Analyse asymptotique d'une interface mince : cas de rigidités comparables

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Outline

1 Introduction

- The Mechanical Problem
- References
- The work of Abdelmoula, Coutris, Marigo

2 Contributions

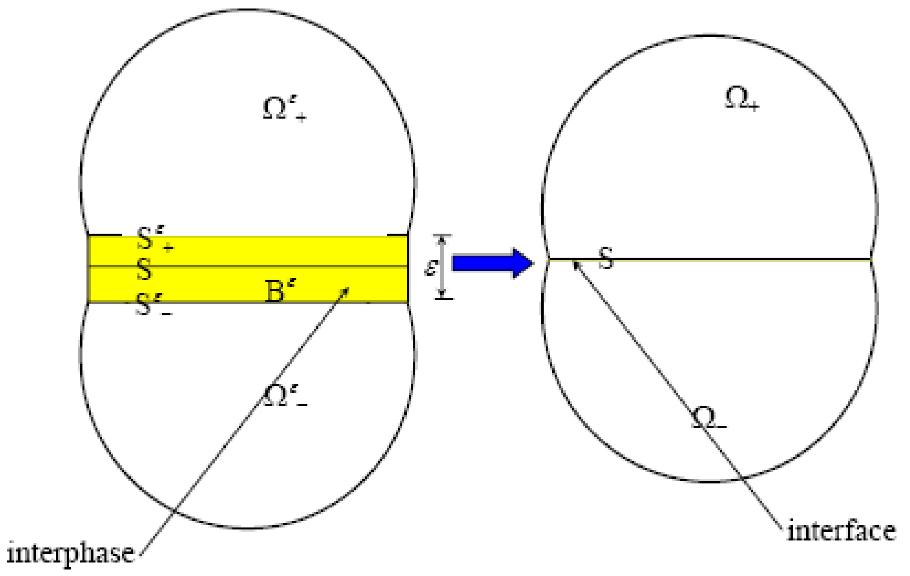
- Γ convergence results
- First order results
- Anisotropy

Motivations

Example: Modeling brick masonries

Sketch of the problem

- Linearized kinematics
- Linear elastic bodies



⇒ The transmission conditions depend on the scaling of the interphase elastic parameters with ε.

Interface models

« Classical » result: Soft interphase $\Rightarrow$ "Spring-type“ interface

$$\sigma \mathrm{n} = \mathbb{K}[\mathrm{u}]$$

Stiff interphase $\Rightarrow$ Different regimes of interface

Soft interface

- Taylor expansion (Bövik, 1994; Hashin, 1990, 1991; Benveniste, 1985; Benveniste and Miloh, 2001)
- Asymptotic expansions and analysis of the stress field (Klarbring, 1991; Bigoni, et. al, 1998; Geymonat et. al. 1999; Lenci, 2000)
- Mathematical analysis (Suquet, 1988; Ganghoffer et. al., 1997; Licht and Michaille, 1997; Geymonat et. al. 1999; Lenci, 2000 ; Aït- Moussa and Zlaïji, 2004)

Stiff interface

- Asymptotic expansions (Nguetseng and Sanchez-Palencia, 1985; Abelmoula, Coutris and Marigo, 1998)
- Mathematical analysis (Caillerie, 1980; Acerbi and Buttazzo, 1986, Acerbi et. al., 1988)

Hypotheses

First hypothesis

The adhesive (interphase) is thin.

Second Hypothesis

Adhesive and adherents have comparable rigidity.

NB: Perfect adhesion

Abdelmoula, Coutris and Marigo, 1998

A two-dimensional stiff interface

- Elastic constant independant on ε
- Isotropic material
- Matched asymptotic expansions
- Transmission conditions at order zero (perfect interface)
- Transmission conditions at order one (imperfect interface)

Abdelmoula, Coutris and Marigo, 1998

Transmission conditions at order zero (Perfect interface)

$$[u^0] = 0\ , \quad [\sigma^0 n] = 0$$

Transmission conditions at order one (Imperfect interface)

$$\begin{aligned}
[u_1^1](x_1) &= \frac{\sigma_{12}^0}{\mu} - \frac{\partial u_2^0}{\partial x_1} \\
[u_2^1](x_1) &= \frac{\sigma_{22}^0}{\lambda + 2\mu} - \frac{\lambda}{\lambda + 2\mu}\frac{\partial u_1^0}{\partial x_1} \\
[\sigma_{22}^1](x_1) &= -\frac{\partial \sigma_{12}^0}{\partial x_1} \\
[\sigma_{12}^1](x_1) &= -\frac{4\mu(\lambda+\mu)}{\lambda + 2\mu}\frac{\partial^2 u_1^0}{\partial x_1^2} - \frac{\lambda}{\lambda + 2\mu}\frac{\partial \sigma_{22}^0}{\partial x_1}
\end{aligned}$$

