha_0)]$ may depend on n.

Pfanzagl's method revisited: Assumptions (3/3)

(A6) Assume that $\widehat{\alpha}_n$ is uniformly consistent at some rate, that is there exists a sequence $\gamma_n \to 0$ such that

$$\sup_{\theta\in\Theta} \mathbb{P}_\theta\left\{ |\widehat{\alpha}_n - \alpha_0| \geq d \right\} \leq \gamma_n,$$

where $d = \inf_{\theta\in\Theta} m(\theta)/8c_W$ with c_W and $m(\theta)$ defined in (A5) and (A2) respectively.

- Comments:
  - The expectations $\mathbb{E}_\theta[M_n'(\alpha)]$ and $\mathbb{E}_\theta[M_n''(\alpha_0)]$ may depend on n.
  - Assumption (A3) ensures that the estimator (approximately) satisfies a kind of first order condition.

Pfanzagl's method revisited: Assumptions (3/3)

(A6) Assume that $\widehat{\alpha}_n$ is uniformly consistent at some rate, that is there exists a sequence $\gamma_n \to 0$ such that

$$\sup_{\theta\in\Theta} \mathbb{P}_\theta\left\{ |\widehat{\alpha}_n - \alpha_0| \geq d \right\} \leq \gamma_n,$$

where $d = \inf_{\theta\in\Theta} m(\theta)/8c_W$ with c_W and $m(\theta)$ defined in (A5) and (A2) respectively.

- Comments:
  - The expectations $\mathbb{E}_\theta[M_n'(\alpha)]$ and $\mathbb{E}_\theta[M_n''(\alpha_0)]$ may depend on n.
  - Assumption (A3) ensures that the estimator (approximately) satisfies a kind of first order condition.
  - The Assumptions (A4) and (A4') are the Berry-Esseen bounds for $M_n'(\alpha_0)$, $M_n''(\alpha_0)$, and for some of their linear combinations.

Pfanzagl's method revisited: Assumptions (3/3)

(A6) Assume that $\widehat{\alpha}_n$ is uniformly consistent at some rate, that is there exists a sequence $\gamma_n \to 0$ such that

$$\sup_{\theta\in\Theta} \mathbb{P}_\theta \{ |\widehat{\alpha}_n - \alpha_0| \geq d \} \leq \gamma_n,$$

where $d = \inf_{\theta\in\Theta} m(\theta)/8c_W$ with c_W and $m(\theta)$ defined in (A5) and (A2) respectively.

- Comments:
  - The expectations $\mathbb{E}_\theta[M_n'(\alpha)]$ and $\mathbb{E}_\theta[M_n''(\alpha_0)]$ may depend on n.
  - Assumption (A3) ensures that the estimator (approximately) satisfies a kind of first order condition.
  - The Assumptions (A4) and (A4') are the Berry-Esseen bounds for $M_n'(\alpha_0)$, $M_n''(\alpha_0)$, and for some of their linear combinations.
  - The identity defining $R_n(\alpha, \alpha')$ in Assumption (A5) is guaranteed by Taylor expansion when the criterion $M_n(\alpha)$ is twice differentiable with respect to α.

Theorem

Under the conditions (A1-A6), $\exists\ C > 0$ *such that* $\forall n \geq 1$,

$$\sup_{\theta\in\Theta}\sup_{u\in\mathbb{R}}\left|\mathbb{P}_\theta\left\{\frac{\sqrt{n}}{\tau(\theta)}\left(\widehat{\alpha}_n-\alpha_0\right)\leq u\right\}-\Gamma(u)\right|\leq C\left(\frac{1}{\sqrt{n}}+\sqrt{n}r_n+\omega_n+\gamma_n\right)$$

with $\tau(\theta) := \sigma_1(\theta)/m(\theta)$.

Theorem

Under the conditions (A1-A6), $\exists\ C > 0$ such that $\forall n \geq 1$,

$$\sup_{\theta\in\Theta}\sup_{u\in\mathbb{R}}\left|\mathbb{P}_\theta\left\{\frac{\sqrt{n}}{\tau(\theta)}(\widehat{\alpha}_n - \alpha_0) \leq u\right\} - \Gamma(u)\right| \leq C\left(\frac{1}{\sqrt{n}} + \sqrt{n}r_n + \omega_n + \gamma_n\right)$$

with $\tau(\theta) := \sigma_1(\theta)/m(\theta)$.

Comments:

- To derive the Berry-Esseen bound of classical order $O(n^{-1/2})$, we need $\gamma_n = O(n^{-1/2})$, $r_n = O(n^{-1})$ and $\omega_n = O(n^{-1/2})$.
- Usually c_n in the definition of the $M-$estimator has to decrease at the rate $n^{-3/2}$.

