Line energies in micromagnetism

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ANR - MicroMANIP

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Some 2D micromagnetic models

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$$\boxed{\mathrm{E}_{\varepsilon,\beta}(m) := \frac{\varepsilon}{2}\int_\omega |\nabla m|^2 + \frac{1}{\beta}\int_\omega m_3^2 + \frac{1}{2\varepsilon}\int_{\mathbf{R}^2} |\nabla u|^2.}$$

Stray field: $\nabla u := \nabla \Delta^{-1}\{\nabla \cdot \{m' \mathbf{1}_\omega\}\}$. $\quad \beta << \varepsilon << 1$.

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Prediction (by van der Berg): Limiting Configurations

$$\begin{cases} m_3 = 0, \quad |m| = 1, \quad \text{i. e. } m(x) \in S^1 \\ \nabla \cdot m = 0 \quad \text{in } \omega, \qquad m \cdot \nu = 0 \quad \text{on } \partial\omega. \end{cases} \tag{1}$$

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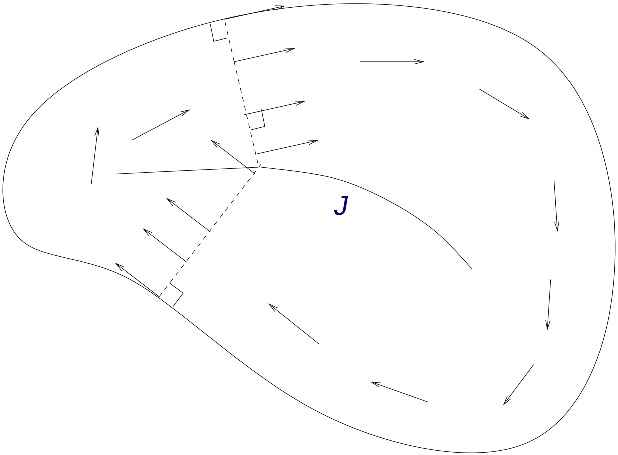
Let $m = \nabla^{\perp}\psi$. Then

$$|\nabla\psi| = 1 \text{ in } \omega \quad \text{and} \quad \psi = 0 \text{ on } \partial\omega.$$

$\Rightarrow$ NO smooth solution (jump lines or vortices pour m)

A special solution: the Landau state: $m_0 := \nabla^{\perp} \operatorname{dist}(x, \partial\omega)$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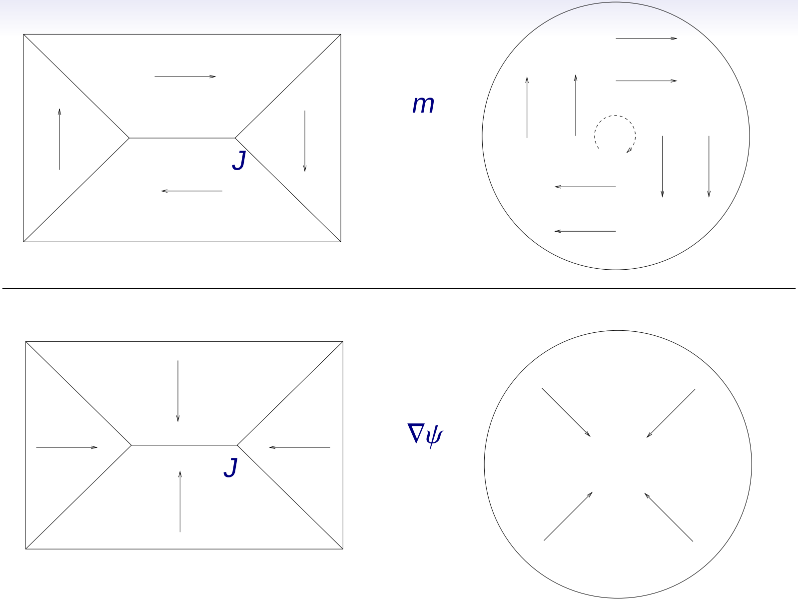

