Measure concentration, functional inequalities, and curvature of metric measure spaces

M. Ledoux

Institut de Mathématiques de Toulouse, France

circle of ideas

between analysis, geometry and probability theory

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concentration of measure phenomenon

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(Ricci) curvature bounds on metric measure spaces

concentration of measure phenomenon

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Dvoretzky's theorem on spherical sections of convex bodies

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every convex body $K \subset \mathbb{R}^n$

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$K \cap F$

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normalized volume $\mu(\cdot) = \dfrac{\mathrm{vol}\,(\cdot)}{\mathrm{vol}\,(\mathbb{S}^n)}$

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metric measure space (X, d, μ)

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concentration function

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Gauss space : curvature 1 dimension ∞

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infinite dimensional analysis (Wiener space)

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complementary to PDE and calculus of variations viewpoint

geometric description : Brunn-Minkowski inequality

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$u = \chi_A, \ \ v = \chi_B$ multiplicative form of Brunn-Minkowski

$$\mathrm{vol}_n\big(\theta A + (1-\theta)B\big) \geq \mathrm{vol}_n(A)^\theta \mathrm{vol}_n(B)^{1-\theta}$$

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extension : $\quad d\mu = e^{-V} dx, \quad V'' \geq c > 0$

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$\int \frac{|\nabla f|^2}{f} \, d\gamma$ Fisher information

functional description : logarithmic Sobolev inequality

$$f \geq 0 \quad \text{smooth on } \mathbb{R}^n, \ \int f \, d\gamma = 1$$

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A. Stam (1959), L. Gross (1975)

concentration via the logarithmic Sobolev inequality **(I. Herbst)**

$F : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz, $\int F \, d\gamma = 0$

$f = e^{\lambda F} / \int e^{\lambda F} d\gamma, \quad \lambda \in \mathbb{R}$

differential inequality on $\int e^{\lambda F} d\gamma$

$$\int e^{\lambda F} d\gamma \le e^{\lambda^2/2}, \quad \lambda \in \mathbb{R}$$

$$\gamma\big(\{F < r\}\big) \ge 1 - e^{-r^2/2}, \quad r > 0$$

measure/information theoretic description :

transportation cost inequality

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ν probability measure on $\mathbb{R}^n$, $\nu \ll \gamma$

$$W_2(\nu, \gamma)^2 \leq H(\nu \,|\, \gamma)$$

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relative entropy

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$$W_2(\nu, \gamma)^2 = \inf_{\nu \leftarrow \pi \rightarrow \gamma} \iint \frac{1}{2} \, |x - y|^2 d\pi(x, y)$$

Kantorovich-Rubinstein-Wasserstein distance

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Kantorovich-Rubinstein-Wasserstein distance

M. Talagrand (1996)

concentration via the transportation cost inequality **(K. Marton)**

$$A, B \subset \mathbb{R}^n, \quad d(A,B) \geq r > 0$$

$$\gamma_A = \gamma(\cdot\,|A), \quad \gamma_B = \gamma(\cdot\,|B)$$

$$W_2(\gamma_A, \gamma_B) \leq \Big(\log \frac{1}{\gamma(A)}\Big)^{1/2} + \Big(\log \frac{1}{\gamma(B)}\Big)^{1/2}$$

$$\frac{r}{\sqrt{2}} \leq W_2(\gamma_A, \gamma_B) \leq \Big(\log \frac{1}{\gamma(A)}\Big)^{1/2} + \Big(\log \frac{1}{\gamma(B)}\Big)^{1/2}$$

$\gamma(A) \geq 1/2, \quad B =$ complement of A_r

$$\gamma(A_r) \geq 1 - e^{-r^2/4}, \quad r \geq r_0$$

Prékopa-Leindler inequality

logarithmic Sobolev inequality

transportation cost inequality

Prékopa-Leindler inequality

$$\int w \, d\gamma \ \geq \ \Big(\int u \, d\gamma \Big)^{\theta} \Big(\int v \, d\gamma \Big)^{1-\theta}$$

logarithmic Sobolev inequality

$$\int f \log f \, d\gamma \ \leq \ \frac{1}{2} \int \frac{|\nabla f|^2}{f} \, d\gamma$$

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$$W_2(\nu, \gamma)^2 \ \leq \ H(\nu \,|\, \gamma)$$

Prékopa-Leindler inequality

$$\int w \, d\mu \, \geq \Big(\int u \, d\mu \Big)^{\theta} \Big(\int v \, d\mu \Big)^{1-\theta}$$

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$$\int f \log f \, d\mu \, \leq \, \frac{C}{2} \int \frac{|\nabla f|^2}{f} \, d\mu$$

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$$W_2(\nu, \mu)^2 \, \leq \, C \, H(\nu \,|\, \mu)$$

μ probability measure on $\mathbb{R}^n$

or more general spaces

hierarchy

Prékopa-Leindler inequality

$\Downarrow$

logarithmic Sobolev inequality

$\Downarrow$

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transportation cost inequality

F. Otto, C. Villani (2000)

PDE and transportation methods

hypercontractivity of Hamilton-Jacobi equations

hierarchy

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dimension free measure concentration

tools to establish these inequalities

hierarchy

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tools to establish these inequalities

parametrisation methods

heat kernel parametrisation

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equivalent to a curvature condition

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$$\frac{1}{2}\,\Delta\big(|\nabla f|^2\big) - \nabla f \cdot \nabla(\Delta f) \;=\; \mathrm{Ric}(\nabla f, \nabla f) + \|\mathrm{Hess}\,(f)\|^2$$

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equivalent to $\quad |\nabla P_t f| \,\leq\, e^{-ct} P_t\big(|\nabla f|\big)$ **D. Bakry (1986)**

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◇ manifolds with weights

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extensions

◇ second order differential operators

D. Bakry, M. Emery (1985)

extensions

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Γ_2 curvature principle

analogue of Bochner's formula for Markov operator

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Markov chains, discrete structures

Y. Ollivier (2008)

parametrisation by optimal transportation

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Riemannian geometry of $(\mathcal{P}_2, W_2)$ **F. Otto (2001), C. Villani (2003, 09)**

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manifold case **R. McCann (1995)**

parametrisation $\quad T_\theta = (1-\theta)\,\mathrm{Id} + \theta\, T, \quad \theta \in [0,1]$

$$(\, T_0\gamma = \gamma, \quad T_1\gamma = T\gamma = \nu)$$

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Monge-Ampère equation

$$e^{-|x|^2/2} = f_\theta \circ T_\theta\, e^{-|T_\theta|^2/2} \det\big((1-\theta)\,\mathrm{Id} + \theta\,\phi''\big)$$

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non-smooth analysis, PDE methods

mass transportation method

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optimal parametrisation and entropy

J. Lott - C. Villani, K. Th. Sturm (2006-09)

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H relative entropy, W_2 Wasserstein distance

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reference measure $d\mu = e^{-V}dx$

extends to (weighted) manifolds

characterizes $\mathrm{Ric} + \mathrm{Hess}(V) \geq c$

K. Th. Sturm (2005)

notion of Ricci curvature bound

in a metric measure space (length space) (X, d, μ)

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J. Lott - C. Villani, K.-Th. Sturm (2006-09)

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◇ generalizes Ricci curvature in Riemannian manifolds

J. Lott - C. Villani, K.-Th. Sturm (2006-09)

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◇ generalizes Ricci curvature in Riemannian manifolds

◇ allows for geometric and functional inequalities

(transportation cost, logarithmic Sobolev inequalities)

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