- Consider
  - E a measurable space with a countably generated σ-field $\mathcal{E}$
  - Θ some general parameter space.
  - $\{X_n\}_{n\geq 0}$ a Markov chain with state space E, transition kernels $\{Q_\theta(x,\cdot),\ x\in E\}$, $\theta\in\Theta$, and initial distribution μ.
  - μ is either given (e.g. Dirac measure) or depends on θ. Suppose $\mu(V)<\infty$ or $\sup_{\theta\in\Theta}\mu(V)<\infty$.

- Consider
  - E a measurable space with a countably generated σ-field $\mathcal{E}$
  - Θ some general parameter space.
  - $\{X_n\}_{n\geq 0}$ a Markov chain with state space E, transition kernels $\{Q_\theta(x,\cdot),\ x \in E\}$, $\theta \in \Theta$, and initial distribution μ.
  - μ is either given (e.g. Dirac measure) or depends on θ. Suppose $\mu(V) < \infty$ or $\sup_{\theta\in\Theta} \mu(V) < \infty$.

- **Assumption ($\mathcal{M}$).** $\forall\ \theta \in \Theta$, there exists a Q_θ-invariant probability distribution π_θ and an unbounded $V : E \to [1, \infty)$ such that

- Consider
  - E a measurable space with a countably generated σ-field $\mathcal{E}$
  - Θ some general parameter space.
  - $\{X_n\}_{n\geq 0}$ a Markov chain with state space E, transition kernels $\{Q_\theta(x,\cdot),\ x \in E\}$, $\theta \in \Theta$, and initial distribution μ.
  - μ is either given (e.g. Dirac measure) or depends on θ. Suppose $\mu(V) < \infty$ or $\sup_{\theta\in\Theta} \mu(V) < \infty$.

- **Assumption ($\mathcal{M}$).** $\forall\ \theta \in \Theta$, there exists a Q_θ-invariant probability distribution π_θ and an unbounded $V : E \to [1,\infty)$ such that

(VG1) $\sup_{\theta\in\Theta} \pi_\theta(V) < \infty$

(VG2) For all $\gamma \in (0,1]$, there exist real numbers $\kappa_\gamma < 1$ and $C_\gamma \geq 0$ such that we have, for each $\theta \in \Theta$ and for all $n \geq 1$ and $x \in E$

$$\sup_{|f|\leq V^\gamma} |Q_\theta^n f(x) - \pi_\theta(f)| \leq C_\gamma\, \kappa_\gamma^n\, V(x)^\gamma.$$

- Consider
  - E a measurable space with a countably generated σ-field $\mathcal{E}$
  - Θ some general parameter space.
  - $\{X_n\}_{n\geq 0}$ a Markov chain with state space E, transition kernels $\{Q_\theta(x,\cdot),\ x\in E\}$, $\theta\in\Theta$, and initial distribution μ.
  - μ is either given (e.g. Dirac measure) or depends on θ. Suppose $\mu(V)<\infty$ or $\sup_{\theta\in\Theta}\mu(V)<\infty$.

- **Assumption ($\mathcal{M}$).** $\forall\ \theta\in\Theta$, there exists a Q_θ-invariant probability distribution π_θ and an unbounded $V : E\to[1,\infty)$ such that

(VG1) $\sup_{\theta\in\Theta}\ \pi_\theta(V)<\infty$

(VG2) For all $\gamma\in(0,1]$, there exist real numbers $\kappa_\gamma<1$ and $C_\gamma\geq 0$ such that we have, for each $\theta\in\Theta$ and for all $n\geq 1$ and $x\in E$

$$\sup_{|f|\leq V^\gamma}\ |Q_\theta^n f(x)-\pi_\theta(f)|\leq C_\gamma\,\kappa_\gamma^n\,V(x)^\gamma.$$

- Remarks:
  - (VG1)-(VG2) $\Rightarrow\forall\gamma\in(0,1],\ \forall\theta\in\Theta$, Q_θ is V^γ-geometrically ergodic.
  - The constants C_γ and κ_γ do not depend on θ.

- Let $\alpha_0 = \alpha_0(\theta) \in \mathcal{A} \subset \mathbb{R}$ be the parameter of interest

- Let $\alpha_0 = \alpha_0(\theta) \in \mathcal{A} \subset \mathbb{R}$ be the parameter of interest
- Let $\xi(\alpha, x, y)$ such that $\mathbb{E}_{\theta,\pi_\theta}[\xi(\alpha_0, X_0, X_1)] = 0$

- Let $\alpha_0 = \alpha_0(\theta) \in \mathcal{A} \subset \mathbb{R}$ be the parameter of interest
- Let $\xi(\alpha, x, y)$ such that $\mathbb{E}_{\theta,\pi_\theta}[\xi(\alpha_0, X_0, X_1)] = 0$
- Let $S_n(\alpha) = \sum_{k=1}^{n} \xi(\alpha, X_{k-1}, X_k)$

