10ème Colloque Franco-Roumain de Mathématiques Appliquées, 26-31 Août 2010, Poitiers, France

A duality variational approach to time-dependent nonlinear diffusion equations

Gabriela Marinoschi
Institute of Mathematical Statistics and Applied Mathematics,
Bucharest, Romania

Presentation outline

1. *Overview*

   – *Problem presentation*
   – *Problem history*
   – *Connection with physical processes*

2. *Main results*

   – *Functional framework*
   – *Continuous case with polynomial growth*
   – *General continuous case*
   – *Discontinuous case*

3. *Comments and applications*

1 Overview

Formulation of nonlinear time-dependent diffusion problems

as convex minimization problems

Problem presentation - diffusion equation

$$
\begin{aligned}
\frac{\partial y}{\partial t} - \Delta\beta^*(t,x,y) &\ni f && \text{in } Q = (0,T)\times\Omega,\\
-\frac{\partial \beta^*(t,x,y)}{\partial \nu} &= \alpha\beta^*(t,x,y) && \text{on } \Sigma = (0,T)\times\Gamma,\quad \alpha > 0,\\
y(0,x) &= y_0 && \text{in } \Omega
\end{aligned}
\qquad (ODP)
$$

Problem presentation - diffusion equation

$$
\begin{aligned}
\frac{\partial y}{\partial t} - \Delta\beta^*(t,x,y) &\ni f && \text{in } Q = (0,T)\times\Omega, \\
-\frac{\partial \beta^*(t,x,y)}{\partial \nu} &= \alpha\beta^*(t,x,y) && \text{on } \Sigma = (0,T)\times\Gamma, \quad \alpha > 0, \\
y(0,x) &= y_0 && \text{in } \Omega
\end{aligned}
\qquad (ODP)
$$

$\beta^*(t,x,\cdot)$ is a maximal monotone graph on $\mathbf{R}$, a.e. $(t,x)\in Q$,

$$\partial j(t,x,\cdot) = \beta^*(t,x,\cdot) \text{ a.e. } (t,x)\in Q.$$

Problem presentation

$$j(t,x,\cdot) : \mathbf{R} \to (-\infty, \infty] \text{ proper convex l.s.c. a.e. } (t,x) \in Q,$$

$$(t,x) \to j(t,x,r) \text{ is measurable on } Q \text{ for any } r \in \mathbf{R}.$$

Problem presentation

$$j(t,x,\cdot):\mathbf{R}\to(-\infty,\infty] \text{ proper convex l.s.c. a.e. } (t,x)\in Q,$$

$$(t,x)\to j(t,x,r) \text{ is measurable on } Q \text{ for any } r\in\mathbf{R}.$$

Conjugate of j

$$j^*(t,x,\omega)=\sup_{r\in\mathbf{R}}(\omega r-j(t,x,r)) \text{ a.e. on } Q$$

Problem presentation

$$j(t,x,r)+j^*(t,x,\omega)\geq r\omega \text{ for any } r,\omega\in\mathbf{R},\ \text{ a.e. } (t,x)\in Q,$$

$$j(t,x,r)+j^*(t,x,\omega)=r\omega \text{ if and only if } \omega\in\partial j(t,x,r), \text{ a.e. } (t,x)\in Q.$$

W. Fenchel, 1953

Problem history

$*$ **Brezis-Ekeland principle** (C.R. Acad. Sci. Paris, 1976)

$$\frac{du}{dt} + \partial\varphi(u) \ni f \text{ a.e. } t, \quad u(0) = u_0$$

$$\Updownarrow$$

Minimize $J(v)$ for all $v \in K$

$$J(v) = \int_0^T \left\{ \varphi(v) + \varphi^* \left(f - \frac{dv}{dt} \right) - (f, v) \right\} dt + \frac{1}{2} \| v(T) \| \, ,$$

Problem history

$\ast$ **Brezis-Ekeland principle** (C.R. Acad. Sci. Paris, 1976)

- G. Auchmuty (Nonlinear Anal. 1988)
- N. Ghoussoub, L. Tzou (Math. Ann. 2004)
- A. Visintin (Adv. Math. Sci. Appl. 2008)

Problem presentation

(ODP) can be reduced to a minimization problem

$$\Downarrow \qquad \Uparrow$$

Minimize $\displaystyle\int_Q (j(t,x,y(t,x)) + j^*(t,x,w(t,x)) - w(t,x)y(t,x))\,dxdt$, for $(y,w) \in U$, $\quad (P)$

y and w connected by a certain equation !

