10eme Colloque Franco-Roumain

Poitiers, France, August 26-31, 2010

EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF HIGHER-ORDER m-POINT BOUNDARY VALUE PROBLEMS

Prof. **Rodica Luca-Tudorache**

Department of Mathematics, Gh. Asachi Technical University
Iasi, Romania

We shall present existence results for positive solutions of two multi-point boundary value problems. The first one is the n-th order nonlinear differential system

$$(S)\qquad \begin{cases} u^{(n)}(t) + \lambda b(t) f(v(t)) = 0, \ \ t \in (0,T) \\ v^{(n)}(t) + \mu c(t) g(u(t)) = 0, \ \ t \in (0,T), \ \ n \geq 2, \end{cases}$$

with the m-point boundary conditions

$$(BC)\qquad \begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ \ u(T) = \displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i) \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, \ \ v(T) = \displaystyle\sum_{i=1}^{m-2} a_i v(\xi_i), \ \ m \geq 3, \end{cases}$$

where $0 < \xi_1 < \cdots < \xi_{m-2} < T$, $a_i > 0$, $i = \overline{1, m-2}$.

The second problem is the n-th order nonlinear differential system

$$(S_1)\qquad \begin{cases} u^{(n)}(t) + b(t) f(v(t)) = 0, \ \ t \in (0,T) \\ v^{(n)}(t) + c(t) g(u(t)) = 0, \ \ t \in (0,T), \ \ n \geq 2, \end{cases}$$

with the m-point boundary conditions

$$(BC_1)\qquad \begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ \ u(T) = \displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i) + b_0 \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, \ \ v(T) = \displaystyle\sum_{i=1}^{m-2} a_i v(\xi_i) + b_0, \ \ m \geq 3, \ \ b_0 > 0. \end{cases}$$

The multi-point boundary value problems for ordinary differential or difference equations have applications in a variety of different areas of applied mathematics and physics. For example the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multi-point boundary value problem (see M. Moshinsky, *Bol. Soc. Mat. Mex.* (1950)); also many problems in the theory of elastic stability can be handled as multi-point problems (see S. Timoshenko, *McGraw-Hill*, 1961). The study of multi-point boundary value problems for second order differential equations was initiated by V.A. Il'in and E.I. Moiseev (see V.A. Il'in, E.I. Moiseev, *Differ. Equ.* 23(7), 23(8) (1987)). Since then such multi-point boundary value problems (continuous or discrete cases) have been studied by many authors (C.P. Gupta, S.I. Trofimchuk, R. Ma, Y. Raffoul, D.R. Anderson, R. Avery, A. Boucherif, W. Cheung, J. Ren, P.W. Eloe, J. Henderson, J.R. Graef, B. Yang, Y. Guo, W. Shan, W. Ge, S.K. Ntouyas, I.K. Purnaras, C. Yu, Y. Ji, W.T. Li, H.R. Sun, R. Song, H. Lu, G. Weigao, X. Chunyan, J.R.L. Webb, etc.), by using different methods, such as fixed point theorems in cones, the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder and coincidence degree theory.

(I) PROBLEM (S), (BC)

Introduction

We firstly consider the n-th order nonlinear differential system

$$(S) \qquad \begin{cases} u^{(n)}(t) + \lambda b(t) f(v(t)) = 0, \;\; t \in (0,T) \\ v^{(n)}(t) + \mu c(t) g(u(t)) = 0, \;\; t \in (0,T), \;\; n \geq 2, \end{cases}$$

with the m-point boundary conditions

$$(BC) \qquad \begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \;\; u(T) = \displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i) \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, \;\; v(T) = \displaystyle\sum_{i=1}^{m-2} a_i v(\xi_i), \;\; m \geq 3, \end{cases}$$

where $0 < \xi_1 < \cdots < \xi_{m-2} < T$, $a_i > 0$, $i = \overline{1, m-2}$.

We shall present sufficient conditions for λ and μ such that positive solutions of (S), (BC) exist. The existence of positive solutions with respect to a cone for the system (S) with $T = 1$ and the three-point boundary conditions $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, $u(1) = \alpha u(\eta)$, $v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0$, $v(1) = \alpha v(\eta)$, where $0 < \eta < 1$, $0 < \alpha\eta^{n-1} < 1$ has been studied in J. Henderson, S.K. Ntouyas, *Electron. J. Qual. Theory Differ. Equ.* (2007).

