Approximation of backward stochastic variational inequalities

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- Euler-type approximations for backward stochastic differential equations were introduced by Bouchard, Touzi (2004) and Zhang (2004).

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First, we remind the problem of the existence and uniqueness of the solution for the backward stochastic differential equation involving an subdifferential operator:

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t \ , \ t \in [0,T] \, , \\ dY_t + F\left(t, X_t, Y_t, Z_t\right) dt \in \partial\varphi\left(Y_t\right) dt + Z_t dW_t \ , \ t \in [0,T] \, , \\ X_0 = x, \ Y_T = g\left(X_T\right). \end{cases} \tag{1}$$

Here $(W_t)_{t\geq 0}$ is a standard Brownian motion defined on a complete right continuous stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$.

We will make the following assumptions on the coefficient functions:

The functions

$$\begin{aligned} & b : \mathbb{R}^d \to \mathbb{R}^d, \ \ \sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}, \\ & F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k, \ \ g : \mathbb{R}^d \to \mathbb{R}^k \end{aligned} \tag{2}$$

are continuous and

there exist $\alpha \in \mathbb{R}$ and $L, \beta, \gamma \geq 0$ such that the coefficients satisfy:

$$\begin{array}{ll} (i) & |b(x) - b(\tilde{x})| \leq L|x - \tilde{x}|, \\ (ii) & ||\sigma(x) - \sigma(\tilde{x})|| \leq L|x - \tilde{x}|, \\ (iii) & \langle y - \tilde{y}, f(t,x,y,z) - f(t,x,\tilde{y},z)\rangle \leq \alpha|y - \tilde{y}|^2, \\ (iv) & |f(t,x,y,z) - f(t,x,y,\tilde{z})| \leq \beta\,||z - \tilde{z}||, \end{array} \tag{3}$$

for all $t \in [0, T]$, $x, \tilde{x} \in \mathbb{R}^d$, $y, \tilde{y} \in \mathbb{R}^k$, $z, \tilde{z} \in \mathbb{R}^{k \times d}$.

There exist some constants $M > 0$ and $p, q \in \mathbb{N}$ such that:

$$\begin{array}{ll} (i) & \big|g(x)\big| \leq M\big(1 + |x|^q\big), \\ (ii) & \big|f(t,x,y,0)\big| \leq M\big(1 + |x|^p + |y|\big), \end{array} \tag{4}$$

for all $t \in [0, T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^k$.

The function $\varphi : \mathbb{R}^k \to (-\infty, +\infty]$ is a proper convex l.s.c. function, and there exist $M > 0$ and $r \in \mathbb{N}$ such that

$$|\varphi(g(x))| \leq M(1 + |x|^r), \ \ \forall x \in \mathbb{R}^d \tag{5}$$

Theorem (see [Pardoux, Răşcanu, 98])

Under the assumptions (2)-(5), there exists a unique triple of $\mathcal{F}_t$-progressively measurable stochastic processes $(Y_t, Z_t, U_t)_{t\in[0,T]}$, *solution of the BSVI:*

$$Y_t + \int_t^T U_s ds = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \qquad \text{for all } t \in [0, T], \ \mathbb{P}\text{-a.s.}, \tag{6}$$

with $(Y_t, U_t) \in \partial\varphi$, $\mathbb{P}(d\omega) \otimes dt$. *Moreover we have the following properties of the solution of Eq.(6):*

1. $Y \in L^2_{ad}(\Omega; C([0, T]; \mathbb{R}^k))$, $Z \in L^2_{ad}(\Omega; L^2([0, T]; \mathbb{R}^{k\times d}))$ *and* $U \in L^2_{ad}(\Omega; L^2([0, T]; \mathbb{R}^k))$.