- Let $\alpha_0 = \alpha_0(\theta) \in \mathcal{A} \subset \mathbb{R}$ be the parameter of interest
- Let $\xi(\alpha, x, y)$ such that $\mathbb{E}_{\theta,\pi_\theta}[\xi(\alpha_0, X_0, X_1)] = 0$
- Let $S_n(\alpha) = \sum_{k=1}^{n} \xi(\alpha, X_{k-1}, X_k)$
- We need to investigate the uniform Berry-Esseen property

$$\sup_{\theta\in\Theta} \sup_{u\in\mathbb{R}} \left| \mathbb{P}_{\theta,\mu} \left\{ \frac{S_n(\alpha_0)}{\sigma(\theta)\sqrt{n}} \leq u \right\} - \Gamma(u) \right| = O(n^{-1/2})$$

where $\sigma^2(\theta)$ is the asymptotic variance

- Let $\alpha_0 = \alpha_0(\theta) \in \mathcal{A} \subset \mathbb{R}$ be the parameter of interest
- Let $\xi(\alpha, x, y)$ such that $\mathbb{E}_{\theta,\pi_\theta}[\xi(\alpha_0, X_0, X_1)] = 0$
- Let $S_n(\alpha) = \sum_{k=1}^{n} \xi(\alpha, X_{k-1}, X_k)$
- We need to investigate the uniform Berry-Esseen property

$$\sup_{\theta\in\Theta}\sup_{u\in\mathbb{R}}\left|\mathbb{P}_{\theta,\mu}\left\{\frac{S_n(\alpha_0)}{\sigma(\theta)\sqrt{n}} \leq u\right\} - \Gamma(u)\right| = O(n^{-1/2})$$

where $\sigma^2(\theta)$ is the asymptotic variance

- The following condition will be required for $m_0 = 1, 2$ or 3: $\exists$ constants $m > m_0 \geq 1$ and $C_\xi > 0$ such that

$$\forall \alpha \in \mathcal{A},\ \forall (x, y) \in E^2,\quad |\xi(\alpha, x, y)|^m \leq C_\xi\,(V(x) + V(y)). \qquad (D_{m_0})$$

- Let $\alpha_0 = \alpha_0(\theta) \in \mathcal{A} \subset \mathbb{R}$ be the parameter of interest
- Let $\xi(\alpha, x, y)$ such that $\mathbb{E}_{\theta,\pi_\theta}[\xi(\alpha_0, X_0, X_1)] = 0$
- Let $S_n(\alpha) = \sum_{k=1}^{n} \xi(\alpha, X_{k-1}, X_k)$
- We need to investigate the uniform Berry-Esseen property

$$\sup_{\theta\in\Theta} \sup_{u\in\mathbb{R}} \left| \mathbb{P}_{\theta,\mu} \left\{ \frac{S_n(\alpha_0)}{\sigma(\theta)\sqrt{n}} \leq u \right\} - \Gamma(u) \right| = O(n^{-1/2})$$

where $\sigma^2(\theta)$ is the asymptotic variance

- The following condition will be required for $m_0 = 1, 2$ or 3: $\exists$ constants $m > m_0 \geq 1$ and $C_\xi > 0$ such that

$$\forall \alpha \in \mathcal{A},\ \forall (x, y) \in E^2, \quad |\xi(\alpha, x, y)|^m \leq C_\xi\,(V(x) + V(y)). \qquad (D_{m_0})$$

- Condition (D_{m_0}) implies

$$\mathbb{E}_{\theta,\pi_\theta}[\,|\xi(\alpha, X_0, X_1)|^m\,] < \infty \qquad (1)$$

Lemma

If ξ is centered and satisfies condition (D_{m_0}) with $m_0 \in \mathbb{N}^*$, then there exists $\beta > 0$ such that $\forall \theta \in \Theta,\ \forall n \geq 1, \forall t \in [-\beta, \beta]$,

$$\mathbb{E}_{\theta,\mu}[e^{itS_n(\alpha_0)}] = \lambda_\theta(t)^n \left[1 + L_\theta(t)\right] + r_{\theta,n}(t),$$

- $\lambda_\theta(\cdot)$, $L_\theta(\cdot)$ *and* $r_{\theta,n}(\cdot)$ *are* $\mathcal{C}^{m_0}([-\beta, \beta], \mathbb{C})$ *functions*
- $\lambda_\theta(0) = 1,\ \lambda'_\theta(0) = 0,\ L_\theta(0) = 0$ *and* $r_{\theta,n}(0) = 0$.

Lemma

If ξ is centered and satisfies condition (D_{m_0}) with $m_0 \in \mathbb{N}^*$, then there exists $\beta > 0$ such that $\forall \theta \in \Theta,\ \forall n \geq 1, \forall t \in [-\beta, \beta]$,

$$\mathbb{E}_{\theta,\mu}[e^{itS_n(\alpha_0)}] = \lambda_\theta(t)^n \left[1 + L_\theta(t)\right] + r_{\theta,n}(t),$$

- $\lambda_\theta(\cdot)$, $L_\theta(\cdot)$ *and* $r_{\theta,n}(\cdot)$ *are* $\mathcal{C}^{m_0}([-\beta,\beta],\mathbb{C})$ *functions*
- $\lambda_\theta(0) = 1,\ \lambda'_\theta(0) = 0,\ L_\theta(0) = 0$ *and* $r_{\theta,n}(0) = 0$.