The existence of positive solutions for (S) with $n=2$ and the boundary conditions $\beta u(0)-\gamma u'(0)=0$, $u(T)=\sum\limits_{i=1}^{m-2} a_i u(\xi_i)+b_0$, $\beta v(0)-\gamma v'(0)=0$, $v(T)=\sum\limits_{i=1}^{m-2} a_i v(\xi_i)+b_0$ has been investigated in L., *Dyn. Contin. Discrete Impuls. Syst.* (2010) for $b_0=0$ and L., *Math. Bohem.* (2010) for $b_0>0$ and $\lambda=\mu=1$. The corresponding discrete case, namely the system with second-order differences

$$\begin{cases} \Delta^2 u_{n-1}+\lambda b_n f(v_n)=0, \ \ n=\overline{1,N-1} \\ \Delta^2 v_{n-1}+\mu c_n g(u_n)=0, \ \ n=\overline{1,N-1}, \end{cases}$$

with the $m+1$-point boundary conditions $\beta u_0-\gamma\Delta u_0=0$, $u_N=\sum\limits_{i=1}^{m-2} a_i u_{\xi_i}$, $\beta v_0-\gamma\Delta v_0=0$, $v_N=\sum\limits_{i=1}^{m-2} a_i v_{\xi_i}$, $m\geq 3$ has been studied in L., *Libertas Math.* (2009). We also mention the paper Y. Ji, Y. Guo, C. Yu, *Appl. Math. Mech.* (2009), where the authors investigated the existence of positive solutions to the n-th order m-point boundary value problem $u^{(n)}(t)+f(t,u,u')=0$, $t\in(0,1)$, $u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0$, $u(1)=\sum\limits_{i=1}^{m-2} k_i u(\xi_i)$.

We shall present the assumptions that we shall use in the sequel

(H1) $0 < \xi_1 < \cdots < \xi_{m-2} < T,\ a_i > 0,\ \ i = \overline{1, m-2},\ d = T^{n-1} - \sum_{i=1}^{m-2} a_i \xi_i^{n-1} > 0.$

(H2) The functions $b, c : [0, T] \to [0, \infty)$ are continuous and each does not vanish identically on any subinterval of $[0, T]$.

(H3) The functions $f,\, g : [0, \infty) \to [0, \infty)$ are continuous and the limits

$$f_0 = \lim_{x \to 0^+} \frac{f(x)}{x},\ \ g_0 = \lim_{x \to 0^+} \frac{g(x)}{x},\ \ f_\infty = \lim_{x \to \infty} \frac{f(x)}{x},\ \ g_\infty = \lim_{x \to \infty} \frac{g(x)}{x}$$

exist and are positive numbers.

We shall firstly present some auxiliary results which investigate a boundary value problem for a n-th order equation (the below problem (1),(2)). Then we shall prove two existence theorems for the positive solutions with respect to a cone for our problem (S), (BC). The proofs of these results are based on the Guo-Krasnoselskii fixed point theorem, presented below.

Theorem 1. *Let X be a Banach space and let $C \subset X$ be a cone in X. Assume Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ and let $\mathcal{A} : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ be a completely continuous operator such that, either*

i) $\|\mathcal{A}u\| \le \|u\|,\ \ u \in C \cap \partial\Omega_1$, and $\|\mathcal{A}u\| \ge \|u\|,\ \ u \in C \cap \partial\Omega_2$, or

ii) $\|\mathcal{A}u\| \ge \|u\|,\ \ u \in C \cap \partial\Omega_1$, and $\|\mathcal{A}u\| \le \|u\|,\ \ u \in C \cap \partial\Omega_2$.

Then $\mathcal{A}$ has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Auxiliary results

We shall present some results from Y. Ji, Y. Guo, C. Yu, *Appl. Math. Mech.* (2009) ([JGY]) and L. (to appear) ([L]), related to the following n-th order differential equation

$$u^{(n)}(t) + y(t) = 0, \;\; 0 < t < T \tag{1}$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \;\; u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i). \tag{2}$$

Lemma 1. ([JGY], [L]) *If* $d = T^{n-1} - \sum_{i=1}^{m-2} a_i \xi_i^{n-1} \neq 0$, $0 < \xi_1 < \cdots < \xi_{m-2} < T$ *and* $y \in C([0,T])$ *then the solution of* (1), (2) *is given by*

$$u(t) = \frac{t^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} y(s)\, ds - \frac{t^{n-1}}{d(n-1)!} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)^{n-1} y(s)\, ds$$
$$- \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} y(s)\, ds, \;\; 0 \le t \le T.$$

Lemma 2. ([JGY], [L]) *If $d \neq 0$, $0 < \xi_1 < \cdots < \xi_{m-2} < T$ then the Green function for the boundary value problem* (1), (2) *is given by*

$$G(t,s) = \begin{cases} \dfrac{t^{n-1}}{d(n-1)!}\left[(T-s)^{n-1} - \displaystyle\sum_{i=j+1}^{m-2} a_i(\xi_i - s)^{n-1}\right] - \dfrac{1}{(n-1)!}(t-s)^{n-1}, \\ \qquad if \ \ \xi_j \le s < \xi_{j+1}, \ \ s \le t, \\ \dfrac{t^{n-1}}{d(n-1)!}\left[(T-s)^{n-1} - \displaystyle\sum_{i=j+1}^{m-2} a_i(\xi_i - s)^{n-1}\right], \\ \qquad if \ \ \xi_j \le s < \xi_{j+1}, \ \ s \ge t, \ \ j = \overline{0, m-3}, \\ \dfrac{t^{n-1}}{d(n-1)!}(T-s)^{n-1} - \dfrac{1}{(n-1)!}(t-s)^{n-1}, \ \ if \ \ \xi_{m-2} \le s \le T, \ \ s \le t, \\ \dfrac{t^{n-1}}{d(n-1)!}(T-s)^{n-1}, \ \ if \ \ \xi_{m-2} \le s \le T, \ \ s \ge t, \ \ (\xi_0 = 0). \end{cases}$$