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2. *There exists some constant $C > 0$, such that, for all $t, \tilde{t} \in [0,T]$, $x, \tilde{x} \in \mathbb{R}^d$:*

$$(i) \quad \mathbb{E} \sup_{t\in[0,T]} |Y_t|^2 \le C(1 + |x|^2)$$

$$(ii) \quad \mathbb{E} \sup_{t\in[0,T]} |Y_t^x - Y_t^{\tilde{x}}|^2 \le C\mathbb{E}\Big[\big|g(X_T^x) - g(X_T^{\tilde{x}})\big|^2 + \int_0^T \big|f(s, X_s^x, Y_s^x, Z_s^x) - f(s, X_s^{\tilde{x}}, Y_s^x, Z_s^x)\big|^2 ds\Big].$$

The existence result for (6) is obtained via Yosida approximations. Define for $\varepsilon > 0$ the convex C^1-function φ_ε by

$$\varphi_\varepsilon(y) := \inf\left\{\frac{1}{2\varepsilon}|y - v|^2 + \varphi(v) : v \in \mathbb{R}^k\right\}$$

We denote by

$$J_\varepsilon(y) := (I + \varepsilon\partial\varphi)^{-1}(y)$$

and we have that

$$\begin{aligned} \nabla\varphi_\varepsilon(y) &= \frac{y - J_\varepsilon y}{\varepsilon},\ \nabla\varphi_\varepsilon(y) \in \partial\varphi(J_\varepsilon y), \\ \varphi_\varepsilon(y) &= \frac{1}{2\varepsilon}|y - J_\varepsilon y|^2 + \varphi(J_\varepsilon y),\ \forall y \in \mathbb{R}^k. \end{aligned} \tag{7}$$

For an arbitrary $(t,x) \in [0,T] \times \mathbb{R}^d$, $\varepsilon \in (0,1)$, by a classical result there exists $(Y_t^\varepsilon, Z_t^\varepsilon)_{t\in[t,T]}$ the unique solution of the following approximating BSDE:

$$\begin{aligned} Y_s^\varepsilon + \int_s^T \nabla\varphi_\varepsilon(Y_r^\varepsilon)dr = g(X_T) &+ \int_s^T f(r, X_r, Y_r^\varepsilon, Z_r^\varepsilon)dr \\ &- \int_s^T Z_r^\varepsilon dW_r\ ,\ \forall s \geq t,\ \mathbb{P}\text{-a.s.}, \end{aligned}$$

solution that converges to the solution of Eq.(6).

We introduce an approximation scheme for the solution of Eq.(6):

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- Let $Y_T := g(X_T^\pi)$ and, for $i = \overline{n-1, 0}$,

$$Y_{t_i}^\varepsilon \sim Y_{t_{i+1}}^\varepsilon - h\left[f(t_i, X_{t_i}^\pi, Y_{t_i}^\varepsilon, Z_{t_i}^\varepsilon) - \nabla\varphi_\varepsilon(Y_{t_i}^\varepsilon)\right] - Z_{t_i}^\varepsilon(W_{t_{i+1}} - W_{t_i}) \qquad (9)$$

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- We take the conditional expectation $\mathbb{E}^{\mathcal{F}_i}$:

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- We multiply (9) with $W_{t_{i+1}} - W_{t_i}$ and we take $\mathbb{E}^{\mathcal{F}_i}$:

$$Z_{t_i}^\varepsilon \sim \frac{1}{h}\mathbb{E}^{\mathcal{F}_{t_i}}(Y_{t_{i+1}}^\varepsilon(W_{t_{i+1}} - W_{t_i}))$$

