Couplings of reflecting Brownian motions and applications

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As an application of the scaling coupling, we will prove a monotonicity of the lifetime of reflecting Brownian motion with killing, which implies the validity of the Hot Spots conjecture of J. Rauch for a certain class of domains.

As applications of the mirror coupling, we will present a proof of the Laugesen-Morpurgo conjecture on the radial monotonicity of the diagonal of the Neumann heat kernel of the unit ball in $\mathbb{R}^n$, and a unifying proof of the results of I. Chavel and W. Kendall on Chavel's conjecture on the domain monotonicity of the Neumann heat kernel.
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In general, a *coupling* is a construction of two processes $X$ and $Y$ on the same (or different) probability space $\Omega$, such that they are dependent in some useful way.
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In general, a coupling is a construction of two processes $X$ and $Y$ on the same (or different) probability space $\Omega$, such that they are dependent in some useful way.

In the present talk we will restrict the attention to couplings of stochastic processes, more precisely to the case of (reflecting or killed) Brownian motions.
**Definition (Brownian motion)**

A **1-dimensional Brownian motion** starting at $x \in \mathbb{R}$ is a continuous stochastic process $(B_t)_{t \geq 0}$ with $B_0 = x$ a.s for which $B_t - B_s$ is a normal random variable $\mathcal{N}(0, t - s)$, independent of the $\sigma$-algebra $\mathcal{F}_s = \sigma(B_r : r \leq s)$, for all $0 \leq s < t$. 

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**Definition (Reflecting Brownian motion)**

Reflecting Brownian motion in a smooth domain $D \subset \mathbb{R}^d$ starting at $x_0 \in D$ is a solution of the stochastic differential equation

$$X_t = x_0 + B_t + Z_t 0 \nu_D(X_s) dL_X_s,$$

where $B_t$ is a $d$-dimensional Brownian motion starting at $B_0 = 0$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, $\nu_D$ is the inward unit vector field on $\partial D$, $L_X_t$ is the local time of $X$ on the boundary of $D$, $X_t$ is $\mathcal{F}_t$-adapted and almost surely $X_t \in D$ for all $t \geq 0$. 

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Preliminaries

Definition (Brownian motion)

A 1-dimensional Brownian motion starting at \( x \in \mathbb{R} \) is a continuous stochastic process \((B_t)_{t \geq 0}\) with \( B_0 = x \) a.s for which \( B_t - B_s \) is a normal random variable \( \mathcal{N}(0, t - s) \), independent of the \( \sigma \)-algebra \( \mathcal{F}_s = \sigma(B_r : r \leq s) \), for all \( 0 \leq s < t \).

If \( B^i_t \) are independent 1-dimensional Brownian motions starting at \( x^i \), \( 1 \leq i \leq d \), then \( B_t = (B^1_t, \ldots, B^d_t) \) is a \( d \)-dimensional Brownian motion starting at \( x = (x^1, \ldots, x^d) \in \mathbb{R}^d \).
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X_t = x_0 + B_t + \int_0^t \nu_D(X_s) \, dL^X_s, \quad t \geq 0,
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where \( B_t \) is a \( d \)-dimensional Brownian motion starting at \( B_0 = 0 \) on \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \), \( \nu_D \) is the inward unit vector field on \( \partial D \), \( L^X_t \) is the local time of \( X \) on the boundary of \( D \), \( X_t \) is \( \mathcal{F}_t \)-adapted and almost surely \( X_t \in \overline{D} \) for all \( t \geq 0 \).
Preliminaries

Definition (Reflecting Brownian motion with killing)

If $X_t$ is a reflecting Brownian motion in a smooth domain $D \subset \mathbb{R}^d$ starting at $x_0 \in \overline{D}$, $S \subset \partial D$ and $\tau = \tau_S = \inf\{t > 0 : X_t \in S\}$ is the hitting time of $S$, then

$$Y_t = \begin{cases} X_t, & t < \tau \\ \dagger, & t \geq \tau \end{cases},$$

is a reflecting Brownian motion in $D$ killed on hitting $S \subset \partial D$ ($\dagger \notin D$ is the cemetery state and $\tau$ is the killing time).
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Proposition (Invariance properties of Brownian motion)