Using the above Green function the solution of problem (1), (2) is expressed as

$$u(t) = \int_0^T G(t,s)y(s)\,ds.$$

Lemma 3. ([JGY], [L]) *If $a_i > 0$ for all $i = \overline{1, m-2}$, $0 < \xi_1 < \cdots < \xi_{m-2} < T$, $d > 0$ and $y \in C([0,T])$, $y(t) \ge 0$ for all $t \in [0,T]$, then the unique solution u of problem* (1), (2) *satisfies $u(t) \ge 0$ for all $t \in [0,T]$.*

Lemma 4. ([L]) *If* $a_i > 0$ *for all* $i = \overline{1, m-2}$, $0 < \xi_1 < \cdots < \xi_{m-2} < T$, $d > 0$, $y \in C([0,T])$, $y(t) \geq 0$ *for all* $t \in [0,T]$, *then the solution of problem* (1), (2) *satisfies*

$$\begin{cases} u(t) \leq \dfrac{T^{n-1}}{d(n-1)!} \displaystyle\int_0^T (T-s)^{n-1} y(s)\, ds, \ \ \forall t \in [0,T], \\ u(\xi_j) \geq \dfrac{\xi_j^{n-1}}{d(n-1)!} \displaystyle\int_{\xi_{m-2}}^T (T-s)^{n-1} y(s)\, ds, \ \ \forall j = \overline{1, m-2}. \end{cases}$$

Lemma 5. ([JGY]) *We assume that* $0 < \xi_1 < \cdots < \xi_{m-2} < T$, $a_i > 0$ *for all* $i = \overline{1, m-2}$, $d > 0$ *and* $y \in C([0,T])$, $y(t) \geq 0$ *for all* $t \in [0,T]$. *Then the solution of problem* (1), (2) *verifies* $\inf\limits_{t \in [\xi_{m-2}, T]} u(t) \geq \gamma \|u\|$, *where*

$$\gamma = \begin{cases} \min\left\{ \dfrac{a_{m-2}(T - \xi_{m-2})}{T - a_{m-2}\xi_{m-2}}, \dfrac{a_{m-2}\xi_{m-2}^{n-1}}{T^{n-1}} \right\}, \ \ if \ \displaystyle\sum_{i=1}^{m-2} a_i < 1, \\ \min\left\{ \dfrac{a_1 \xi_1^{n-1}}{T^{n-1}}, \dfrac{\xi_{m-2}^{n-1}}{T^{n-1}} \right\}, \ \ if \ \displaystyle\sum_{i=1}^{m-2} a_i \geq 1 \end{cases}$$

and $\|u\| = \sup\limits_{t \in [0,T]} |u(t)|$.

Existence of positive solutions of (S), (BC)

Using the Green function given in Lemma 2, a pair $(u(t), v(t))$, $t \in [0, T]$ is a solution of eigenvalue problem (S), (BC) if and only if

$$\begin{cases} u(t) = \lambda \int_0^T G(t,s)b(s)f\left(\mu \int_0^T G(s,\tau)c(\tau)g(u(\tau))\,d\tau\right)\,ds, \ 0 \le t \le T, \\ v(t) = \mu \int_0^T G(t,s)c(s)g(u(s))\,ds, \ 0 \le t \le T. \end{cases}$$

We consider the Banach space $X = C([0,T])$ with supremum norm $\|\cdot\|$ and define the cone $C \subset X$ by

$$C = \{u \in X, \ u(t) \ge 0, \ \forall\, t \in [0,T] \ \text{and} \ \inf_{t \in [\xi_{m-2}, T]} u(t) \ge \gamma \|u\|\},$$

where γ is defined in Lemma 5.

For our first result we define the positive numbers L_1 and L_2 by

$$L_1 = \max\left\{\left(\frac{\gamma^2 \xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^T (T-s)^{n-1} b(s) f_\infty\,ds\right)^{-1}, \left(\frac{\gamma^2 \xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^T (T-s)^{n-1} c(s) g_\infty\,ds\right)^{-1}\right\},$$

$$L_2 = \min\left\{\left(\frac{T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1} b(s) f_0\,ds\right)^{-1}, \left(\frac{T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1} c(s) g_0\,ds\right)^{-1}\right\}.$$

Theorem 2. *Assume the assumptions (H1)–(H3) hold and $L_1 < L_2$. Then for each λ and μ that satisfy λ, $\mu \in (L_1, L_2)$ there exist a positive solution with respect to a cone, $(u(t), v(t))$, $t \in [0, T]$, of problem (S), (BC).*