Problem presentation

This work: (P) optimal control problem in terms of j and j^*

1. Case 1. $j(t,x,\cdot)$ continuous on $\mathbf{R}$, a.e. on Q, and has a polynomial growth

2. Case 2. $j(t,x,\cdot)$ continuous on $\mathbf{R}$, a.e. on Q,

$$\frac{j(t,x,r)}{|r|} \underset{|r|\to\infty}{\longrightarrow} \infty, \quad \frac{j^*(t,x,\omega)}{|\omega|} \underset{|\omega|\to\infty}{\longrightarrow} \infty,$$

3. Case 3. $D(j) = (-\infty, y_s]$ and j is not continuous at y_s.

2 Main results

2.1 Continuous case with polynomial growth

Hypotheses

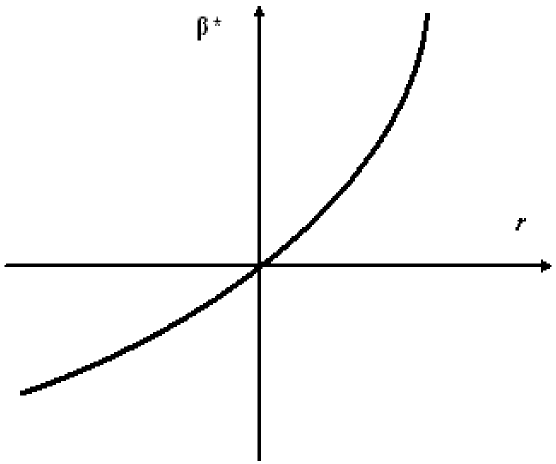
$$j : \mathbf{R} \to (-\infty, \infty] \text{ continuous}$$

$$C_1 \left| r \right|^m + C_1^0 \le j(t, x, r) \le C_2 \left| r \right|^m + C_2^0, \text{ a.e. } (t, x) \in Q, \quad m \ge 2$$

Connection with physical processes

Diffusion with time-dependent diffusion coefficients $\beta(t,x,r)$ in heterogeneous media

$$\beta^*(t,x,r) = \int_0^r \beta(t,x,s)ds, \quad j(t,x,r) = \int_0^r \beta^*(t,x,s)ds, \quad r \in \mathbf{R}$$

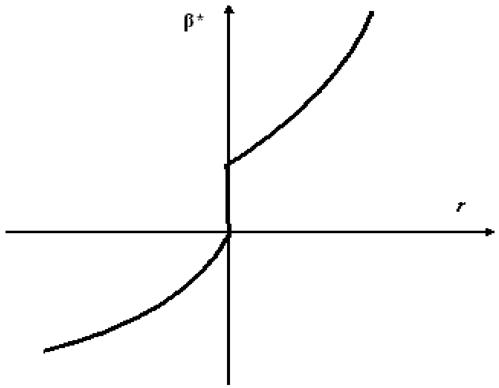


continuous diffusion coefficient $\beta(t,x,\cdot)$

Connection with physical processes

Diffusion with time-dependent diffusion coefficients $\beta(t,x,r)$ in heterogeneous media

$$\beta^*(t,x,r) = \int_0^r \beta(t,x,s)ds, \quad j(t,x,r) = \int_0^r \overset{\circ}{\beta}{}^*(t,x,s)ds, \quad r \in \mathbf{R}$$



discontinuous diffusion coefficient $\beta(t,x,\cdot)$

Functional framework

Let $m > 1 \quad \frac{1}{m'} + \frac{1}{m} = 1$

$$X_m := \left\{ \psi \in W^{2,m}(\Omega),\ \tfrac{\partial \psi}{\partial \nu} + \alpha \psi = 0 \text{ on } \Gamma \right\}$$

$$A_{0,m} : D(A_{0,m}) = X_m \subset L^m(\Omega) \to L^m(\Omega),$$

$$A_{0,m}\psi = -\Delta \psi,$$

Functional framework

Let $m > 1 \quad \frac{1}{m'} + \frac{1}{m} = 1$

$X_m := \left\{ \psi \in W^{2,m}(\Omega),\ \frac{\partial \psi}{\partial \nu} + \alpha \psi = 0 \text{ on } \Gamma \right\}$ $\quad$ X'_m is the dual of X_m

