On the analyticity of the solutions of second grade fluids

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Complex fluids: Basic laws

- Incompressibility of the fluid:

$$\nabla \cdot u = 0 \tag{1}$$

  where u is a vector valued function expressing the velocity of the fluid at a point in space.

- The balance of momentum is

$$\rho \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) = \nabla \cdot T - \nabla p \tag{2}$$

  where ρ is the density, T is the stress tensor and p is an isotropic pressure.

- The stress tensor T represents the forces which the material develops in response to being deformed.

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- We need a constitutive relation relating T to the motion of the fluid.
- The constitutive law for the classical Newtonian fluid is

$$T = \nu \left(\nabla u + (\nabla u)^t \right)$$

where ν is the viscosity. In this case the system (1), (2) becomes the celebrated Navier-Stokes system of equations.

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In the case of Non-Newtonian fluids, of differential type of grade n

$$T = -pI + F(A_1, A_2, A_3, ..., A_n),$$

where F is a polynome of grade n, A_1,A_2,...A_n lare teh n tensors of Rivlin-Erickson :

$$A_1(v) = \nabla v + (\nabla v)^t, \quad A_{k+1}(v) = \frac{d}{dt} A_k(v) + (\nabla v)^t A_k(v) + A_k(v)(\nabla v),$$

where $\frac{d}{dt} = \partial_t + v\nabla$. The equation of second grade fluids is the following:

$$(S_\alpha) \begin{cases} \partial_t(u - \alpha\Delta u) - \nu\Delta u + \text{curl}\,(u - \alpha\Delta u) \times u + \nabla p = f \\ \text{div}\, u = 0 \\ u(0,x) = u_0(x). \end{cases} \tag{3}$$

In the two-dimensional case, the vorticty $\omega = \mathrm{curl}(u - \alpha\Delta u)$ verifies

$$\partial_t \omega + u\nabla\omega - \nu\Delta(I - \alpha\Delta)^{-1}\omega = \mathrm{curl}\, f$$

which is analogue to the 2D Euler with a damped term

$$\partial_t u + u\nabla u + \gamma u = f; \quad \partial_t \omega + u\nabla\omega + \gamma\omega = \mathrm{curl}\, f.$$

- If $u_0 \in H^3(T^2)$ and $f \in L^\infty(R_+; H^1(T^2))$, global existence of $u \in C_b(R_+; H^3(T^2))$
- If $u_0 \in H^4(T^2)$ and $f \in L^\infty(R; H^2(T^2))$, the solution belong to $C(R_+; H^4)$.
- If $f = f(x)$, it exist a global compact attractor A_α in $H^3(T^2)$.

Theorem (P., Raugel, Rekalo): 1) Let $f \in H^m$, $m \in N^*$. Then, for $\alpha > 0$ small enough, the global attractor$\mathcal{A}_\alpha$ is bounded in H^{m+2}.
2) For all $\alpha > 0$ and all $f \in H^{1+\beta}$, the global attractor $\mathcal{A}_\alpha$ is bounded in $H^{3+\beta}$ where $\beta > 0$ depend on α and the norme $\|f\|_{H^1}$.

Sketch of the proof: We split the equation in two non-autonomous equations

$$\partial_t(v_n - \alpha\Delta v_n) - \nu\Delta v_n + \operatorname{curl}(v_n - \alpha\Delta v_n) \times u + \nabla p_n = f, \qquad v_n(s_n, x) = 0 \quad t > s_n,$$

$$\partial_t(w_n - \alpha\Delta w_n) - \nu\Delta w_n + \operatorname{curl}(w_n - \alpha\Delta w_n) \times u + \nabla \tilde{p}_n = 0, \qquad w_n(s_n, x) = u(s_n, x) \quad t > s_n,$$

- by the uniqueness of the solution: $u = v_n + w_n$
- by the propagation of the regularity: $v_n(t)$ is uniformly bounded in H^4, pour $t \in R$
- by the exponential decay in H^3, we have $w_n(t) \to 0$ in H^3, when $s_n \to -\infty$

$$\|w_n(t)\|_{H^3} \leq C_0 \exp(-C_1(t - s_n))\|u(s_n)\|_{H^3}.$$

Analyticity of the solution of the second grade fluids

- We consider analytical data u_0 such that $e^{\tau_0|D|}u_0 \in H^3$ and $e^{\tau_0|D|}f \in L^1(R)$.
- For 2D Euler equation it is well-known that the solution is analytical in space but the radius of the analyticity is exponential decaying in time.