For the above Euler-Yosida approximation scheme, we consider the step of the grid $h = \varepsilon^3$ and we define, for $i = \overline{n-1, 0}$:

$$\begin{cases} Y_{t_i}^{\pi} := \mathbb{E}^{\mathcal{F}_i}(Y_{t_{i+1}}^{\pi}) + h\left[f(t_i, X_{t_i}^{\pi}, Y_{t_i}^{\pi}, Z_{t_i}^{\pi}) - \nabla\varphi_{h^{1/3}}(Y_{t_i}^{\pi})\right], \\ Y_T^{\pi} := g(X_T^{\pi}), \\ Z_{t_i}^{\pi} := \dfrac{1}{h}\mathbb{E}^{\mathcal{F}_{t_i}}(Y_{t_{i+1}}^{\pi}(W_{t_{i+1}} - W_{t_i})), \\ U_{t_i}^{\pi} := \nabla\varphi_{h^{1/3}}(\mathbb{E}^{\mathcal{F}_i}(Y_{t_{i+1}}^{\pi})). \end{cases} \tag{10}$$

Consider now a continuous version of (10). From the martingale representation theorem there exists a square integrable process $\tilde{Z}$ such that

$$Y_{t_{i+1}}^{\pi} = \mathbb{E}^{\mathcal{F}_i}(Y_{t_{i+1}}^{\pi}) + \int_{t_i}^{t_{i+1}} \tilde{Z}_s ds,$$

and, therefore, we define, for $t \in (t_i, t_{i+1}]$

$$Y_t^{\pi} := Y_{t_i}^{\pi} - (t - t_i)\left[f(t_i, X_{t_i}^{\pi}, Y_{t_i}^{\pi}, Z_{t_i}^{\pi}) - \nabla\varphi_{h^{1/3}}(Y_{t_i}^{\pi})\right] + \int_{t_i}^{t_{i+1}} \tilde{Z}_s dW_s.$$

From the isometry property we notice that

$$Z_{t_i}^{\pi} = \frac{1}{h}\mathbb{E}^{\mathcal{F}_i}\left[\int_{t_i}^{t_{i+1}} \tilde{Z}_s dW_s\right], \text{ for } i = \overline{0, n-1}. \tag{11}$$

An error estimate of the approximation is given by the following result.

Theorem

Under the assumptions (2)-(5), there exists $C > 0$ which depends only on the Lipschitz constants of the coefficients, such that:

$$\sup_{t\in[0,T]} \mathbb{E}\left|Y_t - Y_t^{\pi}\right|^2 + \mathbb{E}\int_0^T \left|Z_t - Z_t^{\pi}\right|^2 dt \leq Ch.$$

Proof (sketch).

For $i = \overline{0, n-1}$ let denote: $\delta Y_t := Y_t - Y_t^{\pi}$, $\delta Z_t := Z_t - Z_t^{\pi}$. Applying the Energy equality, using the relation (11) and the properties of the Yosida approximation, we obtain, by standard calculus:

$$\mathbb{E}\left|\delta Y_{t_i}\right|^2 + \frac{1}{2}\int_{t_i}^{t_{i+1}} \mathbb{E}\left|\delta Z_s\right|^2 ds \leq (1+Ch)\left\{h^2 + \mathbb{E}\left|\delta Y_{t_{i+1}}\right|^2 + \int_{t_i}^{t_{i+1}} \mathbb{E}\left|Z_s - \bar{Z}_{t_i}^{\pi}\right|^2 ds\right\}, \tag{12}$$

where

$$\bar{Z}_{t_i}^{\pi} := \frac{1}{h}\mathbb{E}^{\mathcal{F}_i}\left[\int_{t_i}^{t_{i+1}} Z_s ds\right]$$

is the best approximation in L^2 of Z by a process which is constant of each interval $[t_i, t_{i+1})$. □

Proof (cont.)