Sketch of proof. Let λ, $\mu \in (L_1, L_2)$ and we choose a positive number ε such that $\varepsilon < f_\infty$, $\varepsilon < g_\infty$,

$$\max\left\{\left(\frac{\gamma^2\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^{T}(T-s)^{n-1}b(s)(f_\infty-\varepsilon)\,ds\right)^{-1},\right.$$

$$\left.\left(\frac{\gamma^2\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^{T}(T-s)^{n-1}c(s)(g_\infty-\varepsilon)\,ds\right)^{-1}\right\} \le \min(\lambda, \mu)$$

and

$$\max(\lambda, \mu) \le \min\left\{\left(\frac{T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}b(s)(f_0+\varepsilon)\,ds\right)^{-1},\right.$$

$$\left.\left(\frac{T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}c(s)(g_0+\varepsilon)\,ds\right)^{-1}\right\}.$$

We now define the operator $\mathcal{A} : C \to X$, by

$$\mathcal{A}(u)(t) = \lambda \int_0^T G(t,s)b(s)f\left(\mu \int_0^T G(s,\tau)c(\tau)g(u(\tau))\,d\tau\right) ds,\;\; 0 \le t \le T,\;\; u \in C.$$

By Lemma 5, we have $\mathcal{A}(C) \subset C$. By using the Arzela-Ascoli theorem we deduce that the operator $\mathcal{A}$ is completely continuous (compact and continuous). By definitions of f_0 and g_0 there exists $K_1 > 0$ such that

$$f(x) \le (f_0 + \varepsilon)x \quad \text{and} \quad g(x) \le (g_0 + \varepsilon)x,\;\; 0 < x \le K_1.$$

Using (H3) we have $f(0) = g(0) = 0$ and the above inequalities are also valid for $x = 0$.

Let $u \in C$ with $\|u\| = K_1$. Because $v(t) = \mu \int_0^T G(t,s)c(s)g(u(s))\,ds,\;\; t \in [0,T]$ verifies the problem (1), (2) with $y(t) = \mu c(t)g(u(t))$, $t \in [0,T]$, then by Lemma 4 and the above property of g we deduce, after some computations, that $v(t) \le \|u\| = K_1$, for $t \in [0,T]$.

By using once again Lemma 4 and the properties of function f we have $\mathcal{A}(u)(t) \le K_1 = \|u\|$, for all $t \in [0,T]$.

Then $\|\mathcal{A}(u)\| \le \|u\|$, for all $u \in C$ with $\|u\| = K_1$. If we denote by $\Omega_1 = \{u \in C,\;\; \|u\| < K_1\}$, then we obtain $\|\mathcal{A}(u)\| \le \|u\|$ for all $u \in C \cap \partial\Omega_1$.

Next, by definitions of f_∞ and g_∞, there exists $\bar{K}_2 > 0$ such that

$$f(x) \ge (f_\infty - \varepsilon)x \quad \text{and} \quad g(x) \ge (g_\infty - \varepsilon)x,\;\; x \ge \bar{K}_2.$$

We consider now $K_2 = \max\{2K_1, \bar{K}_2/\gamma\}$. For $u \in C$ with $\|u\| = K_2$, we obtain by using Lemma 5, that $u(t) \geq \gamma K_2 \geq \bar{K}_2$, for all $t \in [\xi_{m-2}, T]$.

Then, by using Lemma 4, Lemma 5 and the above relations, we obtain that for $t \geq \xi_{m-2}$, $v(t) \geq \|u\| = K_2$ and $\mathcal{A}(u)(\xi_{m-2}) \geq K_2 = \|u\|$.

Therefore $\|\mathcal{A}(u)\| \geq \mathcal{A}(u)(\xi_{m-2}) \geq \|u\|$, for all $u \in C$ with $\|u\| = K_2$. We denote by $\Omega_2 = \{u \in C, \ \|u\| < K_2\}$. Then $\|\mathcal{A}(u)\| \geq \|u\|$, for all $u \in C \cap \partial\Omega_2$.

We now apply Theorem 1 i) and we deduce that $\mathcal{A}$ has a fixed point $u \in C \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

This element together with $v(t) = \mu \int_0^T G(t,s)c(s)g(u(s))\,ds$, $t \in [0,T]$ represent a positive solution of (S), (BC) with respect to cone C, for the given λ and μ. Q.E.D.

Remark 1. The condition $L_1 < L_2$ from Theorem 2 is equivalent to

$$\frac{\max\left\{\int_0^T (T-s)^{n-1}b(s)f_0\,ds, \int_0^T (T-s)^{n-1}c(s)g_0\,ds\right\}}{\min\left\{\int_{\xi_{m-2}}^T (T-s)^{n-1}b(s)f_\infty\,ds, \int_{\xi_{m-2}}^T (T-s)^{n-1}c(s)g_\infty\,ds\right\}} < \frac{\gamma^2\xi_{m-2}^{n-1}}{T^{n-1}}.$$

For the next existence result we define the positive numbers

$$L_3 = \max\left\{\left(\frac{\gamma\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^T (T-s)^{n-1}b(s)f_0\,ds\right)^{-1}, \left(\frac{\gamma\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^T (T-s)^{n-1}c(s)g_0\,ds\right)^{-1}\right\},$$

$$L_4 = \min\left\{ \left(\frac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} b(s) f_\infty \, ds \right)^{-1}, \left(\frac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} c(s) g_\infty \, ds \right)^{-1} \right\}.$$