Main goal: to prove that the radius of the analyticity is bounded by bellow for all time.

Theorem (P., Vicol) Let $\nu, \alpha > 0$, and assume that $e^{\tau_0|D|}u_0 \in H^3(T^2)$, and $e^{\tau_0|D|}f \in L^1(R)$, $\tau_0 > 0$. There exists a unique global in time Gevrey-class solution $u(t)$, such that $e^{\tau(t)|D|}u(t) \in H^3$, and we have the lower bound

$$\tau(t) \geq \tau_0 e^{-C\gamma^{-1}\|\omega_0\|_{X_{\tau_0}}}.$$

In the three-dimensional case we obtain a analytical local in time solution

$$\tau(t) \geq \tau_0 e^{-C\|\omega_0\|_{X_{\tau_0}} \int_0^t \|\nabla u\|_{L^\infty}} \quad t \in [0, T^*).$$

Let $f = 0$ for simplicity. In the following, the function τ verifies $\dot{\tau}(t) \leq 0$. We remark the imprtant fact that $\tilde{\omega}(t) = e^{\tau(t)|D|}\omega(t)$ verifies a regularized equation (with a damped term)

$$\partial_t \tilde{\omega} - \dot{\tau}(t)|D|\tilde{\omega} + \gamma\tilde{\omega} + e^{\tau(t)|D|}(u\nabla\omega) = 0.$$

By commutators formula

$$|(e^{\tau(t)|D|}(u\nabla\omega) - u\nabla\tilde{\omega}; \tilde{\omega})_{L^2}| \leq C\tau(t)\|\nabla\tilde{u}\|_{L^\infty}\||D|^{\frac{1}{2}}\tilde{\omega}\|^2_{L^2}$$

So, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\omega}\|^2_{L^2} - \dot{\tau}(t)\||D|^{\frac{1}{2}}\tilde{\omega}\|^2_{L^2} + \gamma\|\tilde{\omega}\|^2_{L^2} \leq C\tau(t)\|\nabla\tilde{u}\|_{L^\infty}\||D|^{\frac{1}{2}}\tilde{\omega}\|^2_{L^2}.$$

We chose τ that satisfies

$$\dot{\tau} + C\tau\|\nabla\tilde{u}\|_{L^\infty} = 0, \quad \tau(t) = \tau_0 e^{-\int_0^t \|\nabla\tilde{u}\|_{L^\infty}}$$

then we have

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\omega}\|^2_{L^2} + \gamma\|\tilde{\omega}\|^2_{L^2} \leq 0,$$

and therefore

$$\|\nabla\tilde{u}\|_{L^\infty} \leq \|\tilde{\omega}(t)\|_{L^2} \leq \|\tilde{\omega}_0\|_{L^2}e^{-\gamma t} \qquad \text{and so,} \quad \tau(t) \geq \tau_0 e^{-\gamma^{-1}\|\tilde{\omega}_0\|_{L^2}}.$$

Coupled Navier-Stokes and Q-tensor System

The additional stress tensor and energy dissipation

- In the case of non-Newtonian fluids containing suspensions of liquid crystal molecules the stress has an additional component representing forces due to the liquid crystal molecules.
- On the other hand one should have an equation for the liquid crystals, showing how the flow affects the orientation and distribution of the molecules.
- The additional stress tensor encodes the coupling between the flow and the molecules.
- The form of the additional stress tensor is directly related to energy dissipation. More precisely the "content" of the stress tensor should be such that the total energy of the fluid

$$\underbrace{E(t)}_{\text{total energy}} = \underbrace{\frac{1}{2}\int_{\mathbb{R}^d} |u(x,t)|^2\,dx}_{\text{kinetic energy of the flow}} + \underbrace{\mathcal{F}(t)}_{\text{free energy of the molecules}}$$

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The equations

The flow equations:

- $$\begin{cases} \partial_t u + u\nabla u = \nu\Delta u + \nabla p + \nabla\cdot\tau + \nabla\cdot\sigma \\ \nabla\cdot u = 0 \end{cases}$$

  where we have the symmetric part of the additional stress tensor:

  $$\tau = -\xi\left(Q + \frac{1}{3}Id\right)H - \xi H\left(Q + \frac{1}{3}Id\right)$$
  $$+2\xi(Q + \frac{1}{3}Id)QH - L\left(\nabla Q \odot \nabla Q + \frac{\mathrm{tr}(Q^2)}{3}Id\right)$$

  and an antisymmetric part $\sigma = QH - HQ$ where

  $$H = L\Delta Q - aQ + b[Q^2 - \frac{\mathrm{tr}(Q^2)}{3}Id] - cQ\mathrm{tr}(Q^2)$$