Tangential derivatives

Extension (3D Isotropic)

$$[u_\alpha^1] = \frac{1}{\mu}\sigma_{\alpha 3}^0(\hat{x},0) - u_{3,\alpha}^0(\hat{x},0) - \frac{1}{2}(u_{\alpha,3}^0(\hat{x},0^+) + u_{\alpha,3}^0(\hat{x},0^-)), \quad \alpha = 1,2,$$

$$[u_3^1] = \frac{1}{\lambda+2\mu}\sigma_{33}^0(\hat{x},0) - \frac{\lambda}{\lambda+2\mu}(u_{1,1}^0(\hat{x},0) + u_{2,2}^0(\hat{x},0)) - \frac{1}{2}(u_{3,3}^0(\hat{x},0^+) + u_{3,3}^0(\hat{x},0^-)),$$

$$[\sigma_{13}^1] = -\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}u_{1,11}^0(\hat{x},0) - \mu u_{1,22}^0(\hat{x},0) - \frac{\mu(3\lambda+2\mu)}{\lambda+2\mu}u_{2,21}^0 - \frac{\lambda}{\lambda+2\mu}\sigma_{33,1}^0(\hat{x},0) \boxed{- \frac{1}{2}(\sigma_{13,3}^0(\hat{x},0^+) + \sigma_{13,3}^0(\hat{x},0^-)),}$$

$$[\sigma_{23}^1] = -\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}u_{2,22}^0(\hat{x},0) - \mu u_{2,11}^0(\hat{x},0) - \frac{\mu(3\lambda+2\mu)}{\lambda+2\mu}u_{1,12}^0(\hat{x},0) - \frac{\lambda}{\lambda+2\mu}\sigma_{33,2}^0(\hat{x},0) \boxed{- \frac{1}{2}(\sigma_{23,3}^0(\hat{x},0^+) + \sigma_{23,3}^0(\hat{x},0^-)),}$$

$$[\sigma_{33}^1] = -\sigma_{13,1}^0(\hat{x},0) - \sigma_{23,2}^0(\hat{x},0) \boxed{- \frac{1}{2}(\sigma_{33,3}^0(\hat{x},0^+) + \sigma_{33,3}^0(\hat{x},0^-)),}$$

Saut de la dérivée du vecteur contrainte
à l'interface

Analytical example

$$\phi(x) = \frac{b}{L-\varepsilon}(x-\varepsilon) + c$$

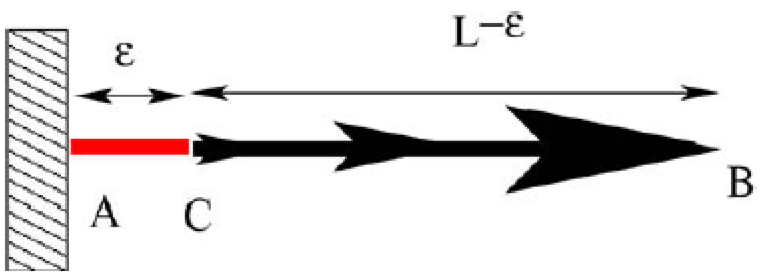


Fig. 4. A composite bar subjected to body forces.

At equilibrium, the displacement field is given by:

$$u(x) = (b/2 + c)(L-\varepsilon)\frac{x}{E_1} \quad \text{if } 0 \leqslant x \leqslant \varepsilon,$$

$$u(x) = -\frac{b(x-\varepsilon)^3}{6(L-\varepsilon)E_2} - \frac{c(x-\varepsilon)^2}{2E_2} + \frac{(b/2+c)(L-\varepsilon)}{E_2}(x-\varepsilon) + (b/2+c)(L-\varepsilon)\frac{x}{E_1} \quad \text{if } L-\varepsilon \leqslant x \leqslant L.$$

The stress, which is not constant, is given by:

$$\sigma(x) = (b/2 + c)(L-\varepsilon) \quad \text{if } 0 \leqslant x \leqslant \varepsilon,$$

$$\sigma(x) = -\frac{b(x-\varepsilon)^2}{2(L-\varepsilon)} - c(x-\varepsilon) + (b/2+c)(L-\varepsilon) \quad \text{if } L-\varepsilon \leqslant x \leqslant L.$$

Analytical example

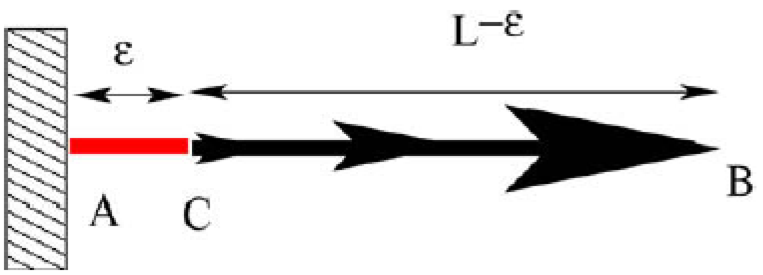


Fig. 4. A composite bar subjected to body forces.