$A_{0,m} : D(A_{0,m}) = X_m \subset L^m(\Omega) \to L^m(\Omega)$ $\quad$ $A_m : D(A_m) = L^m(\Omega) \subset X'_m \to X'_m$

$A_{0,m}\psi = -\Delta \psi$ $\quad$ $\langle A_m y, \psi \rangle_{X'_m, X_{m'}} = (y, A_{0,m'}\psi)_{L^m(Q), L^{m'}(Q)}$

$$V = H^1(\Omega), \quad V' = (H^1(\Omega))'$$

Definition. Let

$$f \in L^{m'}(0,T;X'_{m'}), \qquad y_0 \in V'.$$

A weak solution to (ODP)

$$(y,\eta), \ \ \eta(t,x) \in \partial j(t,x,y(t,x)) \text{ a.e. on } Q,$$

$$y \in L^m(Q) \cap W^{1,m'}([0,T];X'_{m'}), \ \ \eta \in L^{m'}(Q),$$

which satisfies

$$\int_0^T \left\langle \frac{dy}{dt}(t), \psi(t) \right\rangle_{X'_{m'},X_m} dt - \int_Q \eta \Delta\psi dx dt = \int_0^T \langle f(t), \psi(t)\rangle_{X'_{m'},X_m} dt$$

for any $\psi \in L^m(0,T;X_m)$, and the initial condition $y(0,x) = y_0$.

Variational formulation

$$J_0 : L^m(Q) \times L^{m'}(Q) \to (-\infty, \infty]$$

$$J_0(y,w) = \begin{cases} \int_Q (j(t,x,y) + j^*(t,x,w) - wy)\, dxdt, & \text{if } (y,w) \in U_0, \\ +\infty, & \text{otherwise} \end{cases}$$

$$U_0 = \left\{ \begin{array}{l} (y,w); y \in L^m(Q), \quad w \in L^{m'}(Q), \\ \int_Q j(t,x,y)dxdt < \infty, \int_Q j(t,x,y)dxdt < \infty, (y,w) \text{ satisfies } (\,*\,) \end{array} \right\}$$

$$\begin{aligned} \frac{dy}{dt}(t) + A_{m'}w(t) &= f(t) \text{ a.e. } t \in (0,T), \\ y(0) &= y_0. \end{aligned} \qquad (*)$$

Variational formulation

Minimization problem

$$\text{Minimize } J_0(y,w) \text{ for all } (y,w) \in U_0. \qquad (P_0)$$

$$\Updownarrow$$

$$(ODP)$$

Some results

J_0 is proper, convex, l.s.c. on $L^m(Q) \times L^{m'}(Q)$

$$J_0(y,w) = \int_Q (j(t,x,y) + j^*(t,x,w))dxdt - \int_Q wy\,dxdt$$

Some results

J_0 is proper, convex, l.s.c. on $L^m(Q) \times L^{m'}(Q)$

$$J_0(y, w) = \int_Q (j(t, x, y) + j^*(t, x, w))dxdt$$
$$+\frac{1}{2}\left\{\|y(T)\|_{V'}^2 - \|y_0\|_{V'}^2\right\} - \int_0^t \left\langle f(\tau), A_{0,m}^{-1} y(\tau)\right\rangle_{X'_{m'}, X_m} d\tau.$$

Theorem

Problem (P_0) *has at least a solution* (y^*, w^*).