Iterating (12) and summing upon i, we deduce:

$$\mathbb{E}\left|\delta Y_{t_i}\right|^2 + \frac{1}{2}\int_0^T \mathbb{E}\left|\delta Z_s\right|^2 ds \leq C\left\{h + \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \mathbb{E}\left|Z_s - \bar{Z}_{t_i}^{\pi}\right|^2 ds\right\}.$$

Since we have

$$\max_{i=\overline{0,n-1}} \sup_{t\in[t_i,t_{i+1})} \mathbb{E}\left|Y_t - Y_{t_i}\right|^2 + \sum_{i=0}^{n-1} \mathbb{E}\int_{t_i}^{t_{i+1}} \left|Z_t - \bar{Z}_{t_i}^{\pi}\right|^2 dt \leq Ch,$$

the conclusion of the theorem follows. $\square$

Consider now the following generalized backward stochastic variational inequality (in the Markovian case):

$$\begin{cases} dY_t + F(t, X_t, Y_t, Z_t)\, dt + G(t, X_t, Y_t)\, dA_t \in \\ \qquad\qquad\qquad \in \partial\varphi(Y_t)\, dt + Z_t dW_t,\ 0 \le t \le T, \\ Y_T = g(X_T). \end{cases} \tag{13}$$

Throughout this section we suppose that F, G satisfies the same assumption as F from the first section. It is proved (see [Maticiuc, Rășcanu, 2010]) that the above equation admits a unique solution, i.e.

$$\begin{aligned} Y_t + \int_t^T U_s ds = g(X_T) + \int_t^T F(s, X_s, Y_s, Z_s)\, ds \\ + \int_t^T G(s, X_s, Y_s)\, dA_s - \int_t^T Z_s dW_s,\ \text{ for all } t \in [0, T] \text{ a.s.,} \end{aligned}$$

where

$$U_t \in \partial\varphi(Y_t),\ \text{a.e. on } \Omega \times [0, T].$$

More detailed, let $\mathcal{D}$ be a open bounded subset of $\mathbb{R}^d$ of the form

$$\mathcal{D} = \{x \in \mathbb{R}^d : \ell(x) < 0\},\ \ Bd(\mathcal{D}) = \{x \in \mathbb{R}^d : \ell(x) = 0\},$$

where $\ell \in C_b^3(\mathbb{R}^d)$, $|\nabla \ell(x)| = 1$, for all $x \in Bd(\mathcal{D})$.

It follows, (using the paper of [Lions, Sznitman, 1984]), that for each $(t,x) \in \mathbb{R}_+ \times \overline{\mathcal{D}}$ there exists a unique pair of progressively measurable continuous processes

$$(X_s^{t,x}, A_s^{t,x})_{s\geq 0},$$

with values in $\overline{\mathcal{D}} \times \mathbb{R}_+$, solution of the reflected SDE:

$$\begin{cases} X_s^{t,x} = x + \int_t^{s\vee t} b(r, X_r^{t,x})dr + \int_t^{s\vee t} \sigma(r, X_r^{t,x})dW_r - \int_t^{s\vee t} \nabla\ell(X_r^{t,x})dA_r^{t,x}, \\ s \longmapsto A_s^{t,x} \quad \text{is increasing} \\ A_s^{t,x} = \int_t^{s\vee t} \mathbf{1}_{\{X_r^{t,x}\in Bd(\mathcal{D})\}} dA_r^{t,x}. \end{cases} \tag{14}$$

Moreover, it can be proved that

$$\mathbb{E}\left(\sup_{s\in[0,T]} \left|X_s^{t,x} - X_s^{t',x'}\right|^p\right) \leq C\left(\left|x - x'\right|^p + \left|t - t'\right|^{\frac{p}{2}}\right),$$

and for all $\mu > 0$

$$\mathbb{E}\left[e^{\mu A_T^{t,x}}\right] < \infty.$$

Theorem

Under the assumptions (2)-(5), the generalized BSVI (13) admits a unique solution (Y_t, Z_t, U_t) of $\mathcal{F}_t$-progressively measurable processes. Moreover, for any $0 \le s \le t \le T$, we have, for some positive constant C:

$$\begin{aligned}
&(a) \quad \mathbb{E}\left[\int_s^t \left(|Y_r|^2 + ||Z_r||^2\right) dr + \int_s^t |Y_r|^2 \, dA_r\right] \le CM_1 \\
&(b) \quad \mathbb{E} \sup_{s \le r \le t} |Y_r|^2 \le CM_1, \\
&(c) \quad \mathbb{E}\big(\varphi(Y_t)\big) \le CM_2, \\
&(d) \quad \mathbb{E}\left[\int_s^t |U_r|^2 \, dr\right] \le CM_2,
\end{aligned} \tag{15}$$