Theorem 3. *Assume the assumptions (H1)–(H3) hold and $L_3 < L_4$. Then for each λ and μ that satisfy λ, $\mu \in (L_3, L_4)$, there exists a positive solution with respect to a cone, $(u(t), v(t))$, $t \in [0, T]$, of (S), (BC).*

Sketch of proof. Let λ and μ with λ, $\mu \in (L_3, L_4)$. We select a positive number ε such that $\varepsilon < f_0$, $\varepsilon < g_0$ and

$$\max\left\{ \left(\frac{\gamma \xi_{m-2}^{n-1}}{d(n-1)!} \int_{\xi_{m-2}}^T (T-s)^{n-1} b(s)(f_0 - \varepsilon) \, ds \right)^{-1}, \right.$$

$$\left. \left(\frac{\gamma \xi_{m-2}^{n-1}}{d(n-1)!} \int_{\xi_{m-2}}^T (T-s)^{n-1} c(s)(g_0 - \varepsilon) \, ds \right)^{-1} \right\} \le \min(\lambda, \mu)$$

and

$$\max(\lambda, \mu) \le \min\left\{ \left(\frac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} b(s)(f_\infty + \varepsilon) \, ds \right)^{-1}, \right.$$

$$\left. \left(\frac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} c(s)(g_\infty + \varepsilon) \, ds \right)^{-1} \right\}.$$

We also consider the operator $\mathcal{A}$ defined in the proof of Theorem 2. From the definitions of f_0 and g_0, we deduce that there exists $\bar{K}_3 > 0$ such that

$$f(x) \geq (f_0 - \varepsilon)x \quad \text{and} \quad g(x) \geq (g_0 - \varepsilon)x, \;\; 0 < x \leq \bar{K}_3.$$

Using the properties of f and g the above inequalities are also valid for $x = 0$.

In addition, because g is a continuous function with $g_0 > 0$, then $g(0) = 0$ and there exists $K_3 \in (0, \bar{K}_3)$ such that

$$g(x) \leq \frac{\bar{K}_3}{\dfrac{\mu T^{n-1}}{d(n-1)!} \displaystyle\int_0^T (T-s)^{n-1} c(s)\, ds}, \;\; 0 < x \leq K_3.$$

For $u \in C$ with $\|u\| = K_3$, by Lemma 4 and the above inequality, we deduce that $v(t) \leq \bar{K}_3$, for all $t \in [0, T]$.

By using Lemma 4, Lemma 5 and the properties of f, g, we then obtain $\mathcal{A}(u)(\xi_{m-2}) \geq \|u\|$.

Hence $\|\mathcal{A}(u)\| \geq \mathcal{A}(u)(\xi_{m-2}) \geq \|u\|$, for $u \in C$ with $\|u\| = K_3$. We denote by $\Omega_3 = \{u \in C, \; \|u\| < K_3\}$, and then we have $\|\mathcal{A}(u)\| \geq \|u\|$ for all $u \in C \cap \partial\Omega_3$.

We now consider the functions f^*, $g^* : [0,\infty) \to [0,\infty)$ defined by $f^*(x) = \sup\limits_{0\le y\le x} f(y)$, $g^*(x) = \sup\limits_{0\le y\le x} g(y)$. By (H2) we obtain for f^* and g^* the relations $\lim\limits_{x\to\infty} \dfrac{f^*(x)}{x} = f_\infty$, $\lim\limits_{x\to\infty} \dfrac{g^*(x)}{x} = g_\infty$.

We also have $f(x) \le f^*(x)$, $g(x) \le g^*(x)$, for all $x \ge 0$. Then there exists $\bar{K}_4 > 0$ such that

$$f^*(x) \le (f_\infty + \varepsilon)x,\ g^*(x) \le (g_\infty + \varepsilon)x, \text{ for all } x \ge \bar{K}_4.$$

Let $K_4 > \max\{2K_3, \bar{K}_4\}$. Then for u with $\|u\| = K_4$ we obtain $\mathcal{A}(u)(t) \le K_4 = \|u\|$.

So $\|\mathcal{A}(u)\| \le \|u\|$, for all $u \in C$ with $\|u\| = K_4$. If we denote by $\Omega_4 = \{u \in C,\ \|u\| < K_4\}$, then we obtain $\|\mathcal{A}(u)\| \le \|u\|$, for all $u \in C \cap \partial\Omega_4$.

By Theorem 1 ii) we deduce that $\mathcal{A}$ has a fixed point $u \in C \cap (\bar{\Omega}_4 \setminus \Omega_3)$, which together with $v(t) = \mu \int_0^T G(t,s)c(s)g(u(s))\,ds$, $t \in [0,T]$ give us a positive solution of (S), (BC) with respect to cone C, for the chosen values λ and μ. Q.E.D.