Proof sketch: (y_n, w_n) a minimizing sequence

$$d \leq J_0(y_n, w_n) \leq d + \frac{1}{n},$$

$$\Longrightarrow (y_n)_n \text{ and } (w_n)_n \text{ are bounded in } L^m(Q) \text{ and } L^{m'}(Q)$$

-

$y_n \rightharpoonup y^*$ in $L^m(Q)$

$w_n \rightharpoonup w^*$ in $L^{m'}(Q)$

$\frac{dy_n}{dt} \rightharpoonup \frac{dy^*}{dt}$ in $L^{m'}(0,T;X'_{m'})$ $\Longrightarrow$ (y^*, w^*) satisfies $(*)$ $J_0(y^*, w^*) = d$

$\Downarrow$

$y_n(T) \rightharpoonup y^*(T)$ in V' Ascoli-Arzelá

Theorem

Let (y^*, w^*) *be a solution to* (P_0)*. Then,*

$$\inf_{(y,w)\in U_0} (P_0) = J_0(y^*, w^*) = 0$$

and (y^*, w^*) *is the unique weak solution to* (ODP).

Theorem

Let (y^*, w^*) be a solution to (P_0). Then,

$$w^*(t,x) \in \beta^*(t,x,y^*) \ a.e. \ (t,x) \in Q$$

and (y^*, w^*) is the unique weak solution to (ODP).

$$\begin{aligned} \frac{dy^*}{dt}(t) + A_{m'}w^*(t) &= f(t) \text{ a.e. } t \in (0,T), \\ y(0) &= y_0. \end{aligned} \qquad (*)$$

Proof sketch

Let (y^*, w^*) be a solution to (P_0).

$$J_0(y^*, w^*) \leq J_0(y, w) \text{ for any } (y, w) \in U_0.$$

Proof sketch

Let (y^*, w^*) be a solution to (P_0).

$$J_0(y^*, w^*) \leq J_0(y, w) \text{ for any } (y, w) \in U_0.$$

Let $\lambda > 0$

$$y^\lambda = y^* + \lambda Y,\ w^\lambda = w^* + \lambda W$$

$$\begin{aligned} \frac{dY}{dt}(t) + A_{0,m} W(t) &= 0 \text{ a.e. } t \in (0, T), \\ Y(0) &= 0. \end{aligned}$$

Proof sketch

$$-A_{m'}\int_t^T (\eta^* - w^*)(s)ds + (\beta^*)^{-1}(t,x,w^*) - y^* = 0.$$

Proof sketch

$$-A_{m'}\int_t^T(\eta^*-w^*)(s)ds+(\beta^*)^{-1}(t,x,w^*)-y^*=0.$$

We denote

$$p(t,x)=(\beta^*)^{-1}(t,x,w^*)-y^*$$

$$\left.\begin{aligned}-p_t+A_{m'}(w^*-\eta^*)&=0,\\ p(T)&=0.\end{aligned}\right. \Longrightarrow p(t)=0 \text{ for any } t\in[0,T]$$

Uniqueness is proved using (ODP).

2.2 General continuous case

$$\frac{j(t,x,r)}{|r|} \to \infty, \text{ as } |r| \to \infty, \quad \frac{j^*(t,x,\omega)}{|\omega|} \to \infty, \text{ as } |\omega| \to \infty$$

uniformly w.r. $(t,x) \in Q$

Let

$$f \in L^\infty(Q), \quad y_0 \in V'$$

$$\widetilde{A} : L^1(\Omega) \subset X' \to X', \quad X' = (D(A_{0,\infty}))'$$

$$\left\langle \widetilde{A}y, \phi \right\rangle_{X',X} = \langle y, A_{0,\infty}\phi \rangle_{L^1(\Omega), L^\infty(\Omega)}, \text{ for any } \phi \in X = D(A_{0,\infty}).$$

$$J_1(y,w) = \begin{cases} \int_Q (j(t,x,y) + j^*(t,x,w))\,dxdt + \frac{1}{2}\|y(T)\|^2_{V'} - \frac{1}{2}\|y_0\|^2_{V'} \\ -\int_Q yA^{-1}_{0,m'}f\,dxdt & \text{if } (y,w) \in U_1, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$U_1 = \left\{ \begin{array}{l} (y,w); y \in L^1(Q),\ w \in L^1(Q) \\ \int_Q j(t,x,y)dxdt < \infty, \int_Q j^*(t,x,w)dxdt < \infty,\ y(T) \in V', (y,w) \text{ verifies } (*) \end{array} \right\}$$