where

$$M_1 = \mathbb{E}\left[|\xi|^2 + \int_0^T \left(\left|F(s,0,0)\right|^2 ds + \left|G(s,0)\right|^2 dA_s\right)\right],$$

$$M_2 = \mathbb{E}\left(|\xi|^2 + \varphi(\xi)\right).$$

The main idea for the proof of the existence for (13) consists in taking the approximating equation

$$\begin{aligned} & Y_s^{\varepsilon} + \int_s^T \nabla\varphi_{\varepsilon}\left(Y_r^{\varepsilon}\right) dr = g\left(X_T^{t,x}\right) + \int_s^T F\left(r, X_r^{t,x}, Y_r^{\varepsilon}, Z_r^{\varepsilon}\right) dr \\ & + \int_s^T G\left(r, X_r^{t,x}, Y_r^{\varepsilon}\right) dA_r - \int_s^T Z_r^{\varepsilon} dW_r, \ \ \forall s \geq t, \ \ \mathbb{P}\text{-a.s.} \end{aligned} \tag{16}$$

Essential for the estimates of the process $\left(Y^{\varepsilon}\right)_{\varepsilon>0}$ is the stochastic subdifferential inequality given by:

Lemma (see [Pardoux, Rășcanu, 1998])

Let U, Y be k-dimensional c.p.m.s.p. (continuous progressively measurable stochastic processes) and V be a real c.m.s.p. If

- *Y is a semimartingale,*
- *V is a positive bounded variation stochastic process,*
- *$\varphi : \mathbb{R}^k \to \left]-\infty, +\infty\right]$ is a proper convex l.s.c. function,*
- *$U_r \in \partial\varphi\left(Y_r\right)$, then*

$$\int_t^T V_r \left\langle U_r, dY_r \right\rangle + V_t \varphi\left(Y_t\right) + \int_t^T \varphi\left(Y_r\right) dV_r \leq V_T \varphi\left(Y_T\right).$$

Consequently, we can derive, for some constants λ, μ sufficiently large, that

$$
\begin{aligned}
&(a) \quad \mathbb{E}\int_0^T e^{\lambda s+\mu A_s}\left|\nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right)\right|^2 ds \leq CM_2,\\
&(b) \quad \mathbb{E}\int_0^T e^{\lambda s+\mu A_s}\varphi\left(J_\varepsilon\left(Y_s^\varepsilon\right)\right) ds \leq CM_2,\\
&(c) \quad \mathbb{E}e^{\lambda s+\mu A_s}\left|Y_s^\varepsilon - J_\varepsilon\left(Y_s^\varepsilon\right)\right|^2 \leq \varepsilon CM_2,\\
&(d) \quad \mathbb{E}e^{\lambda s+\mu A_s}\varphi\left(J_\varepsilon\left(Y_s^\varepsilon\right)\right) \leq CM_2.
\end{aligned}
$$

and, that $\left(Y^\varepsilon, Z^\varepsilon\right)_{\varepsilon>0}$ is a Cauchy sequence, i.e.:

$$
\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\lambda t+\mu A_t}\left|Y_t^\varepsilon - Y_t^\delta\right|^2\right] + \mathbb{E}\left[\int_0^T e^{\lambda r+\mu A_r}\left|\left|Z_r^\varepsilon - Z_r^\delta\right|\right|^2 dr\right] \leq C\left(\varepsilon+\delta\right).
$$

The unique solution (Y, Z, U) of Eq.(13) is obtained as the limit of the approximating sequence $(Y_s^\varepsilon, Z_s^\varepsilon, \nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right))$.