Furthermore, there exists $\rho \in (0,1)$ such that for $\ell = 0, \ldots, m_0$:

$$G_\ell := \sup_{|t|\leq\beta,\, \theta\in\Theta,\ n\geq 1} \left\{\rho^{-n}\, |r^{(\ell)}_{\theta,n}(t)|\right\} < \infty.$$

The constants β, ρ, G_ℓ, and the following ones (for $\ell = 0, \ldots, m_0$)

$$E_\ell := \sup_{|t|\leq\beta,\, \theta\in\Theta} |\lambda^{(\ell)}_\theta(t)| < \infty, \qquad F_\ell := \sup_{|t|\leq\beta,\, \theta\in\Theta} |L^{(\ell)}_\theta(t)| < \infty,$$

depend on ξ, but only through the constant C_ξ of Condition D_{m_0}.

Proposition

Suppose

- *Assumption* $(\mathcal{M})$ *holds true*
- *the functional* ξ *satisfies condition* (D_1) *and is centered.*

Then,

$$\sup_{\theta\in\Theta}\sup_{n\geq 1}\left|\mathbb{E}_{\theta,\mu}[S_n(\alpha_0)]\right| < \infty \quad \textit{and} \quad \forall\theta\in\Theta,\ \lim_n \mathbb{E}_{\theta,\mu}\left[S_n(\alpha_0)/n\right] = 0.$$

Proposition

Suppose

- *Assumption* $(\mathcal{M})$ *holds true*
- *the functional* ξ *satisfies condition* (D_1) *and is centered.*

Then,

$$\sup_{\theta\in\Theta}\sup_{n\geq 1}\left|\mathbb{E}_{\theta,\mu}[S_n(\alpha_0)]\right| < \infty \quad \textit{and} \quad \forall\theta\in\Theta,\ \lim_n \mathbb{E}_{\theta,\mu}\left[S_n(\alpha_0)/n\right] = 0.$$

If in addition ξ *satisfies condition* (D_2)*, then for each* $\forall\theta\in\Theta$*,*

$$\sigma^2(\theta) := \lim_n \frac{\mathbb{E}_{\theta,\mu}[S_n(\alpha_0)^2]}{n} \in \mathbb{R} \quad \textit{is well defined.}$$

Proposition

Suppose

- *Assumption* $(\mathcal{M})$ *holds true*
- *the functional* ξ *satisfies condition* (D_1) *and is centered.*

Then,

$$\sup_{\theta\in\Theta}\sup_{n\geq 1}\left|\mathbb{E}_{\theta,\mu}[S_n(\alpha_0)]\right| < \infty \quad \textit{and} \quad \forall\theta\in\Theta,\ \lim_n \mathbb{E}_{\theta,\mu}\left[S_n(\alpha_0)/n\right] = 0.$$

If in addition ξ *satisfies condition* (D_2)*, then for each* $\forall\theta\in\Theta$*,*

$$\sigma^2(\theta) := \lim_n \frac{\mathbb{E}_{\theta,\mu}[S_n(\alpha_0)^2]}{n} \in \mathbb{R} \quad \textit{is well defined.}$$

Furthermore, the function $\sigma^2(\cdot)$ *is bounded on* Θ*, and there exists a constant* $C > 0$ *depending only on* C_ξ *and* $\mu(V)$ *such that*

$$\forall\theta\in\Theta,\ \forall n\geq 1,\quad \left|n\sigma^2(\theta) - \mathbb{E}_{\theta,\mu}[S_n(\alpha_0)^2]\right| \leq C.$$

Theorem

Assume that :

- *condition* $(\mathcal{M})$ *holds true;*
- *the functional* ξ *is centered and satisfies condition* (D_3)*;*
- $\sigma_0^2 := \inf_{\theta \in \Theta} \sigma^2(\theta) > 0$.

Then there exists a constant $B(\xi)$ *such that*

$$\forall n \geq 1, \quad \sup_{\theta \in \Theta} \sup_{u \in \mathbb{R}} \left| \mathbb{P}_{\theta,\mu} \left\{ \frac{S_n(\alpha_0)}{\sigma(\theta)\sqrt{n}} \leq u \right\} - \Gamma(u) \right| \leq \frac{B(\xi)}{\sqrt{n}}.$$

Furthermore, the constant $B(\xi)$ *depends on the functional* ξ*, but only through* σ_0 *and the constant* C_ξ *of Condition* (D_3)*.*

- Consider a Markov chain satisfying the condition ($\mathcal{M}$).