- The equation for the liquid crystal molecules, represented by functions with values in the space of Q-tensors (i.e. symmetric and traceless $d \times d$ matrices):

  $$(\partial_t + u\cdot\nabla)Q - S(\nabla u, Q) = \Gamma H$$

  with

  $$S(\nabla u, Q) \stackrel{def}{=} (\xi D + \Omega)(Q + \frac{1}{3}Id) + (Q + \frac{1}{3}Id)(\xi D - \Omega) - 2\xi(Q + \frac{1}{3}Id)\mathrm{tr}(Q\nabla u)$$

Energy dissipation and weak solutions-apriori bounds I

- The total energy

$$E(t) \stackrel{\text{def}}{=} \underbrace{\int_{\mathbb{R}^d} \frac{L}{2}|\nabla Q|^2 + \frac{a}{2}\mathrm{tr}(Q^2) - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}\mathrm{tr}^2(Q^2)\,dx}_{\text{free energy of the liquid crystal molecules}} + \underbrace{\frac{1}{2}\int_{\mathbb{R}^d} |u|^2(t,x)\,dx}_{\text{kinetic energy of the flow}}$$

  is decreasing $\frac{d}{dt}E(t) \leq 0$.

- Simple proof: multiply the first equation in the system to the right by $-H$, take the trace, integrate over $\mathbb{R}^d$ and by parts and sum with the second equation multiplied by u and integrated over $\mathbb{R}^d$ and by parts.
- In the process maximal derivatives are cancelled and you observe suprizing non-trivial cancellations

Energy dissipation and weak solutions-apriori bounds II

- $$\frac{d}{dt}E(t) = -\nu \int_{\mathbb{R}^d} |\nabla u|^2 \, dx$$
$$-\Gamma \int_{\mathbb{R}^d} \mathrm{tr}\left(L\Delta Q - aQ + b[Q^2 - \frac{\mathrm{tr}(Q^2)}{d} Id] - cQ\mathrm{tr}(Q^2)\right)^2 dx \leq 0$$
- Note that this does not readily provide L^p norm estimates.

Proposition

For $d = 2, 3$ there exists a weak solution (Q, u) of the coupled system, with restriction $c > 0$, subject to initial conditions

$$Q(0, x) = \bar{Q}(x) \in H^1(\mathbb{R}^d), \; u(0, x) = \bar{u}(x) \in L^2(\mathbb{R}^d), \nabla \cdot \bar{u} = 0 \;\; in \; \mathcal{D}'(\mathbb{R}^d) \quad (4)$$

The solution (Q, u) is such that $Q \in L^\infty_{loc}(\mathbb{R}_+; H^1) \cap L^2_{loc}(\mathbb{R}_+; H^2)$ and $u \in L^\infty_{loc}(\mathbb{R}_+; L^2) \cap L^2_{loc}(\mathbb{R}_+; H^1)$.

Some other types of complex fluids

- Oldroyd-B:

$$\begin{cases} \partial_t u + u\nabla u - \nu\Delta u + \nabla p = \mu\nabla\cdot\rho \\ \partial_t\rho + u\nabla\rho + a\rho + \rho\Omega - \Omega\rho - b(D\rho + \rho D) = \mu_2 D \end{cases}$$

- The formal energy estimate is:

$$\frac{1}{2}\frac{d}{dt}(\mu_2\|u(t)\|_{L^2} + \mu_1\|\rho(t)\|_{L^2}) + \nu\mu_2\|\nabla u(t)\|_{L^2} + a\mu_1\|\rho(t)\|_{L^2} \leq |b|\|D(t)\|_{L^\infty}\|\rho(t)\|_{L^2}$$

- Smoluchowski Navier-Stokes systems

$$\begin{cases} \frac{\partial v}{\partial t} + v\cdot\nabla v - \nu\Delta v + \nabla p = \nabla\cdot\tau & \text{in} \quad \Omega\times(0,T) \\ \frac{\partial f}{\partial t} + v\nabla f + \nabla_g\cdot(Wf) - \Delta_g f = 0 & \text{in} \quad \Omega\times(0,T) \\ \nabla\cdot v = 0 & \text{in} \quad \Omega\times(0,T), \end{cases}$$

where
$\tau_{ij} = \int_M \gamma_{ij}^{(1)}(m)f(t,x,m)dm + \int_M\int_M \gamma_{ij}^{(2)}(m_1,m_2)f(t,x,m_1)f(t,x,m_2)dm$ and
$W = c_\alpha^{ij}\partial_j v_i$.