$$\begin{aligned} \frac{dy}{dt}(t) + \widetilde{A}w(t) &= f(t) \text{ a.e. } t \in (0,T), \\ y(0) &= y_0. \end{aligned} \qquad (*)$$

$$J_1(y,w) = \begin{cases} \int_Q (j(t,x,y) + j^*(t,x,w))\,dxdt + \frac{1}{2}\|y(T)\|_{V'}^2 - \frac{1}{2}\|y_0\|_{V'}^2 \\ -\int_Q y A_{0,m'}^{-1} f\,dxdt & \text{if } (y,w) \in U_1, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$U_1 = \left\{ \begin{array}{l} (y,w); y \in L^1(Q),\ w \in L^1(Q) \\ \int_Q j(t,x,y)dxdt < \infty, \int_Q j^*(t,x,w)dxdt < \infty,\ y(T) \in V', (y,w) \text{ verifies } (\,*\,) \end{array} \right\}$$

$$\begin{aligned} \frac{dy}{dt}(t) + \widetilde{A}w(t) &= f(t) \text{ a.e. } t \in (0,T), \\ y(0) &= y_0. \end{aligned} \qquad (*)$$

The function J_1 is the closure of the function J_0 in U_0.

Variational formulation

$$\text{Minimize } J_1(y, w) \text{ for all } (y, w) \in U_1. \qquad (P_1)$$

$$\Downarrow$$

A solution to (P_1) is called a *generalized solution* to (ODP)

Theorem

Problem (P_1) has at least a solution (y^*, w^*).
If j is strictly convex the solution is unique.

Theorem

Let (y, w) *be the minimizer such that* $J_1(y, w) = 0$. *If* $y \in [y_m, y_M]$ *then*

$$w(t, x) \in \beta^*(t, x, y(t, x)) \text{ a.e. on } Q$$

and (y, w) *is the weak solution to (*ODP*).*

2.3 Discontinuous case

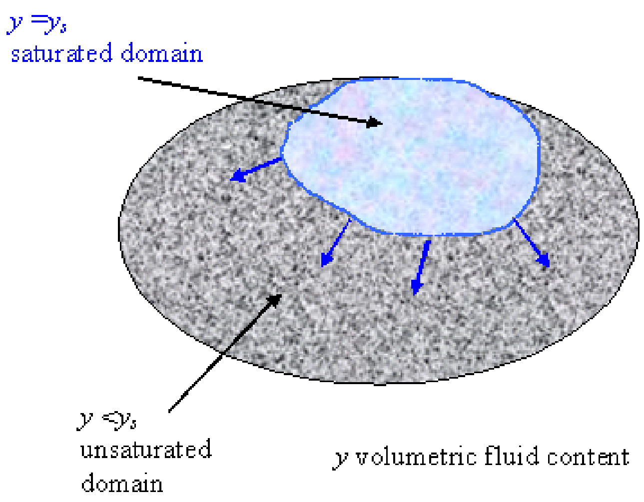
j not continuous

$$\beta : Q \times (-\infty, y_s) \to (0, +\infty)$$

Connection with physical processes

j not continuous$\Longrightarrow$ saturated-unsaturated flow (infiltration in porous media)

$$\beta^*(t,x,r) \begin{cases} = \int_0^r \beta(t,x,s)ds, & r < y_s, \text{ a.e. on } Q, \\ \in [K_s^*(t,x), +\infty), & r = y_s, \text{ a.e. on } Q, \end{cases}$$

