Couplings of reflecting Brownian motions and applications

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As an application of the scaling coupling, we will prove a monotonicity of the lifetime of reflecting Brownian motion with killing, which implies the validity of the Hot Spots conjecture of J. Rauch for a certain class of domains.

As applications of the mirror coupling, we will present a proof of the Laugesen-Morpurgo conjecture on the radial monotonicity of the diagonal of the Neumann heat kernel of the unit ball in $\mathbb{R}^n$, and a unifying proof of the results of I. Chavel and W. Kendall on Chavel's conjecture on the domain monotonicity of the Neumann heat kernel.

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In general, a *coupling* is a construction of two processes X and Y on the same (or different) probability space Ω, such that they are dependent in some useful way.

In the present talk we will restrict the attention to couplings of stochastic processes, more precisely to the case of (reflecting or killed) Brownian motions.

Preliminaries

Definition (Brownian motion)

A 1-dimensional Brownian motion starting at $x \in \mathbb{R}$ is a continuous stochastic process $(B_t)_{t\geq 0}$ with $B_0 = x$ a.s for which $B_t - B_s$ is a normal random variable $\mathcal{N}(0, t-s)$, independent of the σ-algebra $\mathcal{F}_s = \sigma(B_r : r \leq s)$, for all $0 \leq s < t$.

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If B_t^i are independent 1-dimensional Brownian motions starting at x^i, $1 \leq i \leq d$, then $B_t = \left(B_t^1, \ldots, B_t^d\right)$ is a d-dimensional Brownian motion starting at $x = \left(x^1, \ldots, x^d\right) \in \mathbb{R}^d$.

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Definition (Reflecting Brownian motion)

Reflecting Brownian motion in a smooth domain $D \subset \mathbb{R}^d$ starting at $x_0 \in \overline{D}$ is a solution of the stochastic differential equation

$$X_t = x_0 + B_t + \int_0^t \nu_D(X_s)\, dL_s^X, \qquad t \geq 0, \tag{1}$$

where B_t is a d-dimensional Brownian motion starting at $B_0 = 0$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, ν_D is the inward unit vector field on ∂D, L_t^X is the local time of X on the boundary of D, X_t is $\mathcal{F}_t$-adapted and almost surely $X_t \in \overline{D}$ for all $t \geq 0$.

Preliminaries

Definition (Reflecting Brownian motion with killing)

If X_t is a reflecting Brownian motion in a smooth domain $D \subset \mathbb{R}^d$ starting at $x_0 \in \overline{D}$, $S \subset \partial D$ and $\tau = \tau_S = \inf\{t > 0 : X_t \in S\}$ is the hitting time of S, then

$$Y_t = \begin{cases} X_t, & t < \tau \\ \dagger, & t \geq \tau \end{cases}, \tag{2}$$

is a reflecting Brownian motion in D killed on hitting $S \subset \partial D$
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Brownian motion is invariant under is translation, rotation and symmetry.
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This gives rise to the following couplings of Brownian motions:

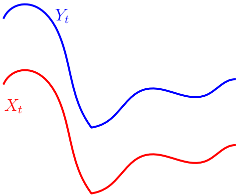
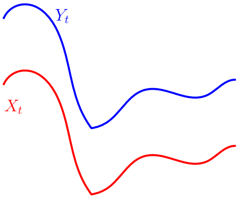


Figure: Translation coupling



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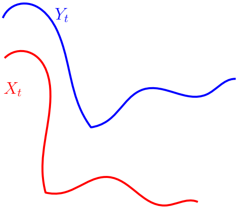

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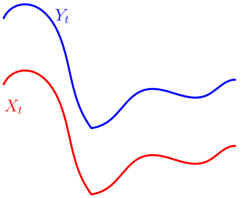

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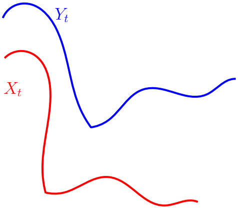

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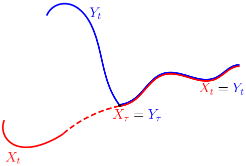
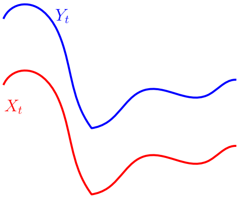


Figure: Mirror coupling



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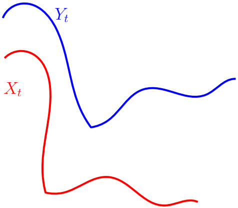

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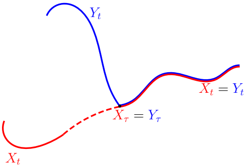

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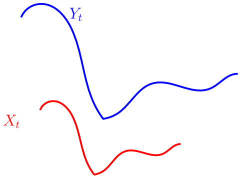

Figure: Scaling coupling

Scaling coupling and applications

Key of the construction: if B_t is a d-dimensional Brownian motion, then

$$\frac{1}{\sup_{s\leq t}\|B_s\|}B_t$$

is a time changed reflecting Brownian motion in the unit ball $\mathbb{U}\subset\mathbb{R}^n$.