- The energy $E(t) = \frac{1}{2}\int_{\mathbb{R}^d}|u(t,x)|^2\,dx + \int_{\mathbb{R}^d}\int_M f\log f\,dx\,dm$ decreases in time.

Regularity difficulties: the maximal derivatives and "the co-rotational parameter"

- Recall the system:

$$\begin{cases} (\partial_t + u\cdot\nabla)Q + \big(\xi D(u) + \Omega(u)\big)(Q + \frac{1}{3}Id) + (Q + \frac{1}{3}Id)\big(\xi D(u) - \Omega(u)\big) \\ -2\xi(Q + \frac{1}{3}Id)\mathrm{tr}(Q\nabla u) = \Gamma H \\ \\ \partial_t u + u\nabla u = \nu\Delta u_\alpha + \nabla p + \nabla\cdot\Big(QH - HQ\Big) \\ -\nabla\cdot\Big(\xi\left(Q + \frac{1}{3}Id\right)H + \xi H\left(Q + \frac{1}{3}Id\right)\Big) \\ +2\xi\nabla\cdot\Big((Q + \frac{1}{3})QH\Big) - L\nabla\cdot\left(\nabla Q\odot\nabla Q + \frac{1}{3}\mathrm{tr}(Q^2)\right) \\ \nabla\cdot u = 0 \end{cases}$$

with $H = L\Delta Q - aQ + b[Q^2 - \frac{\mathrm{tr}(Q^2)}{3}Id] - cQ\mathrm{tr}(Q^2)$.

- Worse than Navier-Stokes
- Where's the difficulty?
- If $\xi = 0$ maximal derivatives only

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Energy dissipation revisited I

- Cancellations that appear in the energy dissipation destroy some high derivatives...
- The cancellation lemma:

Lemma

For any symmetric matrices $Q', Q \in \mathbb{R}^{d\times d}$ and $\Omega_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} - u_{\beta,\alpha}) \in \mathbb{R}^{d\times d}$ (decaying fast enough at infinity so that we can integrate by parts, in the formula below, without boundary terms) we have:

$$\int_{\mathbb{R}^d} tr((\Omega Q' - Q'\Omega)\Delta Q)\, dx - \int_{\mathbb{R}^d} \partial_\beta(Q'_{\alpha\gamma}\Delta Q_{\gamma\beta} - \Delta Q_{\alpha\gamma} Q'_{\gamma\beta}) u_\alpha\, dx = 0$$

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- Idea: differentiate the equations and to the highest derivatives the equations will keep the same structure (as the initial system) plus lower-order derivatives perturbation; thus we can use the high-derivatives cancellations available for the initial system to avoid the maximal derivatives
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Energy dissipation revisited II

- Littlewood-Paley language: take take $\chi \in \mathcal{D}(B(0,1))$ such that $\chi \equiv 1$ on $B(0,1/2)$ and let $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$. Define

$$\Delta_q u = \mathcal{F}^{-1}(\varphi(2^{-q}\xi)\mathcal{F}u), \; S_q u = \mathcal{F}^{-1}(\chi(2^{-q}\xi)\mathcal{F}u)$$

- Then we have in the sense of distributions $u = S_0 u + \sum_{q>0} \Delta_q u$.
- Bony's paraproduct decomposition:

$$\Delta_q(ab) = S_{q-1}a\Delta_q b + \Sigma_{|q'-q|\leq 5}[\Delta_q, S_{q'-1}a]\Delta_{q'} b$$
$$+\Sigma_{|q'-q|\leq 5}(S_{q'-1}a - S_{q-1}a)\Delta_q\Delta_{q'} + \Sigma_{q'>q-5}\Delta_q(S_{q'+2}b\Delta_{q'}a)$$