Brownian motion is invariant under is translation, rotation and symmetry.
It is also (almost) invariant under scaling and composition with conformal maps.
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Proposition (Invariance properties of Brownian motion)

**Brownian motion is invariant under is translation, rotation and symmetry.**

**It is also (almost) invariant under scaling and composition with conformal maps.**

This gives rise to the following couplings of Brownian motions:
Figure: Translation coupling
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Figure: Rotation coupling
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Figure: Mirror coupling
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Figure: Rotation coupling

Figure: Mirror coupling

Figure: Scaling coupling
Scaling coupling and applications

Key of the construction: if $B_t$ is a $d$-dimensional Brownian motion, then

$$\frac{1}{\sup_{s \leq t} \|B_s\|} B_t$$

is a time changed reflecting Brownian motion in the unit ball $U \subset \mathbb{R}^n$. 
Scaling coupling and applications

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**Theorem ([1])**

Let \( X_t \) be a reflecting Brownian motion in \( \mathbb{U} \) starting at \( X_0 = x_0 \in \mathbb{U} - \{0\} \) and \( a \in [\|x_0\|, 1] \). The process \( Y_t \) defined by:

\[
Y_t = \frac{1}{M_{\alpha t}} X_{\alpha t}, \quad t \geq 0,
\]

where \( M_t = a \vee \sup_{s \leq t} \|X_s\| \) and \( \alpha_t^{-1} = A_t = \int_0^t \frac{1}{M_s^2} ds \),

is a \( \mathcal{F}_{\alpha t}^X \)-adapted reflecting Brownian motion in \( \mathbb{U} \) starting at \( Y_0 = \frac{1}{d} x_0 \).
Scaling coupling and applications

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is a time changed reflecting Brownian motion in the unit ball $\mathbb{U} \subset \mathbb{R}^n$.

**Theorem ([1])**

Let $X_t$ be a reflecting Brownian motion in $\mathbb{U}$ starting at $X_0 = x_0 \in \mathbb{U} - \{0\}$ and $a \in [\|x_0\|, 1]$. The process $Y_t$ defined by:

$$Y_t = \frac{1}{M_{\alpha_t}} X_{\alpha_t}, \quad t \geq 0,$$

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is a $\mathcal{F}_{\alpha_t}^X$-adapted reflecting Brownian motion in $\mathbb{U}$ starting at $Y_0 = \frac{1}{a} x_0$.

**Definition**

The pair $X_t, Y_t$ constructed above is called a scaling coupling of reflecting Brownian motions in $\mathbb{U}$ starting at $x_0 \in \mathbb{U} - \{0\}$, respectively $y_0 = \frac{1}{a} x_0 \in \mathbb{U}$.
Figure: Mirror coupling of reflecting Brownian motions in the unit ball $\mathbb{U}$.
Monotonicity of lifetime of the killed RBM in $\mathbb{U}$

$M_t = a \vee \sup_{s \leq t} \|X_s\| \leq 1$

**Figure:** Mirror coupling of reflecting Brownian motions in the unit ball $\mathbb{U}$.
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Figure: Mirror coupling of reflecting Brownian motions in the unit ball $\mathbb{U}$.

$$M_t = a \vee \sup_{s \leq t} \|X_s\| \leq 1 \implies A_t = \int_0^t \frac{1}{M_s^2} ds \geq t$$
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$$M_t = a \vee \sup_{s \leq t} \|X_s\| \leq 1 \implies A_t = \int_0^t \frac{1}{M_s^2} ds \geq t \implies \alpha_t = A_t^{-1} \leq t \implies \tau^X = \alpha \tau^Y \leq \tau^Y$$