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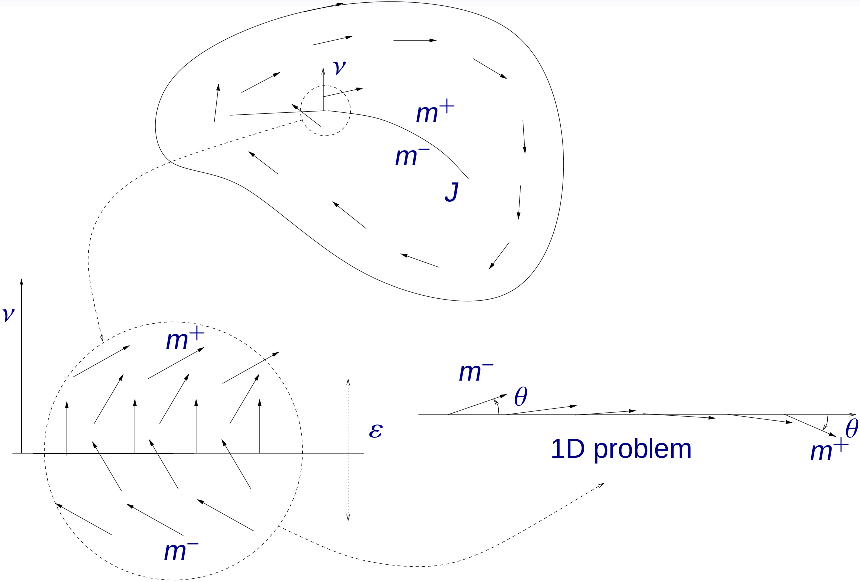

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1D problem

Line Energies

Formally,

$$E_\varepsilon(m_\varepsilon) \xrightarrow{\varepsilon\downarrow 0} \int_J f(|m^+ - m^-|)\, d\mathcal{H}^1.$$

where f depends on the 1D problem.

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Examples :

- $AG \quad \rightsquigarrow \quad f(d) = d^3,$
- $ARS \quad \rightsquigarrow \quad f(d) = \sin\theta - \theta\cos\theta, \quad \text{où} \quad m^\pm = (\cos\theta, \pm\sin\theta).$
- $IM \quad \rightsquigarrow \quad f(d) = d^2,$

Line Energies

Let $f \in C([0,2], \mathbf{R}+)$ be a cost function,

For $m \in BV(\omega, S^1)$ satisfying $\nabla \cdot m \equiv 0$ and $m = \tau$ on $\partial\omega$ we set

$$\boxed{\mathcal{I}_f(m) \;:=\; \int_J f(|m^+ - m^-|)\, d\mathcal{H}^1.}$$

Recall : if $m \in BV$, then

$$\nabla m \;=\; \nabla_d m + (m^+ - m^-) \otimes \nu \mathcal{H}^1 \llcorner J,$$

where $\nabla_d m$ is a diffuse measure, where J is the $\mathcal{H}^1$-rectifiable jump set, where ν is a unit normal vector and where $m^\pm$ are the traces of m on J.

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Question 1 : Prove the existence of minimizers.

Question 2 : Identify the minimizers ?

Existence of minimizers

Usual method :

- Compactness of minimizing sequences

$$\mathcal{I}_f(m_k) \leq \lambda \qquad \Longrightarrow \qquad m_k \overset{k\uparrow\infty}{\longrightarrow} m\,;$$

- The functional $\mathcal{I}_f$ has to be lower semi-continuous for the same topology,

$$\mathcal{I}_f(m) \quad \leq \quad \liminf_{k\uparrow\infty} \mathcal{I}_f(m_k).$$

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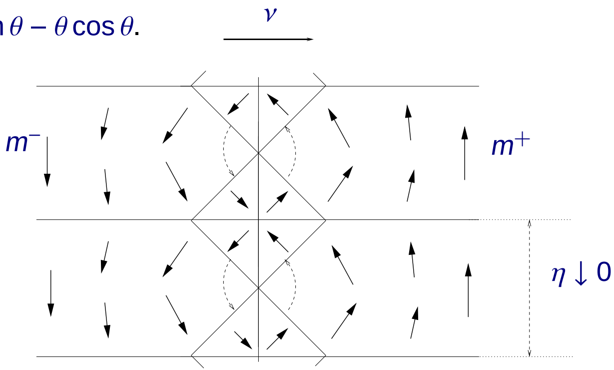
Method : we will extand $\mathcal{I}_f$ as a fonctional $\mathcal{E}$ defined on L^1, l.s.c and such that the sets $\{\mathcal{E} < \lambda\}$ are compact.