- In our case, the equation in Q becomes:

$$\partial_t\Delta_q u - \nu\Delta\Delta_q u = \Delta_q\nabla p + L\nabla\cdot(S_{q-1}Q\Delta_q\Delta Q - \Delta_q\Delta Q S_{q-1}Q)$$
$$-\Delta_q(u\nabla u) - L\nabla\cdot\Delta_q\left(\nabla Q\odot\nabla Q - \frac{1}{3}\mathrm{tr}(\nabla Q\odot\nabla Q)\right)$$
$$+\ \text{perturbative (lower derivatives) terms}$$

The rate of increase of the high norms- the Brezis-Gallouet trick and beyond

- Let $y(t)=\|u(t)\|^2_{H^{1+\varepsilon}}+\|\nabla Q(t)\|^2_{H^{1+\varepsilon}}$ and $f(t)=\|u(t)\|^2_{H^1}+\|\nabla Q(t)\|^2_{H^1}$. An estimate of the form
$$y'(t)\leq f(t)y(t)\log(1+y(t))$$
and $\int_0^t f\leq Ce^t$ would give $y(t)\leq Ce^{e^{e^t}}$.
- The logarithm come from Brezis-Gallouet trick-logarithmic embedding!
$$\|u\|_{L^\infty(\mathbb{R}^2)}\leq\|u\|_{H^1}\Big(1+\ln(e+\|u\|_{H^{1+\varepsilon}})\Big)\sim f(t)\ln(1+y)$$
- This works in the co-rotational case $\xi=0$ after "peeling out" the maximal derivatives.
- If $\xi\neq 0$ turns out that we can obtain an estimate of the form:
$$y'(t)\leq f(t)y(t)\Big(1+C\xi\log(1+y(t))(1+\ln(1+\ln y))\Big)$$

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The rate of increase of the high norms- the Brezis-Gallouet trick and beyond

- Let $y(t) = \|u(t)\|^2_{H^{1+\varepsilon}} + \|\nabla Q(t)\|^2_{H^{1+\varepsilon}}$ and $f(t) = \|u(t)\|^2_{H^1} + \|\nabla Q(t)\|^2_{H^1}$. An estimate of the form
$$y'(t) \le f(t)y(t)\log(1+y(t))$$
and $\int_0^t f \le Ce^t$ would give $y(t) \le Ce^{e^{e^t}}$.
- The logarithm come from Brezis-Gallouet trick-logarithmic embedding!
$$\|u\|_{L^\infty(\mathbb{R}^2)} \le \|u\|_{H^1}\Big(1+\ln(e+\|u\|_{H^{1+\varepsilon}})\Big) \sim f(t)\ln(1+y)$$
- This works in the co-rotational case $\xi = 0$ after "peeling out" the maximal derivatives.
- If $\xi \neq 0$ turns out that we can obtain an estimate of the form:
$$y'(t) \le f(t)y(t)\Big(1 + C\xi\log(1+y(t))(1+\ln(1+\ln y))\Big)$$
- One logarithm is produced through through the logarithmic embedding. But the "double logarithm"?

The regularity result, in $2D$

Theorem

(P., Zarnescu): Let $s > 0$ *and* $(\bar{Q}, \bar{u}) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$. *There exists a global a solution* $(Q(t,x), u(t,x))$ *of the coupled system, with restriction* $c > 0$, *subject to initial conditions*

$$Q(0,x) = \bar{Q}(x),\ u(0,x) = \bar{u}(x)$$

and $Q \in L^2_{loc}(\mathbb{R}_+; H^{s+2}(\mathbb{R}^2)) \cap L^\infty_{loc}(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2))$,
$u \in L^2_{loc}(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2) \cap L^\infty_{loc}(\mathbb{R}_+; H^s)$. *Moreover, we have:*

$$L\|\nabla Q(t,\cdot)\|^2_{H^s(\mathbb{R}^2)} + \|u(t,\cdot)\|^2_{H^s(\mathbb{R}^2)} \leq C\left(e + \|\bar{Q}\|_{H^{s+1}(\mathbb{R}^2)} + \|\bar{u}\|_{H^s(\mathbb{R}^2)}\right)^{e^{e^{e^{Ct}}}} \tag{5}$$

where the constant C *depends only on* $\bar{Q}, \bar{u}, a, b, c, \Gamma$ *and* L. *If* $\xi = 0$ *the increase in time of the norms above can be made to be only doubly exponential.*