At order zero, we obtain:

$$u^0(x) = -\frac{bx^3}{6LE_2} - \frac{cx^2}{2E_2} + \frac{(b/2+c)Lx}{E_1},$$

$$\sigma^0(x) = -\frac{b}{2L}x^2 - cx + (b/2+c)L.$$

It can easily be confirmed that $[u^0] = [\sigma^0] = 0$. At order one, we obtain:

$$u^1(x) = -\frac{b(x^3/L - 3x^2)}{6LE_2} + \frac{cx}{E_2} - \frac{(b/2+c)(L+x)}{E_2} + \frac{(b/2+c)L}{E_1},$$

$$\sigma^1(x) = \frac{b(2x - x^2/L)}{2L} + b/2.$$

$[u^1] = (b/2+c)L\left(\frac{1}{E_1} - \frac{1}{E_2}\right) = \frac{1}{E_1}\sigma^0(0) - u^0_{,x}(0)$ and $[\sigma^1] = c = -\frac{1}{2}(\sigma^0_{,x}(0^+) + \sigma^0_{,x}(0^-))$.

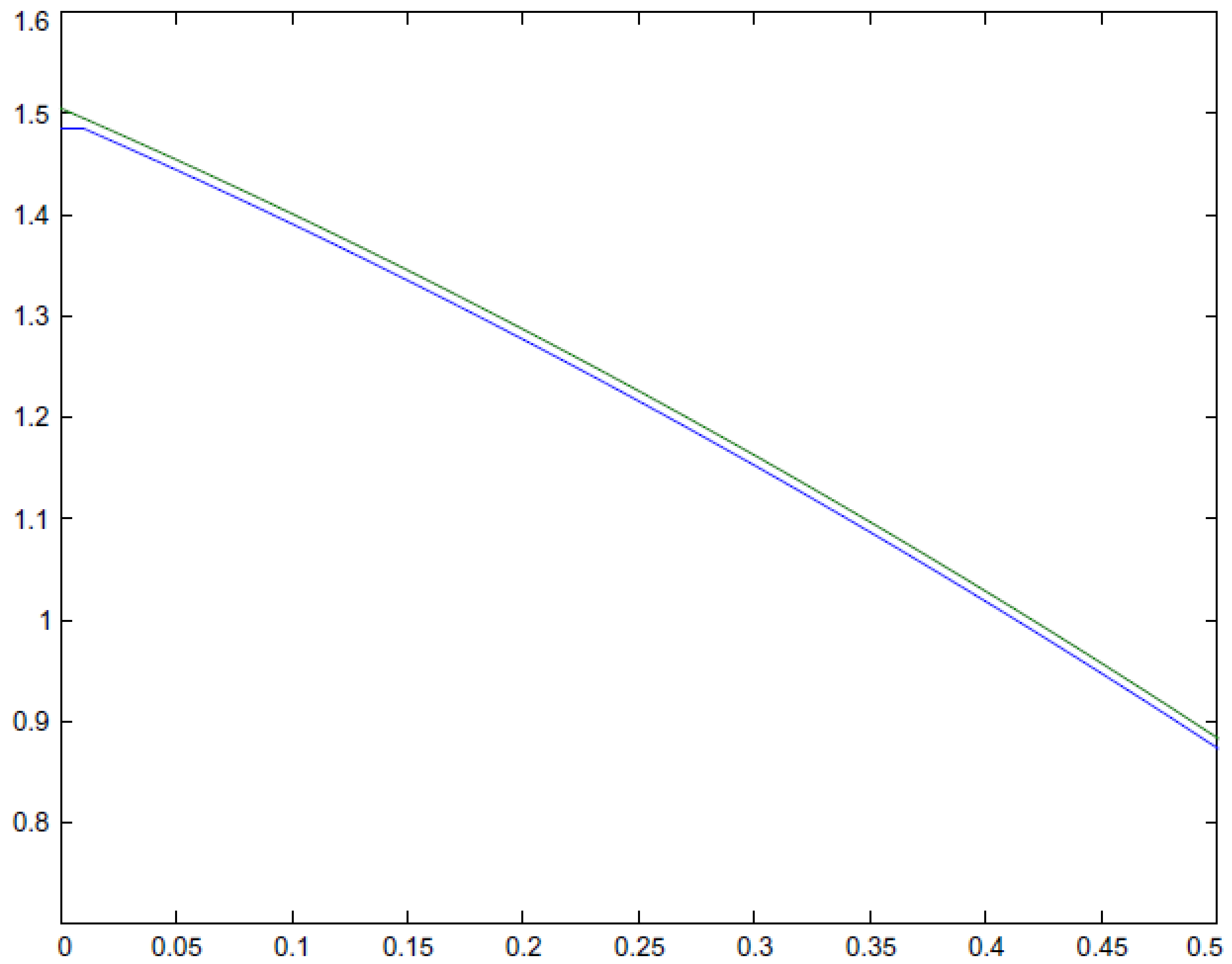


Fig. 5. Exact and order one stresses in an a-dimensionalized case $L = 1$, $\varepsilon = 0.01$, $b = c = 1$, $E_1 = 89$, $E_2 = 210$.