Lower semi-continuity and micro-structure

$f(d) = \sin\theta - \theta\cos\theta.$



The minimal cost of a jump angle θ is

$$\tilde{f}(d) = \begin{cases} \sin\theta - \theta\cos\theta & \text{if } 0 \le \theta \le \pi/4, \\ \sqrt{2} - \left(\frac{\pi}{2} - \theta\right)\cos\theta - \sin\theta & \text{if } \pi/4 < \theta \le \pi/2 \end{cases}$$

If $f(d) = d^p$, $p > 3$ we know that $\mathcal{I}_f$ is not l.s.c. (Ambrosio, DeLellis, Mantegazza '99)

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If $f(d) = d^3$ (ADLM), $\mathcal{I}_f$ is l.s.c.

Question : Are there any favorable micro-structure for the cost $f(d) = d^2$?

Entropies

Definition (DKMO '00)

We say that $\Phi \in C^\infty(S^1, \mathbf{R}^2)$ *is an entropy if for every* $m \in C^\infty(\omega, \mathbf{R}^2)$,

$$\nabla \cdot m \equiv 0 \quad et \quad |m| = 1 \qquad \Longrightarrow \qquad \nabla \cdot \{\Phi(m)\} \equiv 0.$$

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Proposition

If $m \in BV(\omega, S^1)$ *satisfies* $\nabla \cdot m = 0$, *The entropy production writes,*

$$\mu_{\Phi(m)} = [\Phi(m^+) - \Phi(m^-)] \cdot \nu \mathcal{H}^1 \llcorner J.$$

In particular

$$\int_\omega d\mu_{\Phi(m)} = \int_J [\Phi(m^+) - \Phi(m^-)] \cdot \nu \, d\mathcal{H}^1.$$

For $\alpha \in C_c^\infty(\omega)$, we have :

$$\langle \mu_\Phi, \alpha \rangle = \int_J [\Phi(m^+) - \Phi(m^-)] \cdot \nu(x)\, \alpha(x)\, d\mathcal{H}^1(x).$$

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Definition

Let $S \subset ENT$. For $m \in L^1(\omega, S^1)$ such that $\nabla \cdot m = 0$, we set

$$\mathcal{E}_S(m) := \sup \left\{ \sum_{i=1}^n \langle \mu_{\Phi_i}, \alpha_i \rangle \ : \ (\Phi_i, \alpha_i) \subset S \times C_c^\infty(\omega, \mathbf{R}_+),\ \sum_{i=1}^n \alpha_i \leq 1 \right\}.$$

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For the invariance under isometries, we assume that

- S is symmetric ($S = -S$),
- S is equivariant ($S = R \circ S \circ R^{-1}$ for every roration R).

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where

$$c_S(d) \; := \; \sup\Big\{ \Big[\Phi(z^+) - \Phi(z^-)\Big] \cdot \nu \; : \; \Phi \in S,\; (z^+, z^-, \nu) \in S^1,\; (z^+ - z^-) \cdot \nu = 0,\, |z^+ - z^-| = d \Big\}.$$

Compactness

Proposition

Let $S \subset ENT$. *If* $d^3 = O(c_S(d))$ *then* $\{\mathcal{E}_S(m) < \lambda\}$ *is relatively compact in* L^1. *This leads to the existence of minimizers of* $\mathcal{E}_S$.

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Idea of the proof (DKMO): If (m_k) is such that $\mathcal{E}_S(m_k) < \lambda$, then for every $\Phi \in ENT$, we have

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$$\Phi(m_k) \times \tilde{\Phi}(m_k) \xrightarrow{k\uparrow\infty} \lim_{k\uparrow\infty} \Phi(m_k) \times \lim_{k\uparrow\infty} \tilde{\Phi}(m_k).$$

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Back to the initial cost function

Given a cost-function f, for proving lower semi-continuity of $\mathcal{I}_f$, it is enough to find $S \subset ENT$ such that $c_S = f$.

