WEAK MAJORIZATON PROPERTIES OF STOCHASTIC COMPARISONS

Eugen Păltănea
Universitatea *Transilvania* din Brașov

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Overview on the topic

- *position:* applied probabilities
- *survey:* comparison of (positive) random variables (*lifetimes*)
- *development* exponential growth in the last 45 years
- *interest*: reliability theory, statistics (survival analysis), operations research, financial mathematics,...

A fundamental book:

Shaked & Shanthikumar, *Stochastic Orders*, 2007

Presentation

Aim:

- relevant majorization type orders
- review of some recent advances in *parametric stochastic comparison* of *convolutions* and *order statistics*
- proposed extensions

Usual stochastic order

Let X and Y be two positive random variables with distribution functions F_X and F_Y, survival functions $\overline{F}_X$ and $\overline{F}_Y$, and densities f_X and f_Y, respectively.

- X is greater than Y in *the usual stochastic order*, denoted by $X \geq_{st} Y$, if

$$P\{X > t\} = \overline{F}_X(t) \geq \overline{F}_Y(t) = P\{Y > t\}, \;\; \forall\ t \geq 0.$$

The hazard rate order

- *The hazard rate order*: $X \geq_{hr} Y$ if

  $$P\{X-t>u|X>t\} \geq P\{Y-t>u|Y>t\} \text{ (i.e. } \overline{F}_X/\overline{F}_Y \uparrow),$$

  for all $t,u \geq 0$.
- *The hazard rate function* of a r.v. X: $h_X = \frac{f_X}{\overline{F}_X}$.
- Equivalent definition:

  $$X \geq_{hr} Y \;\Leftrightarrow\; h_X(t) \leq h_Y(t),\ \forall\ t \geq 0.$$

- X is *IFR (Increasing Failure Rate)*
  if $h_X \uparrow$ ($\overline{F}$ - logconcave)

Other stochastic orders

- *the likelihood ratio order:* $X \geq_{lr} Y$
  if $f_X(t)/f_Y(t)$ is increasing in $t \geq 0$;
- *the mean residual life order:* $X \geq_{mrl} Y$
  if $\frac{1}{\overline{F}_X(t)} \int_t^\infty \overline{F}_X(u)\, du \geq \frac{1}{\overline{F}_Y(t)} \int_t^\infty \overline{F}_Y(u)\, du$, for all $t \geq 0$;
- *increasing convex order*: $X \geq_{icx} Y$
  if $\int_t^\infty \overline{F}_X(u)du \geq \int_t^\infty \overline{F}_Y(u)du, \ \forall \ t \geq 0$.

Other stochastic orders

- *the dispersive order*: $X \geq_{disp} Y$ if $F_X^{-1}(v) - F_X^{-1}(u) \geq F_Y^{-1}(v) - F_Y^{-1}(u)$, for all $0 \leq u \leq v \leq 1$.

Classical relations

The following relations hold:

- $X \geq_{lr} Y \Rightarrow X \geq_{hr} Y \Rightarrow X \geq_{mrl,\ st} Y \Rightarrow X \geq_{ics} Y$;
- $X \geq_{disp} Y$ and X (or Y) is $IFR \Rightarrow X \geq_{hr} Y$
- $X \leq_{disp} Y \Rightarrow Var(X) \leq Var(Y)$.

Majorization

Definition

Let $\lambda_{(1)} \leq \lambda_{(2)} \leq \cdots \leq \lambda_{(n)}$ *be the increasing arrangement of the components of* $\overline{\lambda} = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$.