Analytical example

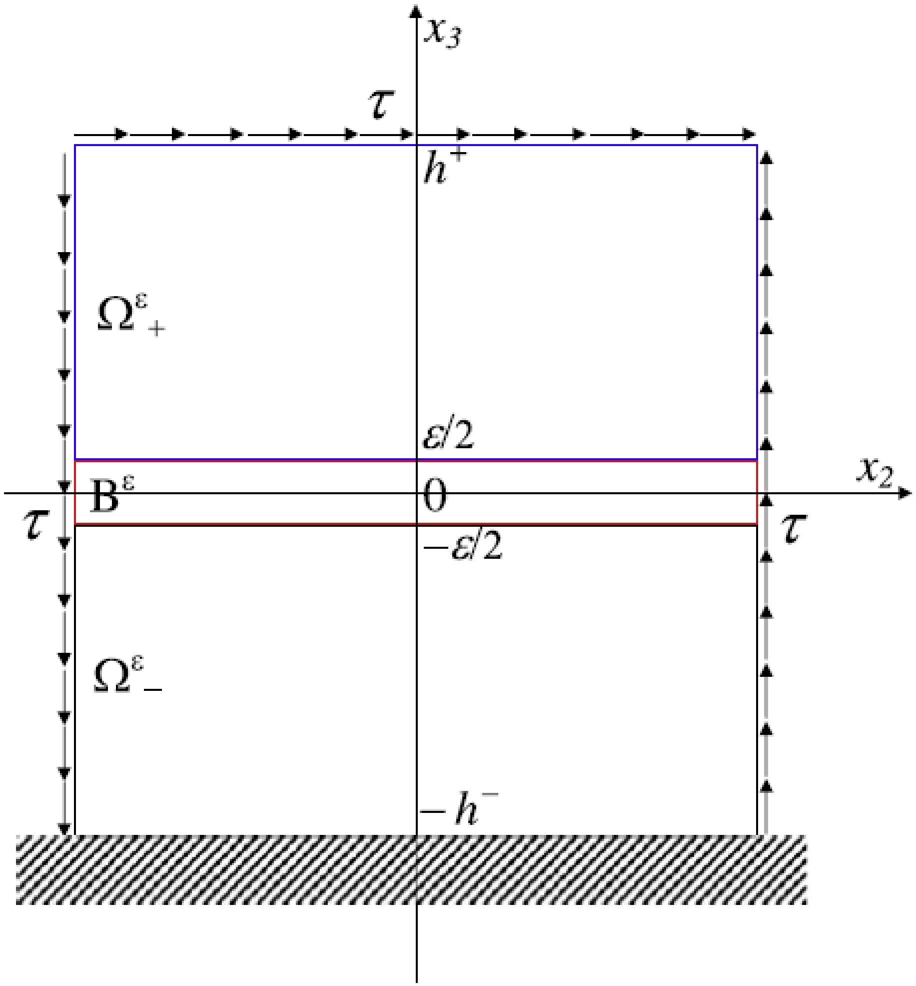


Fig. 6. Composite block subjected to tangential loading.

Analytical example

$$u^\varepsilon = \begin{cases} \left(\frac{\tau}{\mu_-}(x_3 + h^- + \frac{\varepsilon}{2})\right)e_2 & \text{in } \Omega_-^\varepsilon, \\ \left(\frac{\tau}{\mu}(x_3 + \frac{\varepsilon}{2}) + \frac{\tau}{\mu_-}h^-\right)e_2 & \text{in } B^\varepsilon, \\ \left(\frac{\tau}{\mu_+}(x_3 - \frac{\varepsilon}{2}) + \frac{\tau}{\mu}\varepsilon + \frac{\tau}{\mu_-}h^-\right)e_2 & \text{in } \Omega_+^\varepsilon, \end{cases}$$

$$u^0 = \begin{cases} \left(\frac{\tau}{\mu_-}(x_3 + h^-)\right)e_2 & \text{in } \Omega_-, \\ \left(\frac{\tau}{\mu_+}x_3 + \frac{\tau}{\mu_-}h^-\right)e_2 & \text{in } \Omega_+, \end{cases}$$

$$u^1 = \begin{cases} \frac{\tau}{2\mu_-}e_2 & \text{in } \Omega_-, \\ \left(-\frac{\tau}{2\mu_+} + \frac{\tau}{\mu}\right)e_2 & \text{in } \Omega_+. \end{cases}$$

The limit u^1 is discontinuous at $x_3 = 0$ and

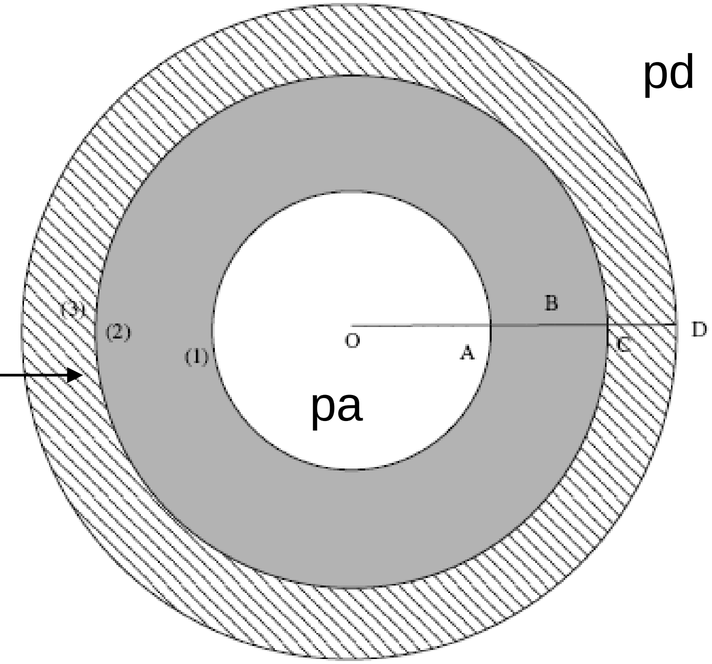
$$[u_1^1] = 0,$$

$$[u_2^1] = \frac{\tau}{\mu} - \frac{1}{2}\left(\frac{\tau}{\mu_+} + \frac{\tau}{\mu_-}\right),$$