Thus, we are looking for a family $S =< S_0 >$, such that,

$$f(2\sin\theta) = \sup_{\Phi \in S} \left| \left[\Phi(z^+) - \Phi(z^-) \right] \cdot e_1 \right|,$$

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Set $\lambda = \varphi + \frac{d}{d\theta^2}\varphi$. *We have for* $z^\pm = (\cos\theta, \pm\sin\theta)$,

$$[\Phi(z^+) - \Phi(z^-)] \cdot e_1 = \lambda \star \left\{ \mathbf{1}_{(-\theta,\theta)} \sin \right\}.$$

Conclusion

- AG : $f(d) = d^3$, Ok with

$$\Phi(m_1, m_2) = (m_1^3/3 + m_2^2 m_1,\ m_2^3/3 + m_2^2 m_2)$$

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- $f(d) = d^2$, Ok with a family

$$\{\Phi_\theta,\; 0 \le \theta \le \pi/2\}\,.$$

$\rightsquigarrow$ no micro-structure.

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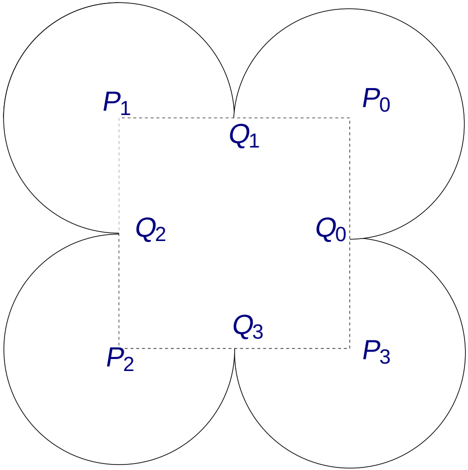
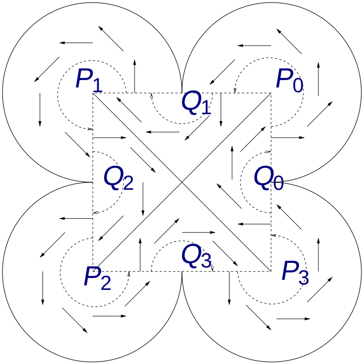
Proposition (Ignat, M. '10) : There exists ω non convex such that $f(d) \gneqq 0$ sur $[0, \sqrt{2}/2]$, m_0 is not a minimizer.

Construction of ω



m_0



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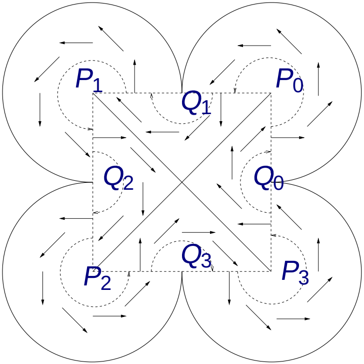
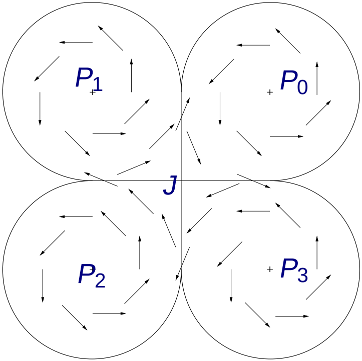
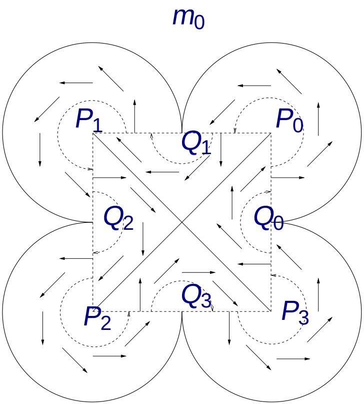
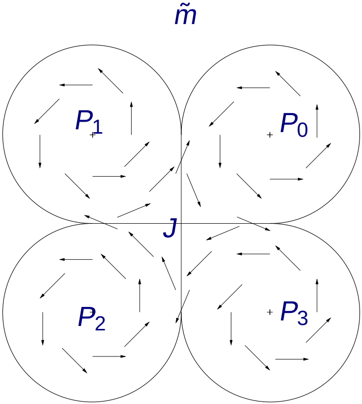