($\tau^X, \tau^Y$ denotes the lifetime of $X_t, Y_t$ killed on a hyperplane through origin).
Corollary

For any $t > 0$, $P(\tau^x > t)$ is a radially increasing function ($\tau^x$ is the lifetime of RBM in $U$ starting at $x$, killed on a hyperplane through origin).
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Using the asymptotics $P(\tau^x > t) \approx e^{-\lambda_1 t} \psi_1^2(x) = e^{-\mu_2 t} \varphi_2^2(x)$, we obtain:

**Theorem**

If $\varphi$ is a second Neumann eigenfunction of the Laplacian on $\mathbb{U}$ which is antisymmetric with respect to a hyperplane through the origin, then $\varphi$ is a radially monotone function. In particular, the maximum and the minimum of $\varphi$ over $\mathbb{U}$ are attained only at the boundary of $U$, that is the Hot Spots conjecture holds for $\varphi$. 
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Theorem

If $\varphi$ is a second Neumann eigenfunction of the Laplacian on $U$ which is antisymmetric with respect to a hyperplane through the origin, then $\varphi$ is a radially monotone function.
In particular, the maximum and the minimum of $\varphi$ over $\overline{U}$ are attained only at the boundary of $U$, that is the Hot Spots conjecture holds for $\varphi$.

Corollary

The Hot Spots conjecture holds for the unit ball $U \subset \mathbb{R}^n$, that is

$$\min_{\partial U} \varphi = \min_U \varphi < \max_U \varphi = \max_{\partial U} \varphi,$$

for any second Neumann eigenfunction $\varphi$ of $U$. 
An application to the Hots Spots conjecture

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Using conformal invariance of Brownian motion and the geometric characterization of a convex map, the same arguments can be applied to any smooth bounded domain \( D \subset \mathbb{R}^2 \) in order to obtain the following:

**Theorem ([1])**

> If \( D \subset \mathbb{R}^2 \) is a convex \( C^{1,\alpha} \) domain \( (0 < \alpha < 1) \), and at least one of the following hypothesis hold,

1. \( D \) is symmetric with respect to both coordinate axes;
2. \( D \) is symmetric with respect to the horizontal axis and the diameter to width ratio \( d_D/l_D \) is larger than \( \frac{4j_0}{\pi} \approx 3.06 \);

then Hot Spots conjecture holds for the domain \( D \).
Mirror coupling and applications

Mirror coupling was introduced by Kendall ([7]), and developed by Burdzy et. al ([1], [2], [3]).
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\begin{align*}
X_t &= x + B_t + \int_0^t \nu_D (X_s) \, dL_s^X \\
Y_t &= y + Z_t + \int_0^t \nu_D (X_s) \, dL_s^Y \\
Z_t &= B_t - 2 \int_0^t \frac{X_s - Y_s}{\|X_s - Y_s\|^2} (X_s - Y_s) \cdot dB_s
\end{align*}

and proved pathwise uniqueness and strong uniqueness for $t < \tau = \inf \{s > 0 : X_s = Y_s\}$.
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and proved pathwise uniqueness and strong uniqueness for $t < \tau = \inf \{ s > 0 : X_s = Y_s \}$. We let $X_t = Y_t$ for $t \geq \tau$, and refer to $X_t, Y_t$ as a **mirror coupling** in $D$ starting at $x, y \in \overline{D}$.
Mirror coupling and applications

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$$X_t = x + B_t + \int_0^t \nu_D (X_s) \, dL^X_s$$  \hspace{1cm} (4)$$

$$Y_t = y + Z_t + \int_0^t \nu_D (X_s) \, dL^Y_s$$  \hspace{1cm} (5)$$

$$Z_t = B_t - 2 \int_0^t \frac{X_s - Y_s}{||X_s - Y_s||^2} (X_s - Y_s) \cdot dB_s$$  \hspace{1cm} (6)$$