Remark 2. The condition $L_3 < L_4$ is equivalent to

$$\frac{\max\left\{\int_0^T (T-s)^{n-1} b(s) f_\infty\, ds,\ \int_0^T (T-s)^{n-1} c(s) g_\infty\, ds\right\}}{\min\left\{\int_{\xi_{m-2}}^T (T-s)^{n-1} b(s) f_0\, ds,\ \int_{\xi_{m-2}}^T (T-s)^{n-1} c(s) g_0\, ds\right\}} < \frac{\gamma \xi_{m-2}^{n-1}}{T^{n-1}}.$$

Example 1

As in Example 4.1 in J. Henderson, S.K. Ntouyas, I.K. Purnaras, *Neural, Parallel, Scient. Comp.* (2008), let us consider the functions

$$\begin{cases} f(x) = p_2|\sin x| + p_1 x e^{-1/x}, & x \in [0,\infty), \\ g(x) = q_2|\sin x| + q_1 x e^{-1/x}, & x \in [0,\infty), \end{cases}$$

with $p_1,\, p_2,\, q_1,\, q_2 > 0$.

We have $\lim\limits_{x\to 0^+} \dfrac{f(x)}{x} = p_2,\ \lim\limits_{x\to\infty} \dfrac{f(x)}{x} = p_1,\ \lim\limits_{x\to 0^+} \dfrac{g(x)}{x} = q_2,\ \lim\limits_{x\to\infty} \dfrac{g(x)}{x} = q_1.$

Let $T=1,\ n=3,\ m=4,\ b(t)=b_0 t,\ c(t)=c_0 t,\ t\in[0,1]$, with $b_0, c_0 > 0$ and $\xi_1 = \frac{1}{3},\ \xi_2 = \frac{2}{3},\ a_1 = 1,\ a_2 = \frac{1}{2}$.

We consider the third-order differential system

$$(S_0) \qquad \begin{cases} u'''(t) + \lambda b_0 t \left[p_2|\sin v(t)| + p_1 v(t) e^{-1/v(t)}\right] = 0, & t \in (0,1) \\ v'''(t) + \mu c_0 t \left[q_2|\sin u(t)| + q_1 |u(t)| e^{-1/u(t)}\right] = 0, & t \in (0,1), \end{cases}$$

with the boundary conditions

$$(BC_0) \qquad \begin{cases} u(0) = u'(0) = 0, \ \ u(1) = u(\frac{1}{3}) + \frac{1}{2}u(\frac{2}{3}) \\ v(0) = v'(0) = 0, \ \ v(1) = v(\frac{1}{3}) + \frac{1}{2}v(\frac{2}{3}). \end{cases}$$

We also have $d = 1 - \sum_{i=1}^{2} a_i \xi_i^2 = \dfrac{2}{3} > 0$, $\sum_{i=1}^{2} a_i = \dfrac{3}{2} > 1$ and $\gamma = \min\{a_1\xi_1^2, \xi_2^2\} = \frac{1}{9}$. The condition $L_1 < L_2$ or the equivalent form given in Remark 3 is $\dfrac{\max\{b_0p_2, c_0q_2\}}{\min\{b_0p_1, c_0q_1\}} < \dfrac{4}{6561}$.

Therefore if the above condition is verified, then by Theorem 2 we deduce that for all numbers λ, $\mu \in (L_1, L_2)$ the problem (S_0), (BC_0) has positive solutions.

(II) PROBLEM (S_1), (BC_1)

Introduction

We consider the nonlinear n-th order differential system

$$(S_1) \qquad \begin{cases} u^{(n)}(t) + b(t)f(v(t)) = 0, \;\; t \in (0,T) \\ v^{(n)}(t) + c(t)g(u(t)) = 0, \;\; t \in (0,T), \;\; n \geq 2, \end{cases}$$

with the m-point boundary conditions

$$(BC_1) \quad \begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \;\; u(T) = \displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i) + b_0 \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, \;\; v(T) = \displaystyle\sum_{i=1}^{m-2} a_i v(\xi_i) + b_0, \;\; m \geq 3, \;\; b_0 > 0. \end{cases}$$

We shall investigate the existence and nonexistence of positive solutions of (S_1), (BC_1).

We shall suppose that the following conditions are verified

(A1) $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < T$, $a_i > 0$ for $i = \overline{1, m-2}$, $d = T^{n-1} - \displaystyle\sum_{i=1}^{m-2} a_i \xi_i^{n-1} > 0$, $b_0 > 0$.

(A2) The functions $b,\, c : [0,T] \to [0,\infty)$ are continuous and there exist $t_0,\, \widetilde{t}_0 \in [\xi_{m-2}, T)$ such that $b(t_0) > 0,\ c(\widetilde{t}_0) > 0$.

(A3) The functions $f,\, g : [0,\infty) \to [0,\infty)$ are continuous and satisfy the conditions

a) There exists $c_0 > 0$ such that $f(u) < \dfrac{c_0}{L},\ g(u) < \dfrac{c_0}{L}$, for all $u \in [0, c_0]$.

b) $\lim\limits_{u\to\infty} \dfrac{f(u)}{u} = \infty,\ \ \lim\limits_{u\to\infty} \dfrac{g(u)}{u} = \infty$,

where $L = \max\left\{ \dfrac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} b(s)\, ds,\ \dfrac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} c(s)\, ds \right\}.$

Main results

First we shall present an existence result for the positive solutions of (S_1), (BC_1).