The majorization order: $\overline{\lambda} \overset{m}{\succeq} \overline{\mu}$ *if*

$$\sum_{i=1}^{k} \lambda_{(i)} \leq \sum_{i=1}^{k} \mu_{(i)} \text{ for } 1 \leq k \leq n-1, \quad \text{and} \quad \sum_{i=1}^{n} \lambda_{(i)} = \sum_{i=1}^{n} \mu_{(i)}.$$

Schur convexity, weak majorization

Definition

1. *A permutation symmetric function* $f : D \subset \mathbb{R}^n \to \mathbb{R}$ *is* *Schur convex if*

$$\overline{\lambda} \overset{m}{\succeq} \overline{\mu} \;\Rightarrow\; f(\overline{\lambda}) \geq f(\overline{\mu}).$$

2. *Weak (increasing) majorization order:* $\overline{\lambda} \overset{w\uparrow}{\succeq} \overline{\mu}$ *if*

$$\sum_{i=1}^{k} x_{(i)} \leq \sum_{i=1}^{k} y_{(i)} \quad \text{for } k = 1, \cdots, n.$$

Weak majorization type orders

Definition

1. *p-larger order* (Bon & Păltănea, 1999)
   Assume $\overline{\lambda}, \overline{\mu} \in (0,\infty)^n$. Then $\overline{\lambda} \overset{p}{\succeq} \overline{\mu}$ if
   $$\prod_{i=1}^{k} \lambda_{(i)} \le \prod_{i=1}^{k} \mu_{(i)} \quad \text{for } k = 1, \cdots, n.$$
2. *reciprocal majorization* (Zhao & Balakrishnan, 2009b)
   Assume $\overline{\lambda}, \overline{\mu} \in (0,\infty)^n$. Then $\overline{\lambda} \overset{rm}{\succeq} \overline{\mu}$ if
   $$\sum_{i=1}^{k} \frac{1}{\lambda_{(i)}} \ge \sum_{i=1}^{k} \frac{1}{\mu_{(i)}} \quad \text{for } k = 1, \cdots, n.$$

Relations between the majorization type orders

Theorem

1. *(Khaledi & Kochar, 2002)*

$$\overline{\lambda} \overset{m}{\succeq} \overline{\mu} \;\Rightarrow\; \overline{\lambda} \overset{p}{\succeq} \overline{\mu}$$

2. *(Kochar & Xu, 2010)*

$$\overline{\lambda} \overset{p}{\succeq} \overline{\mu} \;\Rightarrow\; \overline{\lambda} \overset{rm}{\succeq} \overline{\mu}$$

3.

$$\overline{\lambda} \overset{w\uparrow}{\succeq} \overline{\mu} \;\Rightarrow\; \overline{\lambda} \overset{p}{\succeq} \overline{\mu}$$

PHR model

Definition

Let $\overline{F}$ be a survival function (of some positive r. v. X). Then independent r. v.'s $X_1, \cdots, X_n$ follow the PHR model (Proportional Hazard Rates) if there exists $\overline{\lambda} = (\lambda_1, \cdots, \lambda_n) > 0$ such that:

$$\overline{F}_{X_i}(t) = \left(\overline{F}(t)\right)^{\lambda_i}, \; t \geq 0, \; i = 1, \cdots, n.$$

Equivalent definition of *PHR model*:

$$h_{X_i}(t) = \lambda_i h(t), \; t \geq 0, \; i = 1, \cdots, n,$$

where h is the hazard rate of X.

PRV model

Definition

Let $\overline{F}$ be a survival function (of some positive r. v. X). Then independent r. v.'s $X_1, \cdots, X_n$ follow the PRV model (Proportional Random Variables) if there exists $\overline{\lambda} = (\lambda_1, \cdots, \lambda_n) > 0$ such that:

$$\overline{F}_{X_i}(t) = \overline{F}(\lambda_i t), \; t \geq 0, \; i = 1, \cdots, n.$$

Equivalent definition of *PRV model*:

$$X_i = U_i/\lambda_i, \; t \geq 0, \; i = 1, \cdots, n,$$

where $U_1, \cdots, U_n$ are iid, with $U_i \sim X$.

Example

Definition

$X \sim Exp(\lambda)$ (exponential distribution of parameter λ) if $\overline{F}_X(t) = e^{-\lambda t}$, $t \geq 0$, where $\lambda > 0$.