- Consider a Markov chain satisfying the condition ($\mathcal{M}$).
- Let

$$M_n(\alpha) = \frac{1}{n}\sum_{k=1}^{n} F(\alpha, X_{k-1}, X_k)$$

where $\alpha \in \mathcal{A} \subset \mathbb{R}$ is the parameter of interest, $F : \mathcal{A} \times E^2 \to \mathbb{R}$.

- Consider a Markov chain satisfying the condition $(\mathcal{M})$.
- Let

$$M_n(\alpha) = \frac{1}{n}\sum_{k=1}^{n} F(\alpha, X_{k-1}, X_k)$$

where $\alpha \in \mathcal{A} \subset \mathbb{R}$ is the parameter of interest, $F : \mathcal{A} \times E^2 \to \mathbb{R}$.
- Assume that F satisfies condition (D_1) and let

$$M_\theta(\alpha) = \lim_{n\to\infty} \mathbb{E}_{\theta,\mu}[M_n(\alpha)] = \mathbb{E}_{\theta,\pi_\theta}[F(\alpha, X_0, X_1)].$$

- Consider a Markov chain satisfying the condition $(\mathcal{M})$.
- Let

$$M_n(\alpha) = \frac{1}{n}\sum_{k=1}^{n} F(\alpha, X_{k-1}, X_k)$$

where $\alpha \in \mathcal{A} \subset \mathbb{R}$ is the parameter of interest, $F : \mathcal{A} \times E^2 \to \mathbb{R}$.

- Assume that F satisfies condition (D_1) and let

$$M_\theta(\alpha) = \lim_{n\to\infty} \mathbb{E}_{\theta,\mu}[M_n(\alpha)] = \mathbb{E}_{\theta,\pi_\theta}[F(\alpha, X_0, X_1)].$$

- For each $\theta \in \Theta$ there exists a unique $\alpha_0 = \alpha_0(\theta) \in \mathcal{A}$, the "true" value of the parameter of interest, such that

$$M_\theta(\alpha) > M_\theta(\alpha_0), \ \forall \alpha \neq \alpha_0.$$

- Consider a Markov chain satisfying the condition $(\mathcal{M})$.
- Let
$$M_n(\alpha) = \frac{1}{n}\sum_{k=1}^{n} F(\alpha, X_{k-1}, X_k)$$
where $\alpha \in \mathcal{A} \subset \mathbb{R}$ is the parameter of interest, $F : \mathcal{A} \times E^2 \to \mathbb{R}$.
- Assume that F satisfies condition (D_1) and let
$$M_\theta(\alpha) = \lim_{n\to\infty} \mathbb{E}_{\theta,\mu}[M_n(\alpha)] = \mathbb{E}_{\theta,\pi_\theta}[F(\alpha, X_0, X_1)].$$
- For each $\theta \in \Theta$ there exists a unique $\alpha_0 = \alpha_0(\theta) \in \mathcal{A}$, the "true" value of the parameter of interest, such that
$$M_\theta(\alpha) > M_\theta(\alpha_0), \ \forall \alpha \neq \alpha_0.$$
- Consider a $M-$estimator $\widehat{\alpha}_n$, that is
$$M_n(\widehat{\alpha}_n) \leq \min_{\alpha\in\mathcal{A}} M_n(\alpha) + c_n,$$
where $c_n \downarrow 0$.

- $\forall (x, y) \in E^2$, the map $\alpha \mapsto F(\alpha, x, y)$ is twice continuously differentiable on $\mathcal{A}$.

- $\forall (x, y) \in E^2$, the map $\alpha \mapsto F(\alpha, x, y)$ is twice continuously differentiable on $\mathcal{A}$.
- Let $F'(\cdot, \cdot, \cdot)$ and $F''(\cdot, \cdot, \cdot)$ be the first and second order partial derivatives w.r.t. α.

- $\forall (x,y) \in E^2$, the map $\alpha \mapsto F(\alpha, x, y)$ is twice continuously differentiable on $\mathcal{A}$.
- Let $F'(\cdot,\cdot,\cdot)$ and $F''(\cdot,\cdot,\cdot)$ be the first and second order partial derivatives w.r.t. α.
- Let

$$M_n'(\alpha) = \frac{1}{n}\sum_{k=1}^{n} F'(\alpha, X_{k-1}, X_k), \qquad M_n''(\alpha) = \frac{1}{n}\sum_{k=1}^{n} F''(\alpha, X_{k-1}, X_k).$$

Assumptions (1/2)

We impose sufficient conditions guaranteeing assumptions (A1)-(A6)

(V0) F' and F'' satisfy Condition (D_3);

Assumptions (1/2)

We impose sufficient conditions guaranteeing assumptions (A1)-(A6)

(V0) F' and F'' satisfy Condition (D_3);

(V1) $\forall\theta \in \Theta, \;\; \mathbb{E}_{\theta,\pi_\theta}[F'(\alpha_0, X_0, X_1)] = 0$ and $\alpha_0 = \alpha_0(\theta)$ is unique with this property;