Theorem 4. *Assume that the assumptions (A1), (A2), (A3)a hold. Then the problem (S_1), (BC_1) has at least one positive solution for $b_0 > 0$ sufficiently small.*

Sketch of proof. We consider the problem

$$\left\{ \begin{array}{l} h^{(n)}(t) = 0,\ \ t \in (0,T) \\ h(0) = h'(0) = \cdots = h^{(n-2)}(0) = 0,\ \ h(T) = \displaystyle\sum_{i=1}^{n-2} a_i h(\xi_i) + 1. \end{array} \right. \tag{3}$$

The solution $h(t),\ t \in (0,T)$ of problem (3) is $h(t) = \dfrac{t^{n-1}}{d},\ t \in [0,T]$.

We define the functions $x(t),\ y(t),\ t \in [0,T]$ by

$x(t) = u(t) - b_0 h(t),\ y(t) = v(t) - b_0 h(t),\ t \in [0,T]$.

Then (S_1), (BC_1) can be equivalently written as

$$\begin{cases} x^{(n)}(t) + b(t) f(y(t) + b_0 h(t)) = 0 \\ y^{(n)}(t) + c(t) g(x(t) + b_0 h(t)) = 0, \ t \in (0,T) \end{cases} \tag{4}$$

with the boundary conditions

$$\begin{cases} x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \ x(T) = \displaystyle\sum_{i=1}^{m-2} a_i x(\xi_i) \\ y(0) = y'(0) = \cdots = y^{(n-2)}(0) = 0, \ y(T) = \displaystyle\sum_{i=1}^{m-2} a_i y(\xi_i). \end{cases} \tag{5}$$

Using the Green function given in Lemma 2, a pair $(x(t), y(t))$ is a solution of problem (4), (5) if and only if

$$\begin{cases} x(t) = \displaystyle\int_0^T G(t,s) b(s) f\left(\int_0^T G(s,\tau) c(\tau) g(x(\tau) + b_0 h(\tau))\, d\tau + b_0 h(s) \right) ds, \\ y(t) = \displaystyle\int_0^T G(t,s) c(s) g(x(s) + b_0 h(s))\, ds, \ 0 \le t \le T, \end{cases} \tag{6}$$

where $h(t),\ t \in [0,T]$ is the solution of (3).

We consider the Banach space $X = C([0,T])$ with supremum norm $\|\cdot\|$ and we define the set

$$K = \{x \in C([0,T]),\ 0 \le x(t) \le c_0,\ \forall\, t \in [0,T]\} \subset X.$$

We also define the operator $\mathcal{B} : K \to X$ by

$$\mathcal{B}(x)(t) = \int_0^T G(t,s)b(s)f\left(\int_0^T G(s,\tau)c(\tau)g(x(\tau)+b_0h(\tau))d\tau + b_0h(s)\right)ds,$$
$$0 \le t \le T.$$

For sufficiently small $b_0 > 0$, by $(A3)a$ we deduce

$$f(y(t)+b_0h(t)) \le \frac{c_0}{L},\ \ g(x(t)+b_0h(t)) \le \frac{c_0}{L},\ \ \forall\, x,\, y \in K,\ \ \forall\, t \in [0,T].$$

Then for any $x \in K$ we have, by using Lemma 3, that $\mathcal{B}(x)(t) \ge 0,\ \forall\, t \in [0,T]$. By Lemma 4 we also have $y(s) \le c_0,\ \forall\, s \in [0,T]$ and $\mathcal{B}(x)(t) \le c_0,\ \forall\, t \in [0,T]$. Therefore $\mathcal{B}(K) \subset K$.

Using standard arguments we deduce that $\mathcal{B}$ is completely continuous (continuous and compact). By the Schauder fixed point theorem, we conclude that $\mathcal{B}$ has a fixed point $x \in K$. This element together with y given by (6) represent a solution for (4) and (5). This shows that our problem (S_1), (BC_1) has a positive solution $u = x+b_0h,\ v = y+b_0h$ for sufficiently small b_0. Q.E.D.

In what follows we shall present sufficient conditions for nonexistence of positive solutions of (S_1), (BC_1).

Theorem 5. *Assume that the assumptions (A1),(A2),$(A3)b$ hold. Then the problem (S_1), (BC_1) has no positive solution for b_0 sufficiently large.*

Sketch of proof. We suppose that (u, v) is a positive solution of (S_1), (BC_1) . Then $x = u - b_0 h$, $y = v - b_0 h$ is solution for (4), (5), where h is the solution of problem (3). By Lemma 3 we have $x(t) \geq 0$, $y(t) \geq 0$, $\forall\, t \in [0, T]$, and by $(A2)$ we deduce that $\|x\| > 0$, $\|y\| > 0$. Using Lemma 5 we also have $\inf\limits_{t \in [\xi_{m-2}, T]} x(t) \geq \gamma \|x\|$ and $\inf\limits_{t \in [\xi_{m-2}, T]} y(t) \geq \gamma \|y\|$.