The difference between $\xi = 0$ (co-rotational) and $\xi \neq 0$

- Recall the system:

$$\begin{cases} (\partial_t + u\cdot\nabla)Q - \big(\xi D(u) + \Omega(u)\big)(Q + \frac{1}{3}Id) + (Q + \frac{1}{3}Id)\big(\xi D(u) - \Omega(u)\big) \\ -2\xi(Q + \frac{1}{3}Id)\mathrm{tr}(Q\nabla u) = \Gamma H \\ \\ \partial_t u + u\nabla u = \nu\Delta u_\alpha + \nabla p + \nabla\cdot\Big(QH - HQ\Big) \\ -\nabla\cdot\Big(\xi\left(Q + \frac{1}{3}Id\right)H + \xi H\left(Q + \frac{1}{3}Id\right)\Big) \\ +2\xi\nabla\cdot\Big((Q + \frac{1}{3})QH\Big) - L\nabla\cdot\big(\nabla Q\odot\nabla Q + \frac{1}{3}\mathrm{tr}(Q^2)\big) \\ \nabla\cdot u = 0 \end{cases}$$

with $H = L\Delta Q - aQ + b[Q^2 - \frac{\mathrm{tr}(Q^2)}{3}Id] - cQ\mathrm{tr}(Q^2)$.

A technical trick-how the "double logarithm" appears I

- We want to obtain $y' + \frac{\nu}{2}\|\nabla u\|_{H^s}^2 + \frac{\Gamma L}{2}\|\Delta Q\|_{H^s}^2 \leq y\ln(e+y)\Big(1+\ln\big(e+\ln(e+y)\big)\Big)$ with $y = \|u\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2$.
- Attempt to estimate

$$\mathcal{I} \stackrel{\text{def}}{=} \sum_{|q'-q|\leq 5} \|S_{q'-1}(\nabla Q)\cdot\Delta_{q'}u\|_{L^2}\|Q\|_{L^\infty}\|\Delta\Delta_q Q\|_{L^2}$$

$(\mathcal{I} \sim \Delta_q(\text{ worst term})$ so we want to estimate $2^{2qs}\mathcal{I} \sim y)$

- We have

$$|\mathcal{I}| \leq \sum_{|q'-q|\leq 5} \|S_{q'-1}\nabla Q\|_{L^{\frac{2}{\varepsilon}}}\|Q\|_{L^\infty}\|\Delta_{q'}u\|_{L^{\frac{2}{1-\varepsilon}}}\|\Delta\Delta_q Q\|_{L^2} \tag{6}$$

- Using the interpolation inequality

$$\|f\|_{L^{2p}} \leq C\sqrt{p}\|f\|_{L^2}^{\frac{1}{p}}\|\nabla f\|_{L^2}^{1-\frac{1}{p}}$$

with $p = \frac{1}{1-\varepsilon} \in [1,2]$, we obtain:

$$|\mathcal{I}| \leq C\sum_{|q'-q|\leq 5} \|S_q\nabla Q\|_{L^{\frac{2}{\varepsilon}}}\|Q\|_{L^\infty}\|\Delta_q u\|_{L^2}^{1-\varepsilon}\|\Delta_q\nabla u\|_{L^2}^{\varepsilon}\|\Delta_q\Delta Q\|_{L^2},$$

where $C > 0$ is constant independent of $\varepsilon \in (0, \frac{1}{2})$.

- Using Young's inequality we obtain:

$$|\mathcal{I}| \leq C\sum_{|q'-q|\leq 5}\left(\left(\|S_q\nabla Q\|_{L^{\frac{2}{\varepsilon}}}\|Q\|_{L^\infty}\right)^{\frac{2}{1-\varepsilon}}\|\Delta_q u\|_{L^2}^2 + \frac{\nu}{100}\|\Delta_q\nabla u\|_{L^2}^2 + \frac{\Gamma L^2}{100}\|\Delta_q\Delta Q\|_{L^2}^2\right)$$

$$\leq 2^{-2qs}\left(C\left(\|S_q\nabla Q\|_{L^{\frac{2}{\varepsilon}}}\|Q\|_{L^\infty}\right)^{\frac{2}{1-\varepsilon}}\|u\|_{H^s}^2 + \frac{\nu}{100}\|\nabla u\|_{H^s}^2 + \frac{\Gamma L^2}{100}\|\Delta Q\|_{H^s}\right)$$