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Theorem ([1])

Let X_t be a reflecting Brownian motion in $\mathbb{U}$ starting at $X_0=x_0\in\overline{\mathbb{U}}-\{0\}$ and $a\in[\|x_0\|,1]$. The process Y_t defined by:

$$Y_t=\frac{1}{M_{\alpha_t}}X_{\alpha_t},\qquad t\geq 0, \tag{3}$$

where $M_t=a\vee\sup_{s\leq t}\|X_s\|$ and $\alpha_t^{-1}=A_t=\int_0^t\frac{1}{M_s^2}ds$,
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Definition

The pair X_t,Y_t constructed above is called a scaling coupling of reflecting Brownian motions in $\mathbb{U}$ starting at $x_0\in\overline{\mathbb{U}}-\{0\}$, respectively $y_0=\frac{1}{a}x_0\in\overline{\mathbb{U}}$.

Monotonicity of lifetime of the killed RBM in $\mathbb{U}$

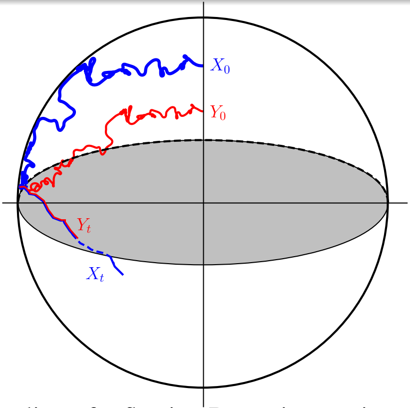

Figure: Mirror coupling of reflecting Brownian motions in the unit ball $\mathbb{U}$.

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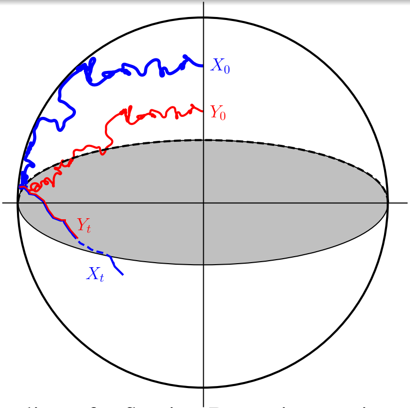

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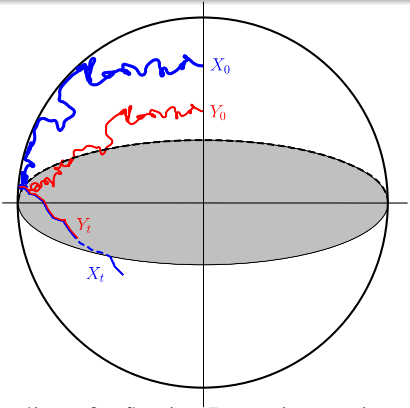

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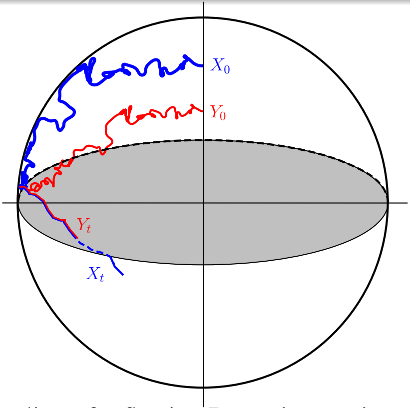

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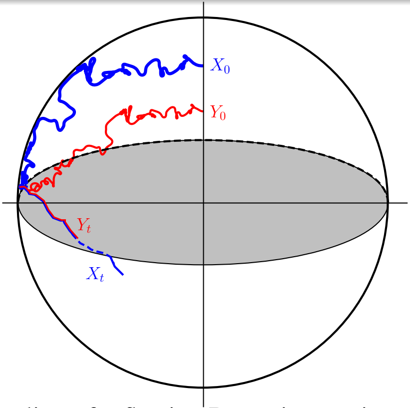

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(τ^X, τ^Y denotes the lifetime of X_t, Y_t killed on a hyperplane through origin).