$$[u_3^1] = 0,$$

Analytical study

Thin layer



$$
\begin{cases}
u_1(r) &= \dfrac{p_a a^2 - p_b b^2}{b^2 - a^2} \dfrac{r}{2(\lambda_1 + \mu_1)} + \dfrac{b^2 a^2 (p_a - p_b)}{2\mu_1 (b^2 - a^2) r} \\
u_2(r) &= \dfrac{p_b b^2 - p_c c^2}{c^2 - b^2} \dfrac{r}{2(\lambda_2 + \mu_2)} + \dfrac{c^2 b^2 (p_b - p_c)}{2\mu_2 (c^2 - b^2) r} \\
u_3(r) &= \dfrac{p_c c^2 - p_d d^2}{d^2 - c^2} \dfrac{r}{2(\lambda_3 + \mu_3)} + \dfrac{d^2 c^2 (p_c - p_d)}{2\mu_3 (d^2 - c^2) r}
\end{cases}
$$

Analytical study

$$\begin{cases} p_b &= \dfrac{1}{\Delta}\left(K_a p_a (M_d + M_{22}) + K_d p_d N_{21}\right) \\ p_c &= \dfrac{1}{\Delta}\left(K_d p_d (M_a + M_{21}) + K_a p_a N_{22}\right) \end{cases}$$