Remark

1. *For the exponential distribution, PHR and PRV are equivalent models:*

$$\overline{F}(t) = e^{-t} \Rightarrow \left(\overline{F}(t)\right)^{\lambda} = \overline{F}(\lambda t) = e^{-\lambda t}.$$

2. *$Exp(\lambda)$ has:*
   - *memoryless property ($\mapsto$ Markov processes)*
   - *constant hazard rate λ*

Classical results

Theorem

Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ be two sets of independent exponential r. v.'s with respective parameters vectors $\overline{\lambda} = (\lambda_1, \cdots, \lambda_n)$ and $\overline{\mu} = (\mu_1, \cdots, \mu_n)$.

1. *(Proschan & Sethuraman, 1976)*

$$\overline{\lambda} \overset{m}{\succeq} \overline{\mu} \;\Rightarrow\; \sum_{i=1}^{n} X_i \geq_{st} \sum_{i=1}^{n} Y_i .$$

2. *(Boland, El-Neweihi & Proschan, 1994)*

$$\overline{\lambda} \overset{m}{\succeq} \overline{\mu} \;\Rightarrow\; \sum_{i=1}^{n} X_i \geq_{lr} \sum_{i=1}^{n} Y_i .$$

Extensions

Theorem

Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ be indep. exp. r. v.'s with parameters $\overline{\lambda} = (\lambda_1, \cdots, \lambda_n)$ and $\overline{\mu} = (\mu_1, \cdots, \mu_n)$.
Denote $X_{(i)}$ the r.v. corresponding to parameter $\lambda_{(i)}$
$Y_{(i)}$ the r.v. corresponding to parameter $\mu_{(i)}$.
Then:

1. $\overline{\lambda} \overset{w\uparrow}{\succeq} \overline{\mu} \quad \Leftrightarrow \quad \sum_{i=1}^{k} X_{(i)} \geq_{lr} \sum_{i=1}^{k} Y_{(i)}$ $(1 \leq k \leq n)$;
2. $\overline{\lambda} \overset{p}{\succeq} \overline{\mu} \quad \Leftrightarrow \quad \sum_{i=1}^{k} X_{(i)} \geq_{hr,\ st,\ disp} \sum_{i=1}^{k} Y_{(i)}$ $(1 \leq k \leq n)$;
3. $\overline{\lambda} \overset{rm}{\succeq} \overline{\mu} \quad \Leftrightarrow \quad \sum_{i=1}^{k} X_{(i)} \geq_{mrl,\ icx} \sum_{i=1}^{k} Y_{(i)}$ $(1 \leq k \leq n)$.

References

For the proof, we make use of some recent characterizations.
More specifically,
Proof of (1): Zhao & Balakrishnan, 2009a;
Proof of (2): Bon & Păltănea, 1999;
Khaledi and Kochar, 2004;
Zhao & Balakrishnan, 2010 (to appear);
Proof of (3): Zhao & Balakrishnan, 2009c;
Zhao & Balakrishnan, 2010.

Case iid

Corollary

Assume $\mu_i = \mu,\ i = 1, \cdots, n.$ *Then*

1. $$\mu \geq \frac{\sum_{i=1}^n \lambda_i}{n} \quad \Leftrightarrow \quad \sum_{i=1}^n X_i \geq_{lr} \sum_{i=1}^n Y_i;$$

2. $$\mu \geq \sqrt[n]{\prod_{i=1}^n \lambda_i} \quad \Leftrightarrow \quad \sum_{i=1}^n X_i \geq_{hr,\ st,\ disp} \sum_{i=1}^n Y_i;$$

3. $$\mu \geq \frac{n}{\sum_{i=1}^n \frac{1}{\lambda_i}} \quad \Leftrightarrow \quad \sum_{i=1}^n X_i \geq_{mrl,\ icx} \sum_{i=1}^n Y_i.$$

Discrete random variables

Definition

X is a geometric random variable with parameter $p \in (0,1)$ if its probability mass function is

$$f_X(k) = P(X = k) = p(1-p)^k, \; k = 0, 1, 2, \cdots$$

Remark

Geometric distribution has some analogous properties with the exponential distribution:

1. *constant hazard rate*
2. *memoryless property*

Sums of independent geometric random variables

Theorem

Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ be two sets of independent geometric random variables with respective parameters $\overline{p} = (p_1, \cdots, p_n)$ and $\overline{q} = (q_1, \cdots, q_n)$. Let $X_{(i)}$ have the parameter $p_{(i)}$ and $Y_{(i)}$ have the parameter $q_{(i)}$. Then:

1. $\sum_{i=1}^{k} X_{(i)} \geq_{lr} \sum_{i=1}^{k} Y_{(i)}$ $(1 \leq k \leq n)$ $\Leftrightarrow$ $\overline{p} \overset{w\uparrow}{\succeq} \overline{q}$
2. $\sum_{i=1}^{k} X_{(i)} \geq_{hr,\ st} \sum_{i=1}^{k} Y_{(i)}$ $(1 \leq k \leq n)$ $\Leftrightarrow$ $\overline{p} \overset{p}{\succeq} \overline{q}$
3. $\sum_{i=1}^{k} X_{(i)} \geq_{mrl} \sum_{i=1}^{k} Y_{(i)}$ $(1 \leq k \leq n)$ $\Leftrightarrow$ $\overline{p} \overset{rm}{\succeq} \overline{q}$

References

The proof is based on the papers: Zhao & Hu, 2009; Zhao & Balakrishnan, 2009c and on the following theorem:

Theorem

Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ be two sets of independent heterogeneous geometric random variables with parameters $p_1, \cdots, p_n$ and $q_1, \cdots, q_n$, respectively.

- *If $(p_1, \cdots, p_n) \overset{rm}{\succeq} (q_1, \cdots, q_n)$, then $\sum_{i=1}^n X_i \geq_{mrl} \sum_{i=1}^n Y_i$*
- *If $\sum_{i=1}^n X_i \geq_{mrl} \sum_{i=1}^n Y_i$, then $p_{(1)} \leq q_{(1)}$ and $\sum_{i=1}^n p_i^{-1} \geq \sum_{i=1}^n q_i^{-1}$.*

Order statistics

Let $X_1, \cdots, X_n$ be independent r.v.'s.
Consider the associated *order statistics*:

$$\min_i X_i = X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n} = \max_i X_i.$$

We intend to characterize:

$$X_{k:n} \geq_{stochastic\ order} Y_{k:n}$$

for a given class of parametric distributions.

Remark

In reliability, $X_{i:n}$ is the lifetime of a $(n+1-i)-out-of-n$ system having independent components X_i.
In particular, $X_{n:n}$ is the lifetime of a parallel system.

Clasical results

Theorem

Assume that $X_1, \cdots, X_n$ *and* $Y_1, \cdots, Y_n$ *follow the PHR model with respective parameters vectors* $\overline{\lambda} = (\lambda_1, \cdots, \lambda_n)$ *and* $\overline{\mu} = (\mu_1, \cdots, \mu_n)$.

Then

1. *(Pledger & Proschan, 1971)*

$$\overline{\lambda} \overset{m}{\succeq} \overline{\mu} \ \Rightarrow \ X_{i:n} \geq_{st} Y_{i:n}, \ i = 1, \cdots, n.$$

2. *(Proschan & Sethuraman, 1976)*

$$\overline{\lambda} \overset{m}{\succeq} \overline{\mu} \ \Rightarrow \ (X_1, \cdots, X_n) \overset{st}{\succeq} (Y_1, \cdots, Y_n).$$

Parallel systems

Theorem (Khaledi & Kochar, 2000)

Assume that $X_1, \cdots, X_n$ *and* $Y_1, \cdots, Y_n$ *are indep. exp. r.v.'s, with respective parameters vectors* $\overline{\lambda} = (\lambda_1, \cdots, \lambda_n)$ *and* $\overline{\mu} = (\mu_1, \cdots, \mu_n)$.
Then

1. $$\overline{\lambda} \overset{p}{\succeq} \overline{\mu} \ \Rightarrow \ X_{n:n} \geq_{st} Y_{n:n}.$$

2. $$\mu_j = \sqrt[n]{\prod_{i=1}^{n} \lambda_i} \ (j = 1, \cdots, n) \ \Rightarrow \ X_{n:n} \geq_{hr,\ disp} Y_{n:n}.$$