Scaling coupling and applications

Corollary

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Theorem

If φ is a second Neumann eigenfunction of the Laplacian on $\mathbb{U}$ which is antisymmetric with respect to a hyperplane through the origin, then φ is a radially monotone function.
In particular, the maximum and the minimum of φ over $\overline{\mathbb{U}}$ are attained only at the boundary of U, that is the Hot Spots conjecture holds for φ.

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Corollary

The Hot Spots conjecture holds for the unit ball $\mathbb{U} \subset \mathbb{R}^n$, that is

$$\min_{\partial\mathbb{U}} \varphi = \min_{\overline{\mathbb{U}}} \varphi < \max_{\overline{\mathbb{U}}} \varphi = \max_{\partial\mathbb{U}} \varphi,$$

for any second Neumann eigenfunction φ of $\mathbb{U}$.

An application to the Hots Spots conjecture

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Theorem ([1])

If $D \subset \mathbb{R}^2$ is a convex $C^{1,\alpha}$ domain ($0 < \alpha < 1$), and at least one of the following hypothesis hold,

i) *D is symmetric with respect to both coordinate axes;*

ii) *D is symmetric with respect to the horizontal axis and the diameter to width ratio d_D / l_D is larger than $\frac{4 j_0}{\pi} \approx 3.06$;*

then Hot Spots conjecture holds for the domain D.

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$$X_t = x + B_t + \int_0^t \nu_D(X_s)\, dL_s^X \quad (4)$$

$$Y_t = y + Z_t + \int_0^t \nu_D(X_s)\, dL_s^Y \quad (5)$$

$$Z_t = B_t - 2\int_0^t \frac{X_s - Y_s}{||X_s - Y_s||^2}(X_s - Y_s) \cdot dB_s \quad (6)$$

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Remark

$G(u)\, v = v - 2(u \cdot v)\, u$ is the mirror image of v wrt hyperplane through 0 perpendicular to u.

$$(6) \iff dZ_t = G\left(\frac{X_t - Y_t}{||X_t - Y_t||}\right) dW_t,$$

(the increments of Z_t and B_t are mirror images wrt hyperplane of symmetry $\mathcal{M}_t$ of X_t and Y_t).

Lemma ("Mirror $\mathcal{M}_t$ moves towards origin", [3])

Let X_t, Y_t be a mirror coupling of RBM in $\mathbb{U}$ starting at $x, y \in \overline{\mathbb{U}}$, and let

$$\tau = \inf\{t > 0 : X_t = Y_t\} \text{ and } \tau_1 = \inf\{t > 0 : 0 \in \mathcal{M}_t\}.$$

For all times $t < \tau \wedge \tau_1$, the mirror $\mathcal{M}_t$ moves towards the origin, in such a way that if a point $P \in \mathbb{U}$ and the origin are separated by $\mathcal{M}_{t_1}$ for $t_1 \in [0, \tau \wedge \tau_1)$, then the point P and the origin are separated by $\mathcal{M}_{t_2}$ for all $t_2 \in [t_1, \tau \wedge \tau_1)$.

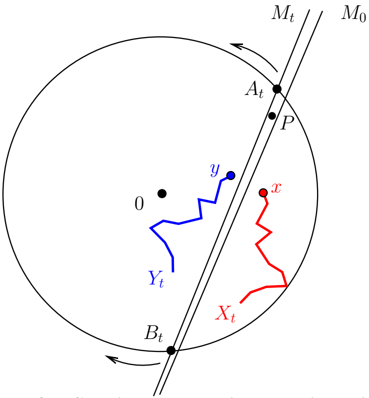

Figure: Mirror coupling of reflecting Brownian motions in the unit disk ($d = 2$).

Inequalities for the Neumann heat kernel of the unit ball

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For any points $x, y, z \in \overline{\mathbb{U}}$ such that $\|y\| \leq \|x\|$ and $\|x - z\| \leq \|y - z\|$, and any $t > 0$ we have:

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Theorem ([3])

For any $x \in \mathbb{U}-\{0\}$, $r \in (0, \min\{\|x\|, 1-\|x\|\})$ and $t>0$ we have:

$$\int_{\partial\mathbb{U}} p_{\mathbb{U}}(t, x+ru, x)\, d\sigma(u) \leq p_{\mathbb{U}}(t, x + r\tfrac{x}{\|x\|}, x) \leq p_{\mathbb{U}}(t, x+r\tfrac{x}{\|x\|}, x + r\tfrac{x}{\|x\|}), \tag{8}$$

where σ is the normalized surface measure on $\partial\mathbb{U}$.

Resolution of the Laugesen-Morpurgo conjecture

Theorem (Resolution of the Laugesen-Morpurgo conjecture)

For any $t > 0$, $p_{\mathbb{U}}(t, x, x)$ is a strictly increasing radial function in $\mathbb{U}$, that is

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for all $x, y \in \overline{\mathbb{U}}$ *with* $\|x\| < \|y\|$.