A technical trick-how the “double logarithm” appears I

- We want to obtain $y' + \frac{\nu}{2}\|\nabla u\|^2_{H^s} + \frac{\Gamma L}{2}\|\Delta Q\|^2_{H^s} \leq y\ln(e+y)\Big(1+\ln\big(e+\ln(e+y)\big)\Big)$ with $y = \|u\|^2_{H^s} + \|\nabla Q\|^2_{H^s}$.
- Attempt to estimate

$$\mathcal{I} \stackrel{\text{def}}{=} \sum_{|q'-q|\leq 5} \|S_{q'-1}(\nabla Q)\cdot\Delta_{q'}u\|_{L^2}\|Q\|_{L^\infty}\|\Delta\Delta_q Q\|_{L^2}$$

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where $C > 0$ is constant independent of $\varepsilon \in (0,\frac{1}{2})$.

- Using Young's inequality we obtain:

$$|\mathcal{I}| \leq C\sum_{|q'-q|\leq 5}\Big(\big(\|S_q\nabla Q\|_{L^{\frac{2}{\varepsilon}}}\|Q\|_{L^\infty}\big)^{\frac{2}{1-\varepsilon}}\|\Delta_q u\|^2_{L^2} + \frac{\nu}{100}\|\Delta_q\nabla u\|^2_{L^2} + \frac{\Gamma L^2}{100}\|\Delta_q\Delta Q\|^2_{L^2}\Big)$$

$$\leq 2^{-2qs}\Big(C\big(\|S_q\nabla Q\|_{L^{\frac{2}{\varepsilon}}}\|Q\|_{L^\infty}\big)^{\frac{2}{1-\varepsilon}}\|u\|^2_{H^s} + \frac{\nu}{100}\|\nabla u\|^2_{H^s} + \frac{\Gamma L^2}{100}\|\Delta Q\|_{H^s}\Big)$$

A technical trick -how the "double logarithm" appears II

- We have obtained:

$$y' \le \left(\|\nabla Q\|_{L^{\frac{2}{\varepsilon}}} \|Q\|_{L^\infty}\right)^{\frac{2}{1-\varepsilon}} y(t) \tag{7}$$

- On the other hand using again the interpolation inequality

$$\|g\|_{L^{2p}} \le C\sqrt{p}\|g\|_{L^2}^{\frac{1}{p}} \|\nabla g\|_{L^2}^{1-\frac{1}{p}}$$

we get:

$$\|\nabla Q\|_{L^{\frac{2}{\varepsilon}}}^{\frac{2}{1-\varepsilon}} \le \left(\frac{1}{\varepsilon}\right)^{\frac{1}{1-\varepsilon}} \|\nabla Q\|_{L^2}^{\frac{2\varepsilon}{1-\varepsilon}} \|\Delta Q\|_{L^2}^2 \le \left(\frac{1}{\varepsilon}\right)^{\frac{1}{1-\varepsilon}} (1 + \underbrace{\|\nabla Q\|_{L^2}^2}_{\overset{\text{def}}{=} f(t)})\|\Delta Q\|_{L^2}^2$$

where for the last inequality we assumed $0 < \varepsilon < \frac{1}{2}$. Then (7) becomes:

$$y'(t) \le C(1 + f(t))\|\Delta Q\|_{L^2}^2 \left[(1 + f(t)) \ln(e + y(t))\right]^{\frac{1}{1-\varepsilon}} \left(\frac{1}{\varepsilon}\right)^{\frac{1}{1-\varepsilon}} y(t)$$

Observing that the constants in the interpolation inequality do not depend on the space L^p that we work with and denoting $N \overset{def}{=} \ln(e + y)$ we choose

$$\varepsilon \overset{def}{=} (1 + \ln N)^{-1}$$

and observing that $[N(1 + \ln N)]^{1+\frac{1}{\ln N}} \le CN(1 + \ln N)$ for some constant C independent of N, the last inequality becomes:

$$\varphi'(t) \le C(1 + f(t))^3 \|\Delta Q\|_{L^2}^2 \varphi(t) \ln\left(e + \varphi(t)\right)\left(1 + \ln(e + \ln(\varphi(t) + e))\right)$$

Current and future work

- Uniqueness of weak solutions
- Asymptotic behaviour
- Non-newtonian effects

Challenging open problems

- The optimal rate of increase of high norms: one, two, three exponentials? No exponential, polynomial growth?
- Are there regimes where the system is “better” than $3D$ Navier-Stokes?

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