As usual, if we consider the deterministic function $u(t,x) = Y_t^{t,x}$, $(t,x) \in [0,T] \times \overline{\mathcal{D}}$, we will obtain a reprezentation for the solution of the following PVI:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \mathcal{L}_t u\,(t,x) + \partial\varphi\big(u(t,x)\big) \ni F\big(t,x,u(t,x),(\nabla u\sigma)(t,x)\big), \\ \hfill t>0,\ x \in \mathcal{D}, \\ \dfrac{\partial u(t,x)}{\partial n} \ni G\big(t,x,u(t,x)\big),\quad t>0,\ x \in Bd\,(\mathcal{D})\,, \\ u(0,x) = g(x),\ x \in \overline{\mathcal{D}}, \end{cases} \tag{17}$$

where the operator $\mathcal{L}_t$ is given by

$$\mathcal{L}_t v(x) = \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^*)_{ij}(t,x)\frac{\partial^2 v(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial v(x)}{\partial x_i},\ \text{for } v \in C^2(\mathbb{R}^d).$$

The functions that appear in (17) satisfy the conditions imposed in the previous slides (for $k=1$).

We have the following result.

Theorem

Under the considered hypotesis on b, σ, F, G and g, the Parabolic Variational Inequality (17) admits at least one viscosity solution $u : [0,\infty) \times \overline{\mathcal{D}} \to \mathbb{R}$, such that $u(0,x) = h(x)$, $\forall\ x \in \overline{\mathcal{D}}$ and

$$u(t,x) \in Dom(\varphi)\,,\ \ \forall (t,x) \in (0,\infty) \times \overline{\mathcal{D}}.$$

Moreover, if

$$r \longrightarrow G(t,x,r)$$

is a non-increasing function, for $t \geq 0$, $x \in Bd(\mathcal{D})$, and there exists a continuous function $\mathbf{m} : [0,\infty) \to [0,\infty)$, $\mathbf{m}(0) = 0$, such that

$$\big|F(t,x,r,p) - F(t,y,r,p)\big| \leq \mathbf{m}\big(\,|x-y|\,(1+|p|)\,\big),$$
$$\forall\ t \geq 0,\ \ x,y \in \overline{\mathcal{D}},\ \ p \in \mathbb{R}^d,$$

then the uniqueness holds.

For the generalized system considered above, we propose a mixed Euler-Yosida type approximation scheme.

For the simplicity of the presentation, we consider the case $\varphi \equiv 0$:

- Consider the grid of $[0, T] : \pi = \{t_i = ih,\ i \leq n\}, \quad$ with $h := T/n,\ n \in \mathbb{N}^*$,

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  - $\hat{X}_{t_{i+1}}^\pi = X_{t_i}^\pi + b(t_i, X_{t_i}^\pi)(t_{i+1} - t_i) + \sigma(t_i, X_{t_i}^\pi)(W_{t_{i+1}} - W_{t_i}),$

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  - $X_{t_{i+1}}^\pi = \begin{cases} \hat{X}_{t_{i+1}}^\pi\ , & \hat{X}_{t_{i+1}}^\pi \in \overline{\mathcal{D}}, \\ \Pr_{\overline{\mathcal{D}}}(\hat{X}_{t_{i+1}}^\pi)\ , & \hat{X}_{t_{i+1}}^\pi \notin \overline{\mathcal{D}}, \end{cases}$ and

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  - $A_{t_{i+1}}^\pi = \begin{cases} A_{t_i}^\pi\ , & \hat{X}_{t_{i+1}}^\pi \in \overline{\mathcal{D}}, \\ A_{t_i}^\pi + \|\Pr_{\overline{\mathcal{D}}}(\hat{X}_{t_{i+1}}^\pi) - \hat{X}_{t_{i+1}}^\pi\|\ , & \hat{X}_{t_{i+1}}^\pi \notin \overline{\mathcal{D}}; \end{cases}$