Proof.

$$\frac{d}{d\|x\|} p_{\mathbb{U}}(t,x,x) = \lim_{r \searrow 0} \frac{p_{\mathbb{U}}(t, x + r\frac{x}{\|x\|}, x + r\frac{x}{\|x\|}) - p_U(t,x,x)}{r}$$

Resolution of the Laugesen-Morpurgo conjecture

Theorem (Resolution of the Laugesen-Morpurgo conjecture)

For any $t > 0$, $p_{\mathbb{U}}(t,x,x)$ is a strictly increasing radial function in $\mathbb{U}$, that is

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&= 0.
\end{aligned}$$

Recently ([3]), the author extended the construction of the mirror coupling to the case when the two RBM live in different domains $D_1, D_2 \subset \mathbb{R}^n$ such that $D_1 \cap D_2$ is a convex domain and $D_{1,2}$ have non-tangential boundaries.

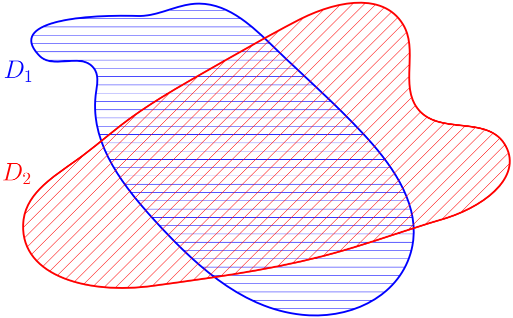

Figure: Typical domains for the extended mirror coupling.

A unifying proof of Chavel's conjecture

Conjecture (Chavel's conjecture on domain monotonicity of Neumann heat kernel, 1986)

If $D_1 \subset D_2$ are convex domains then for all $t > 0$ and $x, y \in D_1$ we have

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When combined, the above results show the following:

Theorem

If $D_1 \subset D_2$ are convex domains then for all $t > 0$ and $x, y \in D_1$ we have

$$p_{D_1}(t, x, y) \geq p_{D_2}(t, x, y),$$

whenever there exists a ball B centered at either x or y such that $D_1 \subset B \subset D_2$.

Sketch of the proof

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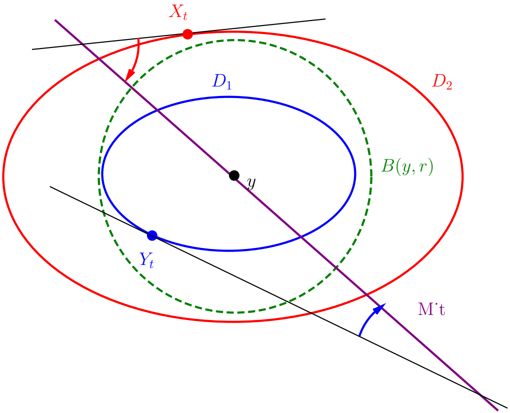

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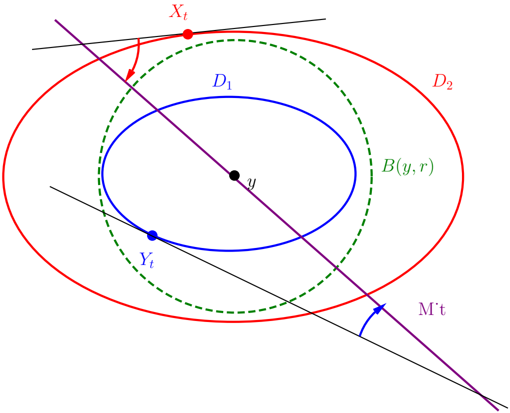

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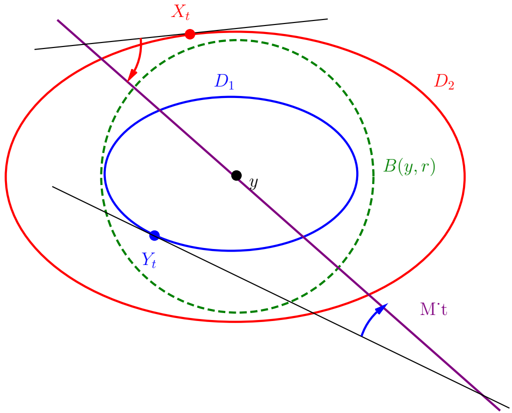

$$\|Y_t - y\| \le \|X_t - y\|, \qquad t > 0 \qquad \Longrightarrow \qquad p_{D_1}(t,x,y) > p_{D_2}(t,x,y). \qquad \square$$

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