Using now the expression for h, we deduce that $\inf\limits_{t \in [\xi_{m-2}, T]} h(t) \geq \gamma \|h\|$.

Then $\inf\limits_{t \in [\xi_{m-2}, T]} (x(t) + b_0 h(t)) \geq \gamma \|x + b_0 h\|$ and $\inf\limits_{t \in [\xi_{m-2}, T]} (y(t) + b_0 h(t)) \geq \gamma \|y + b_0 h\|$.

We now consider

$$R = \frac{d(n-1)!}{\gamma \xi_{m-2}^{n-1}} \left(\min \left\{ \int_{\xi_{m-2}}^{T} (T-s)^{n-1} c(s)\, ds, \int_{\xi_{m-2}}^{T} (T-s)^{n-1} b(s)\, ds \right\} \right)^{-1} > 0.$$

By $(A3)b$, for R defined above we deduce that there exists $M > 0$ such that $f(u) > 2Ru$, $g(u) > 2Ru$, for all $u \geq M$.

We consider $b_0 > 0$ sufficiently large such that

$$\inf_{t\in[\xi_{m-2},T]}(x(t)+b_0h(t)) \geq M \quad \text{and} \quad \inf_{t\in[\xi_{m-2},T]}(y(t)+b_0h(t)) \geq M.$$

By using Lemma 4 and the above considerations, we have $y(\xi_{m-2}) \geq 2\|x+b_0h\| \geq 2\|x\|$.

And then we obtain

$$\|x\| \leq \frac{1}{2}y(\xi_{m-2}) \leq \frac{1}{2}\|y\|. \tag{7}$$

In a similar manner we deduce $x(\xi_{m-2}) \geq 2\|y+b_0h\| \geq 2\|y\|$ and so

$$\|y\| \leq \frac{1}{2}x(\xi_{m-2}) \leq \frac{1}{2}\|x\|. \tag{8}$$

By (7) and (8) we obtain $\|x\| \leq \frac{1}{2}\|y\| \leq \frac{1}{4}\|x\|$, which is a contradiction, because $\|x\| > 0$. Then, when b_0 is sufficiently large, our problem (S_1), (BC_1) has no positive solution. Q.E.D.

Example 2

We consider $T = 1,\ b(t) = bt,\ c(t) = ct,\ t \in [0, 1],\ b,\ c > 0,\ n = 3,\ m = 5,\ \xi_1 = \frac{1}{3},\ \xi_2 = \frac{2}{3},\ a_1 = 1,\ a_2 = \frac{1}{2}$. Then $d = 1 - \sum_{i=1}^{2} a_i \xi_i^2 = \dfrac{2}{3} > 0$.

We also consider the functions $f, g : [0, \infty) \to [0, \infty),\ f(x) = \dfrac{\widetilde{a}x^3}{x+1},\ g(x) = \dfrac{\widetilde{b}x^3}{x+1}$ with $\widetilde{a}, \widetilde{b} > 0$. We have $\lim\limits_{x\to\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to\infty} \dfrac{g(x)}{x} = \infty$. The constant L from $(A3)$ is in this case

$$L = \max\left\{ \frac{1}{2d} \int_0^1 (1-s)^2 bs\, ds, \frac{1}{2d} \int_0^1 (1-s)^2 cs\, ds \right\} = \frac{1}{16} \max\{b, c\}.$$

We choose $c_0 = 1$ and if we select $\widetilde{a}$ and $\widetilde{b}$ satisfying the conditions

$$\widetilde{a} < \frac{2}{L} = \frac{32}{\max\{b, c\}} = 32 \min\left\{\frac{1}{b}, \frac{1}{c}\right\},\ \ \widetilde{b} < \frac{2}{L} = 32 \min\left\{\frac{1}{b}, \frac{1}{c}\right\},$$

then we obtain $f(x) \le \frac{\widetilde{a}}{2} < \frac{1}{L},\ g(x) \le \frac{\widetilde{b}}{2} < \frac{1}{L}$, for all $x \in [0, 1]$.

Thus all the assumptions $(A1) - (A3)$ are verified. By Theorem 4 and Theorem 5 we deduce that the nonlinear third-order differential system

$$\begin{cases} u'''(t) + bt\dfrac{\widetilde{a}v^3(t)}{v(t)+1} = 0 \\ v'''(t) + ct\dfrac{\widetilde{b}u^3(t)}{u(t)+1} = 0, \ \ t \in (0,1) \end{cases}$$

with the boundary conditions

$$\begin{cases} u(0) = u'(0) = 0, \ \ u(1) = u(\frac{1}{3}) + \frac{1}{2}u(\frac{2}{3}) + b_0 \\ v(0) = v'(0) = 0, \ \ v(1) = v(\frac{1}{3}) + \frac{1}{2}v(\frac{2}{3}) + b_0, \end{cases}$$

has at least one positive solution for sufficiently small $b_0 > 0$ and no positive solution for sufficiently large b_0.

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