$$\frac{A}{C} + \varepsilon \frac{BC - AD}{C^2}$$

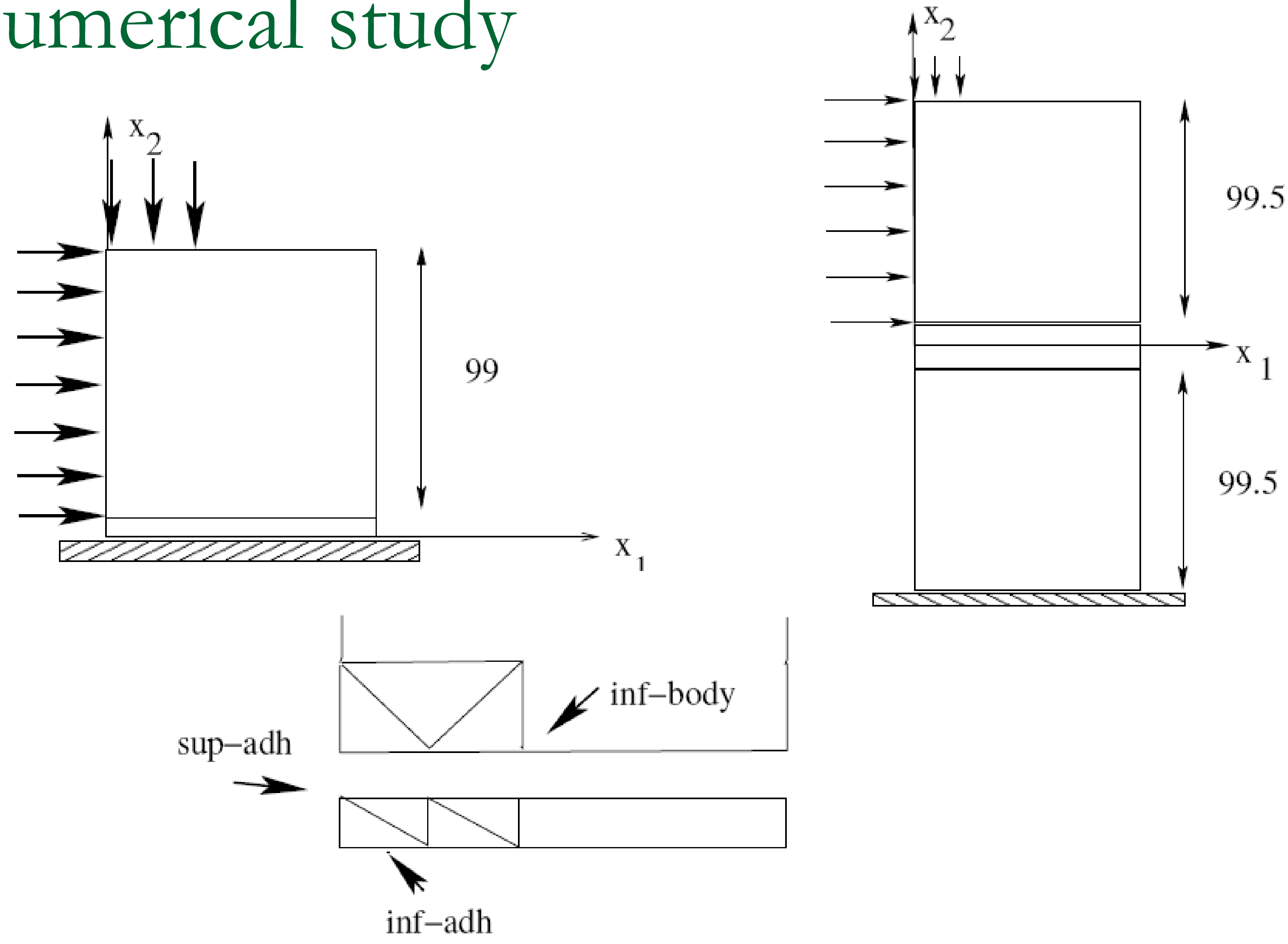
Analytical study

$$[u^0] = u_c^0 - u_b^0 = 0.$$

$$[u_1^1(b)] = \frac{\sigma_{11}^0(b).n}{\lambda_2 + \mu_2} - \frac{\lambda_2 u_1^0(b)}{b(\lambda_2 + \mu_2)}$$

$$\left\{ \begin{aligned}
u(b) &= \frac{p_a a^2 - p_b b^2}{b^2 - a^2} \frac{b}{2(\lambda_1 + \mu_1)} + \frac{ba^2(p_a - p_b)}{2\mu_1(b^2 - a^2)} \\
u(b) &= u_b^0 + \varepsilon u_b^1 + O(\varepsilon^2) \\
u_b^0 &= \frac{p_a a^2 - p_b^0 b^2}{b^2 - a^2} \frac{b}{2(\lambda_1 + \mu_1)} + \frac{ba^2(p_a - p_b^0)}{2\mu_1(b^2 - a^2)} \\
u_b^1 &= -\frac{p_b^1 b^3}{2(b^2 - a^2)(\lambda_1 + \mu_1)} - \frac{ba^2 p_b^1}{2\mu_1(b^2 - a^2)} \\
u(c) &= \frac{p_c c^2 - p_d d^2}{d^2 - c^2} \frac{c}{2(\lambda_3 + \mu_3)} + \frac{cd^2(p_c - p_d)}{2\mu_3(d^2 - c^2)} \\
u(c) &= u_c^0 + \varepsilon u_c^1 + O(\varepsilon^2) \\
u_c^0 &= \frac{p_c^0 c^2 - p_d d^2}{d^2 - b^2} \frac{b}{2(\lambda_3 + \mu_3)} + \frac{bd^2(p_c^0 - p_d)}{2\mu_3(d^2 - b^2)} \\
u_c^1 &= \frac{p_c^0(b^2 d^2 \lambda_3 + d^4 \lambda_3 - b^4 \mu_3 + 4b^2 d^2 \mu_3 + d^4 \mu_3)}{2(b^2 - d^2)^2 \mu_3(\lambda_3 + \mu_3))} \\
&\quad + \frac{p_c^1(-b^3 d^2 \lambda_3 + bd^4 \lambda_3 - b^5 \mu_3 + bd^4 \mu_3)}{2(b^2 - d^2)^2 \mu_3(\lambda_3 + \mu_3))} \\
&\quad + \frac{p_d(-b^2 d^2 \lambda_3 - d^4 \lambda_3 - 2b^2 d^2 \mu_3 - 2d^4 \mu_3)}{2(b^2 - d^2)^2 \mu_3(\lambda_3 + \mu_3)}
\end{aligned} \right.$$