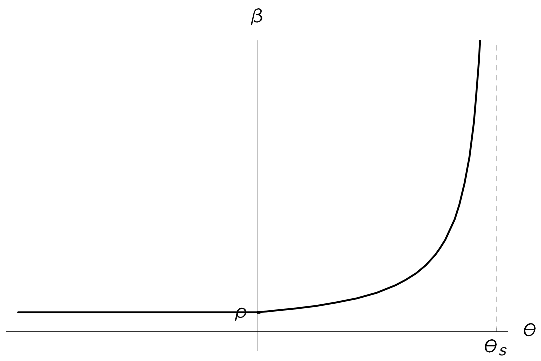


Connection with physical processes

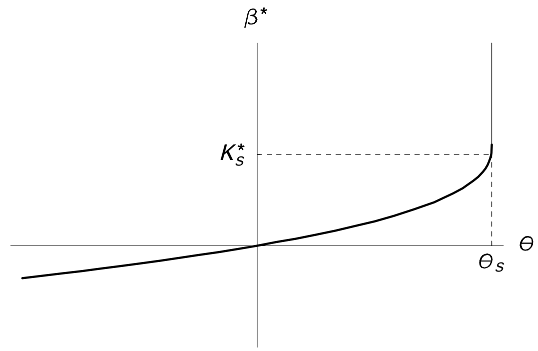
$$j(t,x,r) = \begin{cases} \int_0^r \overset{\circ}{\beta}{}^{*}(t,x,s)ds, & r < y_s, \text{ a.e. on } Q, \\ +\infty, & r = y_s, \text{ a.e. on } Q \end{cases}$$

Connection with physical processes

(a) fast diffusion



diffusion coefficient β

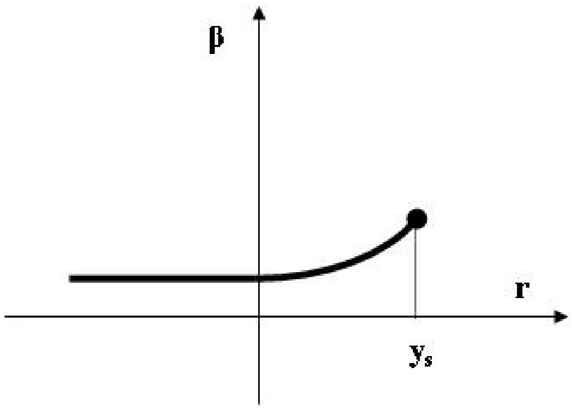


β^*

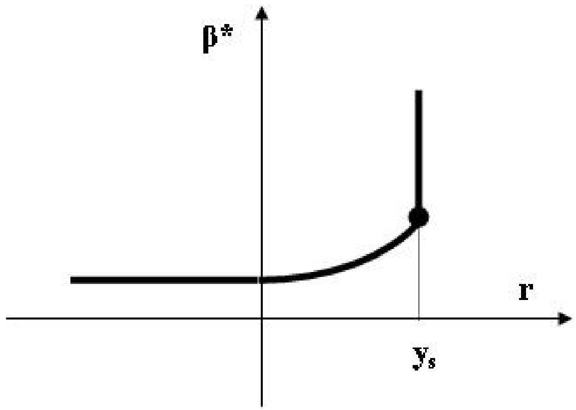
Connection with physical processes

(b) slow diffusion

$$\beta(t,x,y_s) = \beta_s(t,x) \text{ a.e. } (t,x) \in Q$$



diffusion coefficient β



β^*

Preliminaries

$$w \in (L^\infty(Q))', \quad w = w_a + w_s$$

$L^1(Q) \ni w_a =$ continuous component, $w_s =$ singular component

Preliminaries

$$w \in (L^\infty(Q))', \quad w = w_a + w_s$$

$L^1(Q) \ni w_a =$ continuous component, $w_s =$ singular component

$$\varphi : L^\infty(Q) \to (-\infty, \infty], \ \varphi(y) = \int_Q j(t, x, y(t, x))dxdt.$$

Preliminaries

$$w \in (L^\infty(Q))', \quad w = w_a + w_s$$

$L^1(Q) \ni w_a =$ continuous component, $w_s =$ singular component

$$\varphi : L^\infty(Q) \to (-\infty, \infty], \ \varphi(y) = \int_Q j(t,x,y(t,x))dxdt.$$