Assumptions (1/2)

We impose sufficient conditions guaranteeing assumptions (A1)-(A6)

(V0) F' and F'' satisfy Condition (D_3);

(V1) $\forall\theta \in \Theta, \ \ \mathbb{E}_{\theta,\pi_\theta}[F'(\alpha_0, X_0, X_1)] = 0$ and $\alpha_0 = \alpha_0(\theta)$ is unique with this property;

(V2) $m(\theta) := \mathbb{E}_{\theta,\pi_\theta}[F''(\alpha_0, X_0, X_1)]$ satisfies $\inf_{\theta\in\Theta} m(\theta) > 0$;

Assumptions (1/2)

We impose sufficient conditions guaranteeing assumptions (A1)-(A6)

(V0) F' and F'' satisfy Condition (D_3);

(V1) $\forall \theta \in \Theta, \ \ \mathbb{E}_{\theta,\pi_\theta}[F'(\alpha_0, X_0, X_1)] = 0$ and $\alpha_0 = \alpha_0(\theta)$ is unique with this property;

(V2) $m(\theta) := \mathbb{E}_{\theta,\pi_\theta}[F''(\alpha_0, X_0, X_1)]$ satisfies $\inf_{\theta \in \Theta} m(\theta) > 0$;

(V3) $M_n'(\widehat{\alpha}_n)$ satisfies (A3), that is $\forall n \geq 1 \ \exists r_n > 0$ independent of θ such that $\sqrt{n} r_n \to 0$ and $\sup_{\theta \in \Theta} \mathbb{P}_\theta \left(|M_n'(\widehat{\alpha}_n)| \geq r_n\right) = O(n^{-1/2})$.

Assumptions (1/2)

We impose sufficient conditions guaranteeing assumptions (A1)-(A6)

(V0) F' and F'' satisfy Condition (D_3);

(V1) $\forall\theta\in\Theta,\ \ \mathbb{E}_{\theta,\pi_\theta}[F'(\alpha_0,X_0,X_1)]=0$ and $\alpha_0=\alpha_0(\theta)$ is unique with this property;

(V2) $m(\theta):=\mathbb{E}_{\theta,\pi_\theta}[F''(\alpha_0,X_0,X_1)]$ satisfies $\inf\limits_{\theta\in\Theta} m(\theta)>0$;

(V3) $M_n'(\widehat{\alpha}_n)$ satisfies (A3), that is $\forall n\geq 1\ \exists r_n>0$ independent of θ such that $\sqrt{n}r_n\to 0$ and $\sup_{\theta\in\Theta}\mathbb{P}_\theta\left(|M_n'(\widehat{\alpha}_n)|\geq r_n\right)=O(n^{-1/2})$.

Define the asymptotic variances:

$$\sigma_1^2(\theta) \;:=\; \lim_n \frac{1}{n}\,\mathbb{E}_{\theta,\mu}\left[\Big(\sum_{k=1}^{n}F'(\alpha_0,X_{k-1},X_k)\Big)^2\right]$$

$$\sigma_2^2(\theta) \;:=\; \lim_n \frac{1}{n}\,\mathbb{E}_{\theta,\mu}\left[\Big(\sum_{k=1}^{n}F''(\alpha_0,X_{k-1},X_k)-n\,m(\theta)\Big)^2\right].$$

Assumptions (1/2)

We impose sufficient conditions guaranteeing assumptions (A1)-(A6)

(V0) F' and F'' satisfy Condition (D_3);

(V1) $\forall \theta \in \Theta,\ \ \mathbb{E}_{\theta,\pi_\theta}[F'(\alpha_0, X_0, X_1)] = 0$ and $\alpha_0 = \alpha_0(\theta)$ is unique with this property;

(V2) $m(\theta) := \mathbb{E}_{\theta,\pi_\theta}[F''(\alpha_0, X_0, X_1)]$ satisfies $\inf_{\theta \in \Theta} m(\theta) > 0$;

(V3) $M_n'(\widehat{\alpha}_n)$ satisfies (A3), that is $\forall n \geq 1\ \exists r_n > 0$ independent of θ such that $\sqrt{n} r_n \to 0$ and $\sup_{\theta \in \Theta} \mathbb{P}_\theta\left(|M_n'(\widehat{\alpha}_n)| \geq r_n\right) = O(n^{-1/2})$.

Define the asymptotic variances:

$$\sigma_1^2(\theta) := \lim_n \frac{1}{n} \mathbb{E}_{\theta,\mu}\left[\Big(\sum_{k=1}^{n} F'(\alpha_0, X_{k-1}, X_k)\Big)^2\right]$$

$$\sigma_2^2(\theta) := \lim_n \frac{1}{n} \mathbb{E}_{\theta,\mu}\left[\Big(\sum_{k=1}^{n} F''(\alpha_0, X_{k-1}, X_k) - n\, m(\theta)\Big)^2\right].$$

Condition (V0) and Proposition 1 ensure $\sup_{\theta \in \Theta} \sigma_j(\theta) < \infty$ for $j = 1, 2$.