Numerical study

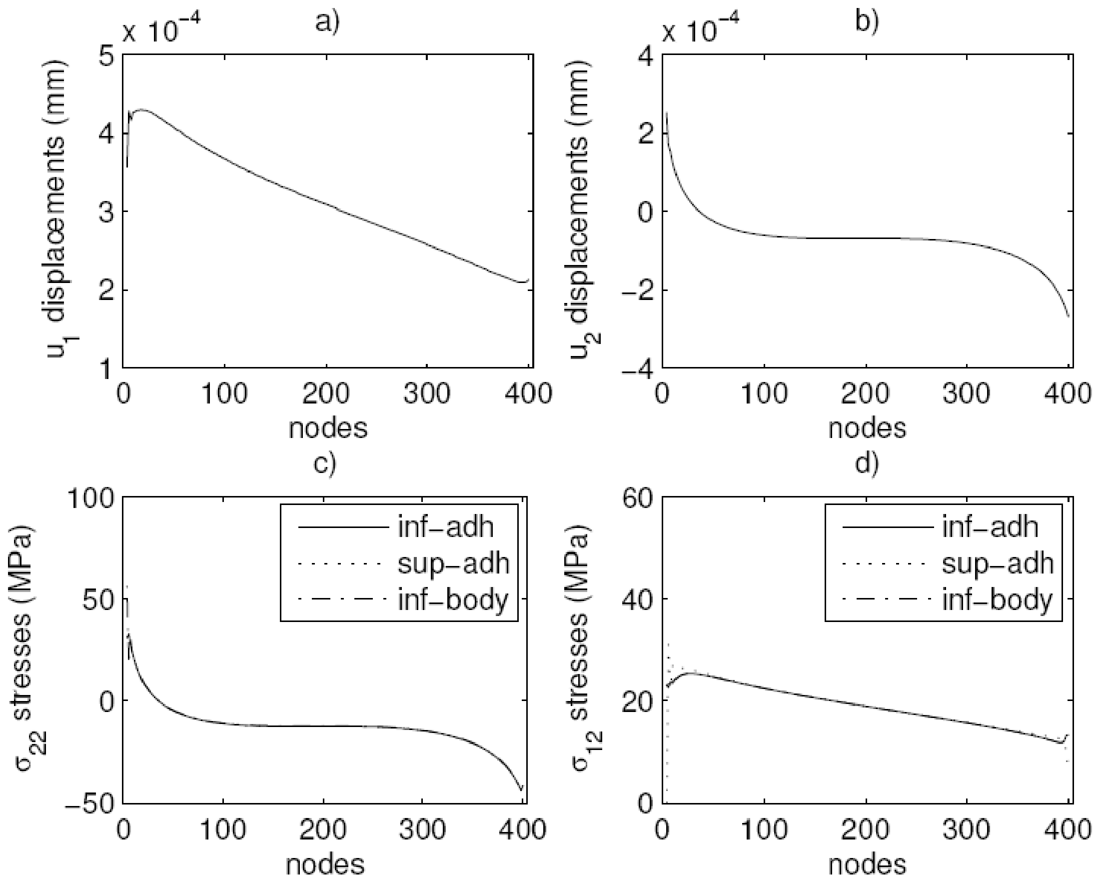


Numerical study

Example	1	2
thickness (mm)	1	1
Substrate: Young's modulus (GPa)	200	200
Substrate: Poisson ratio	0.3	0.3
Thin layer: Young's modulus (GPa)	160	160
Thin layer: Poisson ratio	0.3	0.3
Total x_1-force (N)	1800 (18 nodes)	1800 (18 nodes)
Total x_2-force (N)	-1200 (60 nodes)	-1000 (50 nodes)
Finite element	8-node quadrangle	8-node quadrangle