$$\varphi^* : (L^\infty(Q))' \to (-\infty, \infty]$$

$$\varphi^*(w) = \int_Q j^*(t,x,w_a(t,x))dxdt + \delta_C^*(w_s),$$

$$\delta_C^*(w) = \sup\{w(\psi); \ \psi \in C\}.$$

$$C = \{y \in L^\infty(Q); \varphi(y) < +\infty\}$$

Rockafeller, Pacific J. Math. J., 1971

$$J_\infty : L^m(Q) \times (L^\infty(Q))' \to (-\infty, \infty],$$

$$J_\infty(y,w) = \begin{cases} \int_Q j(t,x,y)dxdt + \frac{1}{2}\|y(T)\|_{V'}^2 - \frac{1}{2}\|y_0\|_{V'}^2 - \int_0^T \langle f(t), A_{0,m}y(t)\rangle_{X'_{m'},X_m} dt \\ \underline{+ \int_Q j^*(t,x,w_a)dxdt + \delta_C^*(w_s)} \text{ if } (y,w) \in U_\infty, \\ +\infty, \qquad \text{otherwise,} \end{cases}$$

$$U_\infty = \left\{ \begin{array}{l} (y,w); y \in L^m(Q),\ y(T) \in V',\ w \in (L^\infty(Q))', \int_Q j(t,x,y)dxdt < \infty, \\ \varphi^*(w) = \int_Q j^*(t,x,w_a)dxdt + \delta_C^*(w_s) < \infty,\ (y,w) \text{ verifies } (*) \end{array} \right\}$$

$$\begin{aligned} -\int_Q y\frac{d\psi}{dt}dxdt + \langle y_0, \psi(0)\rangle_{V',V} + \langle w, A_{0,\infty}\psi\rangle_{(L^\infty(Q))',L^\infty(Q)} &= \int_0^T \langle f(t), \psi(t)\rangle_{X'_m,X_m} dt \quad (*) \\ y(0) &= y_0, \end{aligned}$$

for any $\psi \in W^{1,2}([0,T]; L^{m'}(\Omega)) \cap L^\infty(0,T;X)$, $\psi(T) = 0$.

Variational formulation

$$\text{Minimize } (J_\infty) \text{ for all } (y,w) \in U_\infty \qquad (P_\infty)$$

$$\Downarrow$$

A solution to (P_∞) is called a *generalized solution* to (ODP)

Theorem

Problem (P_∞) has a unique solution (y^*, w^*).

Theorem

Let (y, w) *be the minimizer such that* $J_\infty(y, w) = 0$. *If* $y \in L^\infty(Q)$ *then* (y, w) *is the weak solution to* (ODP),

$$w_a(t, x) \in \beta^*(t, x, y(t, x)) \text{ a.e. } (t, x) \in Q$$

$$w_s \in N_C(y).$$

Interpretation The solution (y, w) satisfies the equations

If $y \in$ int $C = \{y \in L^\infty(Q); ess \sup y(t,x) < y_s$ a.e. on $Q\} \Longrightarrow N_C(y) = \{0\} \Longrightarrow w_s = 0$

$$\frac{dy}{dt} - \Delta\beta^*(t,x,y) \ni f, \text{ a.e. on } Q$$

with initial condition and b.c.

Interpretation The solution (y, w) satisfies the equations

If $y \in$ int $C = \{y \in L^\infty(Q); ess \sup y(t,x) < y_s$ a.e. on $Q\} \Longrightarrow N_C(y) = \{0\} \Longrightarrow w_s = 0$

$$\frac{dy}{dt} - \Delta\beta^*(t,x,y) \ni f, \text{ a.e. on } Q$$

with initial condition and b.c.

If $y \in \partial C = \{y \in L^\infty(Q); ess \sup y(t,x) = y_s$ a.e.$\}$ then there exist

$$Q_s = \{(t,x); y(t,x) = y_s\} \text{ and } Q_u = \{(t,x);\ y(t,x) < y_s\}$$

$$\left(\frac{dy}{dt}\right)_a - \Delta\beta^*(t,x,y) \ni f, \text{ a.e. on } Q_u$$

$$\left(\frac{dy}{dt}\right)_s - \Delta w_s \ni 0, \text{ a.e. on } Q_s.$$

3 Comments and applications

Advantages

- prove of the well-posedness of (ODP) when other methods fail

Advantages

- prove of the well-posedness of (ODP) when other methods fail

$$\begin{aligned} \frac{dy}{dt}(t) + A(t)y(t) &\ni f(t) \text{ a.e. } t \in (0,T), \\ y(0) &= y_0. \end{aligned}$$

Advantages

- prove of the well-posedness of (ODP) when other methods fail

$$\begin{aligned} \frac{dy}{dt}(t) + A(t)y(t) &\ni f(t) \text{ a.e. } t \in (0,T), \\ y(0) &= y_0. \end{aligned}$$

– Theory of evolution equations with m-accretive operators

– Lions theorem

– *M.G. Crandall, A. Pazy, Proc. AMS, 1979* ($f = 0$)

Advantages

- prove of the well-posedness of (ODP) when other methods fail

- solving a minimization convex problem involving a linear state equation

- numerical facilities.