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Remark

$G (u) v = v - 2 (u \cdot v) u$ is the mirror image of $v$ wrt hyperplane through $0$ perpendicular to $u$.

$$(6) \iff dZ_t = G \left( \frac{X_t - Y_t}{||X_t - Y_t||} \right) \, dW_t,$$

(the increments of $Z_t$ and $B_t$ are mirror images wrt hyperplane of symmetry $\mathcal{M}_t$ of $X_t$ and $Y_t$).
Lemma ("Mirror $\mathcal{M}_t$ moves towards origin", [3])

Let $X_t, Y_t$ be a mirror coupling of RBM in $\mathbb{U}$ starting at $x, y \in \overline{\mathbb{U}}$, and let
\[ \tau = \inf\{t > 0 : X_t = Y_t\} \text{ and } \tau_1 = \inf\{t > 0 : 0 \in \mathcal{M}_t\}. \]

For all times $t < \tau \wedge \tau_1$, the mirror $\mathcal{M}_t$ moves towards the origin, in such a way that if a point $P \in \mathbb{U}$ and the origin are separated by $\mathcal{M}_{t_1}$ for $t_1 \in [0, \tau \wedge \tau_1)$, then the point $P$ and the origin are separated by $\mathcal{M}_{t_2}$ for all $t_2 \in [t_1, \tau \wedge \tau_1)$.

**Figure:** Mirror coupling of reflecting Brownian motions in the unit disk ($d = 2$).
Let $p_U(t, x, y)$ denote the Neumann heat kernel of the unit ball $U \subset \mathbb{R}^d (d \geq 1)$. 
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**Theorem ([3])**

For any points $x, y, z \in \overline{U}$ such that $\|y\| \leq \|x\|$ and $\|x - z\| \leq \|y - z\|$, and any $t > 0$ we have:

$$p_U(t, y, z) \leq p_U(t, x, z).$$

(7)
Inequalities for the Neumann heat kernel of the unit ball

Let $p_U(t, x, y)$ denote the Neumann heat kernel of the unit ball $U \subset \mathbb{R}^d$ ($d \geq 1$).

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**Theorem ([3])**

For any $x \in U - \{0\}$, $r \in (0, \min \{\|x\|, 1 - \|x\|\})$ and $t > 0$ we have:

$$\int_{\partial U} p_U(t, x + ru, x) d\sigma(u) \leq p_U(t, x + r\frac{x}{\|x\|}, x) \leq p_U(t, x + r\frac{x}{\|x\|}, x + r\frac{x}{\|x\|}), \tag{8}$$

where $\sigma$ is the normalized surface measure on $\partial U$. 

Theorem (Resolution of the Laugesen-Morpurgo conjecture)

For any $t > 0$, $p_U(t, x, x)$ is a strictly increasing radial function in $\mathbb{U}$, that is

$$p_U(t, x, x) < p_U(t, y, y),$$

(9)

for all $x, y \in \mathbb{U}$ with $\|x\| < \|y\|$.
Resolution of the Laugesen-Morpurgo conjecture

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for all $x, y \in \overline{U}$ with $\|x\| < \|y\|$.

**Proof.**

$$\frac{d}{d\|x\|} p_U(t, x, x) = \lim_{r \searrow 0} \frac{p_U(t, x + r \frac{x}{\|x\|}, x + r \frac{x}{\|x\|}) - p_U(t, x, x)}{r}$$
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\geq \lim_{r \searrow 0} \int_{\partial U} p_U(t, x + ru, x) d\sigma(u) - p_U(t, x, x) / r
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= \int_{\partial U} \lim_{r \searrow 0} \frac{p_U(t, x + ru, x) - p_U(t, x, x)}{r} d\sigma(u) \\
= \int_{\partial U} \nabla p_U(t, x, x) \cdot u d\sigma(u)
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Resolution of the Laugesen-Morpurgo conjecture

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for all \( x, y \in \overline{U} \) with \( ||x|| < ||y|| \).