Elementary symmetrical functions and associated means

Definition

For $n \in \mathbb{N} \setminus \{0\}$ and $j = 1, 2, \cdots, n$, let consider

1. *the k^{th} elementary symmetrical function of n-order:*
   $e_k^{(n)}(\overline{\lambda}) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}, \ \forall \ \overline{\lambda} \in \mathbb{R}^n;$
2. *the k^{th} elementary symmetrical mean of n-order:*

$$m_k^{(n)}(\overline{\lambda}) = \left(\frac{e_k^{(n)}(\overline{\lambda})}{\binom{n}{k}} \right)^{\frac{1}{k}}, \ \forall \ \overline{\lambda} \in (0, \infty)^n$$

Mac Laurin's inequality

Remark

- $m_1^{(n)}(\overline{\lambda}) = arith(\overline{\lambda})$;
- $m_n^{(n)}(\overline{\lambda}) = geom(\overline{\lambda})$.

Theorem (Mac Laurin, 1729)

$$m_k^{(n)}(\overline{x}) \geq m_{k+1}^{(n)}(\overline{x}), \quad k = 1, \cdots, n-1.$$

Extensions

Let $X_{k:n}$ be the k^{th} order statistic of independent exponential r. v.'s $X_1, \cdots, X_n$ with hazard rates $\lambda_1, \cdots, \lambda_n$ and let $Y_{k:n}$ be the k^{th} order statistic of indep. exp. r. v.'s $Y_1, \cdots, Y_n$ with common hazard rate μ

We ask for **the best characterization of the stochastic inequalities**:

$$X_{k:n} \geq_* Y_{k:n}$$

$$X_{k:n} \leq_* Y_{k:n},$$

where "$*$" denotes some stochastic order.

In reliability terms:

*To compare a Markov **k-out-of-n** system having heterogeneous components with a similar system having identical components*

Comparison in the usual stochastic order

Theorem (Bon & Păltănea, 2006)

Let $U_1, U_2, \cdots, U_n$ be iid positive r. v.'s with common distribution function F and hazard rate h.
Consider $\overline{\lambda} = (\lambda_1, \cdots, \lambda_n) \in (0, \infty)^n$ and $\mu > 0$.
Denote $X_i = U_i/\lambda_i$ and $Y_i = U_i/\mu$ (the PRV model).
If
(i) $h(t) = \frac{F'(t)}{1-F(t)}$ *is non-increasing*
(ii) $\frac{F(t)}{t(1-F(t))}$ *is increasing*
(iii) $\mu = m_k^{(n)}(\overline{\lambda})$ *(for a given number $k \in \{1, \cdots, n\}$)*
then

$$X_{k:n} \geq_{st} Y_{k:n}.$$

Demonstration idea

Lemma

Assume $j \in \{1, 2, \cdots, n\}$ *and* $f : (0, \infty)^n \longrightarrow \mathbb{R}$, *such that:*

1. $f \in \mathscr{C}^1$ *is permutation symmetric*
2. *for all* $\overline{x} = (x_1, x_2, \cdots, x_n) \in (0, \infty)^n$, *such that* $x_i \neq x_k$,

$$(x_i - x_k)\left(\frac{\frac{\partial f}{\partial x_i}(\overline{x})}{\frac{\partial e_j^{(n)}}{\partial x_i}(\overline{x})} - \frac{\frac{\partial f}{\partial x_k}(\overline{x})}{\frac{\partial e_j^{(n)}}{\partial x_k}(\overline{x})} \right) > 0.$$

Then, $\forall\ \overline{x} = (x_1, x_2, \cdots, x_n) \in (0, \infty)^n$,

$$f(\overline{x}) \geq f(\underbrace{m_j^{(n)}(\overline{x}), \cdots, m_j^{(n)}(\overline{x})}_{de\ n\ times})$$

Exponential case

Theorem (Bon & Păltănea, 2006)

In the exponential case,

$$X_{k:n} \geq_{st} Y_{k:n} \Leftrightarrow \mu \geq m_k^{(n)}(\overline{\lambda}),$$

for $k = 1, \cdots, n$.