Applications

$$\frac{\partial(u(t,x)\theta)}{\partial t} - \Delta\widetilde{\beta}^*(\theta) \ni f \text{ in } Q$$

- Pollutant propagation in porous saturated media

  u is the absobtion-desorbtion coefficient, θ is the pollutant concentration

Applications

$$\frac{\partial(u(t,x)\theta)}{\partial t} - \Delta\widetilde{\beta}^*(\theta) \ni f \text{ in } Q$$

- Pollutant propagation in porous saturated media

  u is the absobtion-desorbtion coefficient, θ is the pollutant concentration

- Infiltration processes in porous media (water infiltration in soils)

  u is the medium porosity, θ is the water saturation in pores

Applications

$$\frac{\partial(u(t,x)\theta)}{\partial t} - \Delta\widetilde{\beta}^*(\theta) \ni f \text{ in } Q$$

- Pollutant propagation in porous saturated media

  u is the absobtion-desorbtion coefficient, θ is the pollutant concentration

- Infiltration processes in porous media (water infiltration in soils)

  u is the medium porosity, θ is the water saturation in pores

- Previous results (for u very regular)

u constant *G.M. 2005-2006*
$u(t)$ *G.M. 2007*
$u(x)$ degenerate case *A. Favini, G.M. 2008, 2009*

Applications

$$\frac{\partial(u(t,x)\theta)}{\partial t} - \Delta\widetilde{\beta}^*(\theta) \ni f \text{ in } Q$$

$$\Downarrow\, y(t,x) = u(t,x)\theta(t,x), \qquad u > 0$$

$$\frac{\partial y}{\partial t} - \Delta\beta^*(t,x,y) \ni f \text{ in } Q.$$

Future interests

- Case with transport
- Control problems
- Numerical methods

[1] G. Auchmuty, Variational principles for operator equations and initial value problems, Nonlinear Anal. 12 (1988) 531–564.

[2] V. Barbu, Nonlinear Differential Equations of Monotone Type in Banach Spaces, Springer, New York, 2010.

[3] H. Brezis, I. Ekeland, Un principe variationnel associé à certaines equations paraboliques. Le cas independant du temps, C.R. Acad. Sci. Paris Sér. A 282 (1976) 971–974.

[4] H. Brezis, I. Ekeland, Un principe variationnel associé à certaines equations paraboliques. Le cas dependant du temps, C.R. Acad. Sci. Paris Sér. A 282 (1976) 1197–1198.

[5] W. Fenchel, Convex Cones, Sets and Functions., Princeton Univ., 1953.

[6] N. Ghoussoub, L. Tzou, A variational principle for gradient flows, Math. Ann. 330 (2004) 519–549.

[7] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris, 1969.

[8] G. Marinoschi, Functional Approach to Nonlinear Models of Water Flow in Soils, Mathematical Modelling: Theory and Applications, volume 21, Springer, Dordrecht, 2006.

[9] Gabriela Marinoschi, Well-posedness for a non-autonomous model of fast diffusion, Applied Analysis and Differential Equations (Editors: O. Carja, I. Vrabie), 199- 216, World Sci. Publ., Hackensack, NJ, 2007.

[10] R.T. Rockafeller, Pacific J. Math. J, 39, 2, 1971.

[11] J.J. Telega, Extremum principles for nonpotential and initial-value problems, Arch. Mech. 54 (2002) 663–690.

[12] A. Visintin, Extension of the Brezis-Ekeland-Nayroles principle to monotone operators, Adv. Math. Sci. Appl. 18 (2008), 633-650.

Thank you for your attention