Proof.

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\frac{d}{d||x||} p_U(t, x, x) = \lim_{r \searrow 0} \frac{p_U(t, x + r \frac{x}{||x||}, x + r \frac{x}{||x||}) - p_U(t, x, x)}{r} \\
\geq \lim_{r \searrow 0} \int_{\partial U} p_U(t, x + ru, x) d\sigma(u) - p_U(t, x, x) \\
= \int_{\partial U} \lim_{r \searrow 0} \frac{p_U(t, x + ru, x) - p_U(t, x, x)}{r} d\sigma(u) \\
= \int_{\partial U} \nabla p_U(t, x, x) \cdot u d\sigma(u) \\
= 0.
\]
Recently ([3]), the author extended the construction of the mirror coupling to the case when the two RBM live in different domains $D_1, D_2 \subset \mathbb{R}^n$ such that $D_1 \cap D_2$ is a convex domain and $D_{1,2}$ have non-tangential boundaries.

**Figure:** Typical domains for the extended mirror coupling.
Conjecture (Chavel’s conjecture on domain monotonicity of Neumann heat kernel, 1986)

If $D_1 \subset D_2$ are convex domains then for all $t > 0$ and $x, y \in D_1$ we have

$$p_{D_1}(t, x, y) \geq p_{D_2}(t, x, y).$$
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When combined, the above results show the following:

Theorem

If $D_1 \subset D_2$ are convex domains then for all $t > 0$ and $x, y \in D_1$ we have

$$p_{D_1}(t, x, y) \geq p_{D_2}(t, x, y),$$

whenever there exists a ball $B$ centered at either $x$ or $y$ such that $D_1 \subset B \subset D_2$. 
Sketch of the proof

Consider a mirror coupling of RBM $X_t, Y_t$ in $D_2, D_1$, starting at $X_0 = Y_0 = x$. 

For all times $t > 0$, the mirror $M_t$ of the coupling cannot separate the points $Y_t$ and $y$. 

For all $t > 0$, 

$$\| Y_t - y \| \leq \| X_t - y \|.$$ 

$$\Rightarrow p_{D_1}(t, x, y) > p_{D_2}(t, x, y).$$
Sketch of the proof

Consider a mirror coupling of RBM $X_t, Y_t$ in $D_2, D_1$, starting at $X_0 = Y_0 = x$. For all times $t > 0$, the mirror $M_t$ of the coupling cannot separate the points $Y_t$ and $y$. 
Consider a mirror coupling of RBM $X_t, Y_t$ in $D_2, D_1$, starting at $X_0 = Y_0 = x$. For all times $t > 0$, the mirror $\mathcal{M}_t$ of the coupling cannot separate the points $Y_t$ and $y$. 

\[ \| Y_t - y \| \leq \| X_t - y \|, \quad t > 0 \Rightarrow p_{D_1}(t, x, y) > p_{D_2}(t, x, y). \]
Sketch of the proof

Consider a mirror coupling of RBM $X_t, Y_t$ in $D_2, D_1$, starting at $X_0 = Y_0 = x$. For all times $t > 0$, the mirror $M_t$ of the coupling cannot separate the points $Y_t$ and $y$.

$\|Y_t - y\| \leq \|X_t - y\|, \quad t > 0$
Sketch of the proof

Consider a mirror coupling of RBM $X_t, Y_t$ in $D_2, D_1$, starting at $X_0 = Y_0 = x$. For all times $t > 0$, the mirror $M_t$ of the coupling cannot separate the points $Y_t$ and $y$.

$$\|Y_t - y\| \leq \|X_t - y\|, \quad t > 0 \quad \implies \quad p_{D_1}(t, x, y) > p_{D_2}(t, x, y).$$


M. N. Pascu, Mirror coupling of reflecting Brownian motion and an application to Chavel’s conjecture, to appear (http://arxiv.org/abs/1004.2398)