Assumptions (2/2)

The following conditions are also assumed to hold

(V4) $\inf_{\theta\in\Theta} \sigma_j(\theta) > 0$ *for* $j = 1, 2$.

Assumptions (2/2)

The following conditions are also assumed to hold

(V4) $\inf_{\theta\in\Theta}\sigma_j(\theta) > 0$ *for* $j = 1, 2$.

(V5) There exist $\eta \in (0, 1/2)$ and $C > 0$ such that $\forall(\alpha,\alpha') \in \mathcal{A}^2,\ \forall(x,y) \in E^2$,

$$\left|F''(\alpha,x,y) - F''(\alpha',x,y)\right| \leq C\,|\alpha-\alpha'|\,\big(V(x)+V(y)\big)^{\eta}.$$

Assumptions (2/2)

The following conditions are also assumed to hold

(V4) $\inf_{\theta\in\Theta}\sigma_j(\theta) > 0$ *for* $j = 1, 2$.

(V5) There exist $\eta \in (0, 1/2)$ and $C > 0$ such that $\forall(\alpha,\alpha') \in \mathcal{A}^2$, $\forall(x, y) \in E^2$,

$$\left|F''(\alpha, x, y) - F''(\alpha', x, y)\right| \leq C\,|\alpha - \alpha'|\,\big(V(x) + V(y)\big)^{\eta}.$$

(V6) there exists a sequence $\gamma_n \to 0$ such that

$$\sup_{\theta\in\Theta}\mathbb{P}_\theta\big\{\,|\widehat{\alpha}_n - \alpha_0| \geq d\,\big\} \leq \gamma_n,$$

with $d = \inf_{\theta\in\Theta} m(\theta)/8\pi_\theta(V^\eta)$ and η defined in (V5).

Theorem

Assume that the condition $(\mathcal{M})$ holds true, that F satisfies condition (D_1), that conditions (V0) to (V6) are fulfilled. Let $\tau(\theta) := \sigma_1(\theta)/m(\theta)$. Then there exists a positive constant C such that $\forall n \geq 1$,

$$\sup_{\theta\in\Theta}\sup_{u\in\mathbb{R}}\left|\mathbb{P}_{\theta,\mu}\left\{\frac{\sqrt{n}}{\tau(\theta)}(\widehat{\alpha}_n - \alpha_0) \leq u\right\} - \Gamma(u)\right| \leq C\left(\frac{1}{\sqrt{n}} + \sqrt{n}r_n + \gamma_n\right).$$

Estimation of the autoregressive coefficient ρ_0

- Condition $(\mathcal{M})$ holds for $V(x) = (1 + |x|)^p$ with $p > 6$

Estimation of the autoregressive coefficient ρ_0

- Condition $(\mathcal{M})$ holds for $V(x) = (1 + |x|)^p$ with $p > 6$
- $F(\rho, X_{k-1}, X_k) = (X_k - \rho X_{k-1})^2$

Estimation of the autoregressive coefficient ρ_0

- Condition $(\mathcal{M})$ holds for $V(x) = (1 + |x|)^p$ with $p > 6$
- $F(\rho, X_{k-1}, X_k) = (X_k - \rho X_{k-1})^2$
- $F'(\rho, X_{k-1}, X_k) = -2X_{k-1}(X_k - \rho X_{k-1})$
  $F''(\rho, X_{k-1}, X_k) = 2X_{k-1}^2$.

Estimation of the autoregressive coefficient ρ_0

- Condition $(\mathcal{M})$ holds for $V(x) = (1 + |x|)^p$ with $p > 6$
- $F(\rho, X_{k-1}, X_k) = (X_k - \rho X_{k-1})^2$
- $F'(\rho, X_{k-1}, X_k) = -2X_{k-1}(X_k - \rho X_{k-1})$
  $F''(\rho, X_{k-1}, X_k) = 2X_{k-1}^2$.
- By our results, there exists a positive constant C such that

$$\forall n \geq 1, \quad \sup_{\theta \in \Theta} \sup_{u \in \mathbb{R}} \left| \mathbb{P}_{\theta,\delta_0} \left\{ \frac{\sqrt{n}}{\sigma_1(\theta) m(\theta)^{-1}} (\widehat{\rho}_n - \rho_0) \leq u \right\} - \Gamma(u) \right| \leq \frac{C}{\sqrt{n}}.$$

Estimation of the conditional variance coefficient γ_0

- Condition $(\mathcal{M})$ holds for $V(x) = (1 + |x|)^p$ with $p > 12$

Estimation of the conditional variance coefficient γ_0

- Condition $(\mathcal{M})$ holds for $V(x) = (1 + |x|)^p$ with $p > 12$
- Define

$$Q_n(\gamma; r, v) = \frac{1}{n} \sum_{k=1}^{n} \eta_k^2(\gamma, r, v)$$

where

$$\eta_k(\gamma, r, v) = (X_k - rX_{k-1})^2 - v(1 - r^2 - \gamma) - \gamma X_{k-1}^2,$$