- Let $Y_T^\pi := g(X_T^\pi)$ and, for $i = \overline{n-1, 0}$:

$$Y_{t_i} \sim Y_{t_{i+1}} - G(X_{t_{i+1}}^\pi, Y_{t_{i+1}})\Delta A_{t_i}^\pi - Z_{t_i}\Delta W_{t_i}$$

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- We take the conditional expectation $\mathbb{E}^{\mathcal{F}_i}$:

$$Y_{t_i} \sim \mathbb{E}^{\mathcal{F}_i}(Y_{t_{i+1}}) - \mathbb{E}^{\mathcal{F}_i}[G(X_{t_{i+1}}^\pi, Y_{t_{i+1}})\Delta A_{t_i}^\pi]$$

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  This suggest us to define the following approximation scheme:

$$\begin{cases} Y_{t_i}^\pi := \mathbb{E}^{\mathcal{F}_i}[Y_{t_{i+1}}^\pi - G(X_{t_{i+1}}^\pi, Y_{t_{i+1}})\Delta A_{t_i}^\pi], \quad Y_T^\pi := g(X_T^\pi), \\ Z_{t_i}^\pi := \dfrac{1}{h}\mathbb{E}^{\mathcal{F}_{t_i}}[Y_{t_{i+1}}^\pi \Delta W_{t_i} - G(X_{t_{i+1}}^\pi, Y_{t_{i+1}})\Delta A_{t_i}^\pi \Delta W_{t_i}], \end{cases}$$

  The proof of the convergence for the defined approximation scheme is still in progress.

References

- Asiminoaei I., Rășcanu A., *Approximation and simulation of stochastic variational inequalities - splitting up method*, Numerical Functional Analysis and Optimization, 18, pp. 251 - 282, 1997.
- Bouchard B., Touzi N. - *Discrete-time approximation anf Monte-Carlo simulation of BSDEs*, Stochastic Processes and their Applications, 111, pp. 175 - 206, 2004.
- Chitashvili R. J., Lazrieva N. L., *Strong solutions of stochastic differential equations with boundary conditions*, Stochastics, 5, pp. 225 - 309, 1981.
- Ding D., Zhang Y. Y., *A splitting-step algorithm for reflected stochastic differential equations in* $\mathbb{R}^1_+$, Comput. Math. Appl., 55, pp. 2413 - 2425, 2008.
- Lepingle D. - *Euler scheme for reflected stochastic differential equations*, Mathematics and Computers in Simulation, Vol. 38, pp. 119 - 126, 1995.
- Lions P. L., Sznitman S. - *Stochastic differential equations with reflecting boundary conditions*, Commun. on Pure and Appl. Math., Vol. 37, pp. 511 - 537, 1984.
- Maticiuc L., Rășcanu A. - *Backward stochastic generalized variational inequality*, Applied analysis and differential equations, pp. 217 - 226, World Sci. Publ., Hackensack, NJ, 2007.
- Maticiuc L., Rășcanu A., *Stochastic approach for a multivalued Dirichlet-Neumann problem*, Stochastic Processes and their Applications, 120, pp. 777 - 800, 2010.

Menaldi J. L., *Stochastic variational inequality for reflected diffusion*, Indiana University Mathematics Journal, 32, pp. 733 - 744, 1983.

Pardoux E., Răşcanu A. - *Backward stochastic differential equations with subdifferential operator and related variational inequalities*, Stochastic Process. Appl. 76, no. 2, pp. 191 - 215, 1998

Saisho Y. - *Stochastic differential equations for multi-dimensional domain with reflecting boundary*, Probab. Theory and Rel. Fields, Vol. 74, pp. 455 - 477, 1987.

Slominski L. - *On approximation of solutions of multidimensional SDE's with reflecting boundary conditions*, Stochastic Processes and their Applications, Vol. 50, pp. 179 - 219, 1994.

Zhang J. - *A numerical scheme for BSDEs*, Annals of Applied Probability, 14, pp. 459 - 488, 2004.

Thank you for your attention!