Conjecture:

$$X_{k:n} \geq_{hr} Y_{k:n} \Leftrightarrow \mu \geq m_k^{(n)}(\overline{\lambda}),$$

for all k.

Comparison in hazard rate order

Theorem (Păltănea, 2008)

In the exponential case:

$$X_{2:n} \geq_{hr} Y_{2:n} \quad \Leftrightarrow \quad \mu \geq m_2^{(n)}(\overline{\lambda})$$

and

$$X_{2:n} \leq_{hr} Y_{2:n} \quad \Leftrightarrow \quad \mu \leq \frac{\sum_{i=1}^{n} \lambda_i - \max_{1 \leq i \leq n} \lambda_i}{n-1}.$$

Other comparisons

Theorem (exponential case)

1. *(Zhao, Li & Balakrishnan, 2009c)*

$$X_{2:n} \geq_{lr} Y_{2:n} \quad \Leftrightarrow \quad \mu \geq \frac{2\Lambda_1^3 - 3\Lambda_1\Lambda_2 + \Lambda_3}{(2n-1)(\Lambda_1^2 - \Lambda_2)} \; (\Lambda_k := \sum_i \lambda_i^k).$$

2. *(Zhao & Balakrishnan, 2009d)*

$$X_{2:n} \geq_{mrl} Y_{2:n} \quad \Leftrightarrow \mu \geq \frac{2n-1}{n(n-1)\left(\sum_{i=1}^n \frac{1}{\sum_{j\neq i}\lambda_j} - \frac{n-1}{\sum_j \lambda_j}\right)}.$$

3. *(Zhao & Balakrishnan (to appear))*
$X_{2:n} \geq_{disp} Y_{2:n} \quad \Leftrightarrow \quad \mu \geq m_2^{(n)}(\overline{\lambda})$.

Parallel systems with two exponential components

Theorem

1. *(Jo & Mi, 2010)*
   If $\lambda_1 \le \mu_1 \le \mu_2 \le \lambda_2$ *and* $\lambda_1 + \lambda_2 \le \mu_1 + \mu_2$
   then $X_{2:2} \ge_{hr} Y_{2:2}$.
2. *(Zhao & Balakrishnan (to appear))*
   If $\lambda_1 \le \mu_1 \le \mu_2 \le \lambda_2$, *then the following statements are equivalent:*
   (i) $(\lambda_1, \lambda_2) \overset{w\uparrow}{\succeq} (\mu_1, \mu_2)$;
   (ii) $X_{2:2} \ge_{lr} Y_{2:2}$.

Definitions

Denote $F_{k:n}^{(\lambda)}(t)$ the distribution function of $X_{k:n}$ from iid exp. r.v.'s $X_1, \cdots, X_n$ with common hazard rate λ.
Consider two random variables:
U, with the distribution function $F_U = F_{k:n}^{(\mu)}$;
V, with the distribution function F_V:

$$F_V(t) = \sum_{i=1}^{m} p_i F_{k:n}^{(\lambda_i)}(t), \quad t \geq 0,$$

where $p_1, \cdots, p_m > 0$, such that $\sum_{i=1}^{m} p_i = 1$.

Best bounds for mixtures

Theorem

$$V \geq_{hr} U \quad \Leftrightarrow \quad V \geq_{st} U \quad \Leftrightarrow \quad \mu \geq \left(\sum_{i=1}^{m} p_i \lambda_i^k \right)^{\frac{1}{k}}$$

and

$$V \leq_{hr} U \quad \Leftrightarrow \quad V \leq_{st} U \quad \Leftrightarrow \quad \mu \leq \min_{i=1,\cdots,m} \lambda_i .$$

Application:

Theorem

$$X_{k:n} \leq_{st} Y_{k:n} \quad \Leftrightarrow \quad \mu \leq \frac{\sum_{i=1}^{n+1-k} \lambda_{(i)}}{n+1-k} .$$

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