Estimation of the conditional variance coefficient γ_0

- Condition $(\mathcal{M})$ holds for $V(x) = (1 + |x|)^p$ with $p > 12$
- Define

$$Q_n(\gamma; r, v) = \frac{1}{n} \sum_{k=1}^{n} \eta_k^2(\gamma, r, v)$$

where

$$\eta_k(\gamma, r, v) = (X_k - rX_{k-1})^2 - v(1 - r^2 - \gamma) - \gamma X_{k-1}^2,$$

- Let $M_n(\gamma) = Q_n(\gamma; \widehat{\rho}_n, \widehat{\tau}_n^2)$, where $\widehat{\tau}_n^2 = n^{-1} \sum_{k=1}^{n} X_k^2$.

Estimation of the conditional variance coefficient γ_0

- Condition $(\mathcal{M})$ holds for $V(x) = (1 + |x|)^p$ with $p > 12$
- Define

$$Q_n(\gamma; r, v) = \frac{1}{n} \sum_{k=1}^{n} \eta_k^2(\gamma, r, v)$$

where

$$\eta_k(\gamma, r, v) = (X_k - rX_{k-1})^2 - v(1 - r^2 - \gamma) - \gamma X_{k-1}^2,$$

- Let $M_n(\gamma) = Q_n(\gamma; \widehat{\rho}_n, \widehat{\tau}_n^2)$, where $\widehat{\tau}_n^2 = n^{-1} \sum_{k=1}^{n} X_k^2$.
- Define $\widehat{\gamma}_n = \arg\min_{\gamma \in [m_\gamma, M_\gamma]} M_n(\gamma)$

Estimation of the coefficient γ_0 cont'd

- Define
  - $F'(\gamma, X_{k-1}, X_k) = 2(\tau_0^2 - X_{k-1}^2)\eta_k(\gamma, \rho_0, \tau_0^2)$
  - $F''(\gamma, X_{k-1}, X_k) = 2(\tau_0^2 - X_{k-1}^2)^2$
  - $M_n'(\gamma) = \partial Q_n / \partial\gamma(\gamma; \rho_0, \tau_0^2)$
  - $M_n''(\gamma) = \partial^2 Q_n / \partial\gamma^2(\gamma; \rho_0, \tau_0^2)$.

Estimation of the coefficient γ_0 cont'd

- Define
  - $F'(\gamma, X_{k-1}, X_k) = 2(\tau_0^2 - X_{k-1}^2)\eta_k(\gamma, \rho_0, \tau_0^2)$
  - $F''(\gamma, X_{k-1}, X_k) = 2(\tau_0^2 - X_{k-1}^2)^2$
  - $M_n'(\gamma) = \partial Q_n / \partial \gamma(\gamma; \rho_0, \tau_0^2)$
  - $M_n''(\gamma) = \partial^2 Q_n / \partial \gamma^2(\gamma; \rho_0, \tau_0^2)$.
- Check condition (V3) with $r_n = n^{-1} \log n$ and deduce that for some suitable $\tau(\theta)$

$$\forall n \geq 1, \quad \sup_{\theta \in \Theta} \sup_{u \in \mathbb{R}} \left| \mathbb{P}_{\theta, \delta_0} \left\{ \frac{\sqrt{n}}{\tau(\theta)} (\widehat{\gamma}_n - \gamma_0) \leq u \right\} - \Gamma(u) \right| = O\left(\frac{\log n}{\sqrt{n}} \right).$$

Estimation of the coefficient γ_0 cont'd

- Define
  - $F'(\gamma, X_{k-1}, X_k) = 2(\tau_0^2 - X_{k-1}^2)\eta_k(\gamma, \rho_0, \tau_0^2)$
  - $F''(\gamma, X_{k-1}, X_k) = 2(\tau_0^2 - X_{k-1}^2)^2$
  - $M_n'(\gamma) = \partial Q_n / \partial\gamma(\gamma; \rho_0, \tau_0^2)$
  - $M_n''(\gamma) = \partial^2 Q_n / \partial\gamma^2(\gamma; \rho_0, \tau_0^2)$.
- Check condition (V3) with $r_n = n^{-1} \log n$ and deduce that for some suitable $\tau(\theta)$

$$\forall n \geq 1, \quad \sup_{\theta\in\Theta} \sup_{u\in\mathbb{R}} \left| \mathbb{P}_{\theta,\delta_0} \left\{ \frac{\sqrt{n}}{\tau(\theta)} (\widehat{\gamma}_n - \gamma_0) \leq u \right\} - \Gamma(u) \right| = O\left(\frac{\log n}{\sqrt{n}} \right).$$

- The $\log n$ factor could be probably improved – open question...