$\tilde{m}$

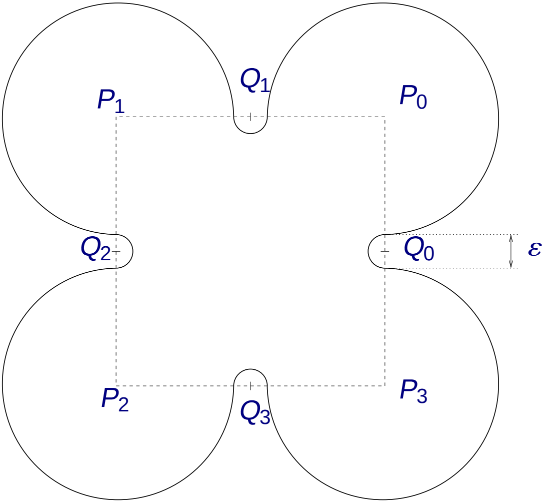




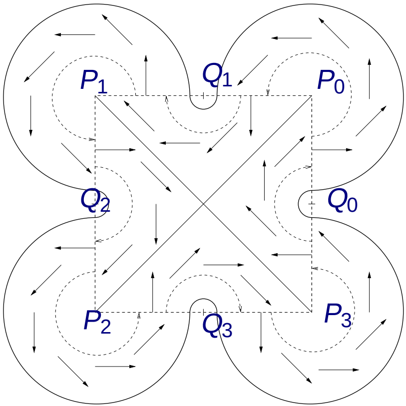


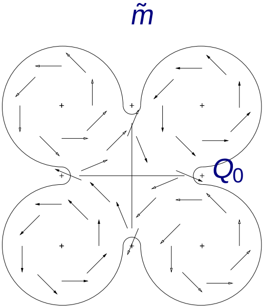
$$\mathcal{I}_f(\tilde{m}) = \frac{\sqrt{2}}{2}\,\mathcal{I}_f(m_0).$$

Construction of $\omega_\varepsilon := \omega \setminus \{\partial\omega + B(0, \varepsilon)\}$

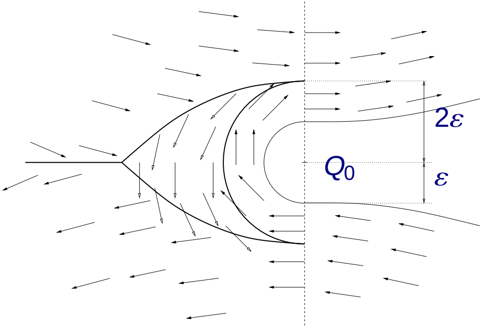


m_0

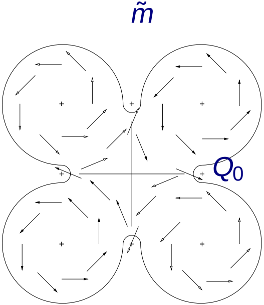


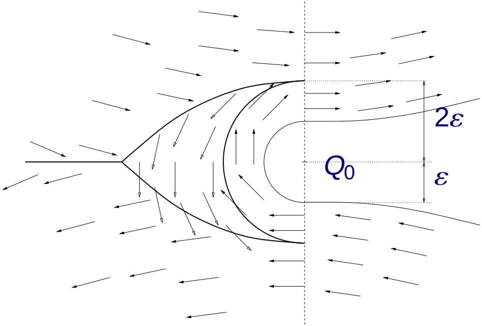


$\tilde{m}$
Q_0



Q_0
2ε
ε





$$\mathcal{I}_f(\tilde{m}) \;=\; \frac{\sqrt{2}}{2}\,\mathcal{I}_f(m_0) + O(\varepsilon).$$

Thank you for your attention !