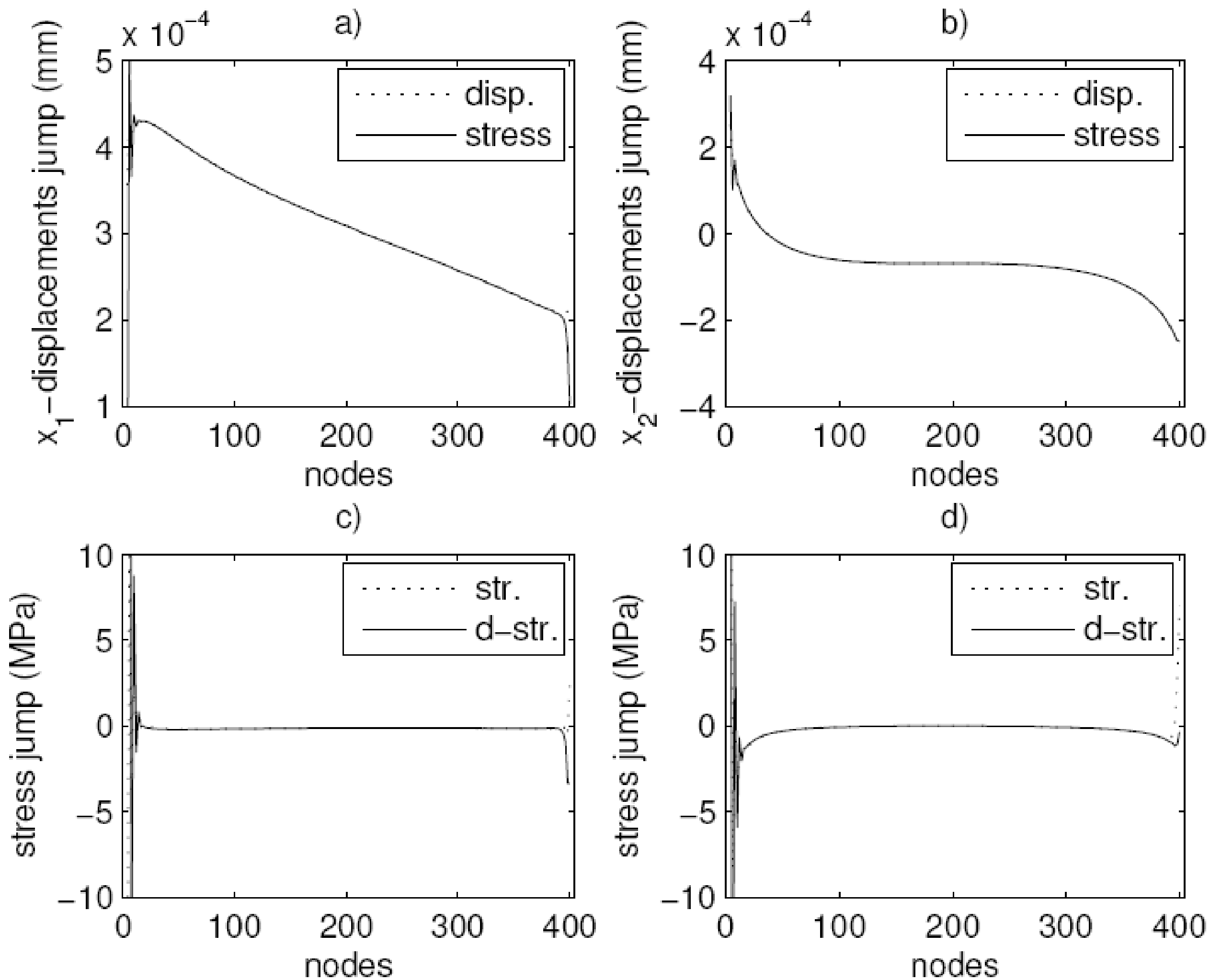
Table 1: Mechanical data

Order zero



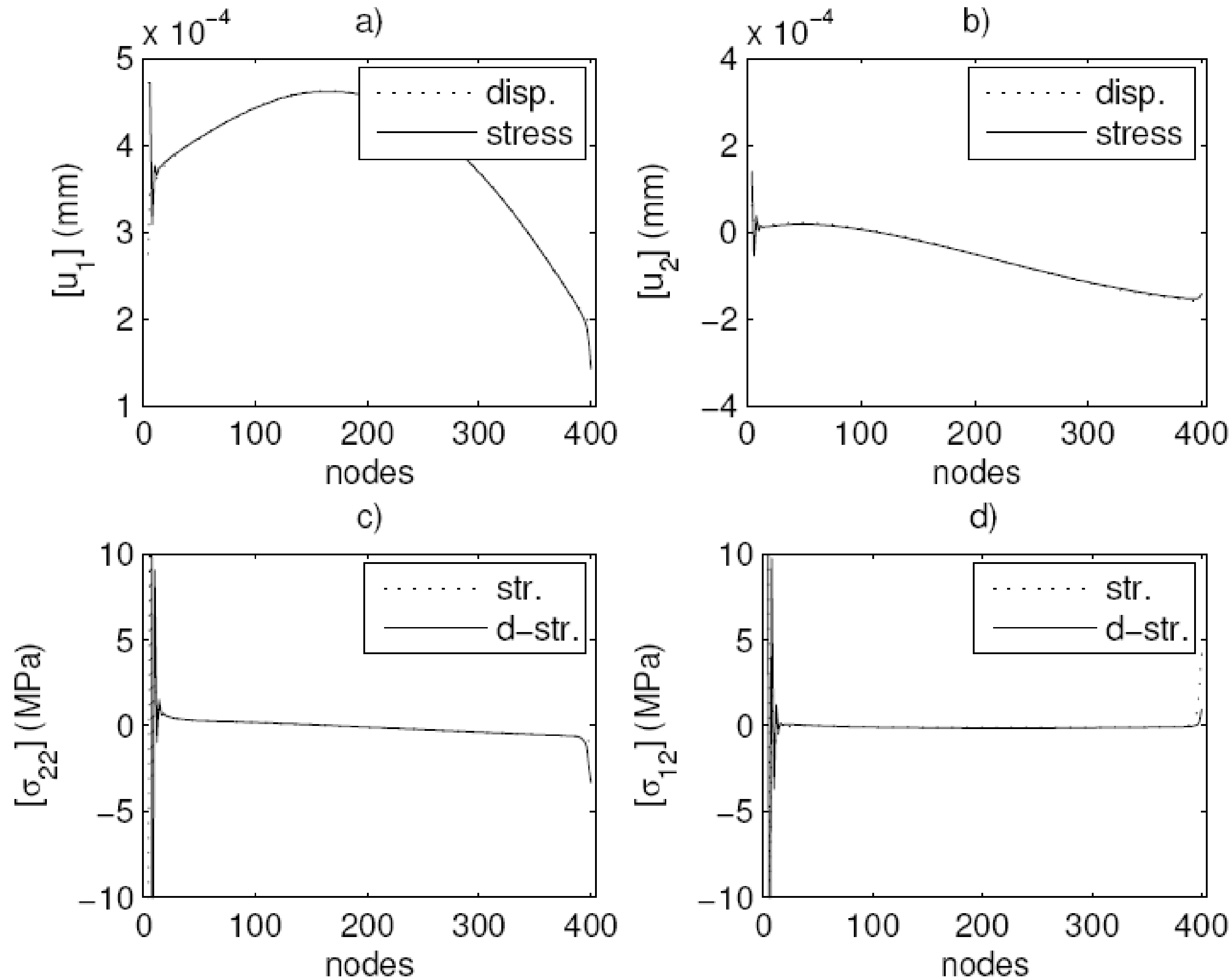
a)
x 10^-4
u1 displacements (mm)
nodes
b)
x 10^-4
u2 displacements (mm)
nodes
c)
σ22 stresses (MPa)
nodes
inf−adh
sup−adh
inf−body
d)
σ12 stresses (MPa)
nodes
inf−adh
sup−adh
inf−body

Order one



a)
x 10^-4
x_1–displacements jump (mm)
disp.
stress
nodes
b)
x 10^-4
x_2–displacements jump (mm)
disp.
stress
nodes
c)
stress jump (MPa)
str.
d–str.
nodes
d)
stress jump (MPa)
str.
d–str.
nodes

Order one



a)
x 10^-4
[u1] (mm)
disp.
stress
nodes
b)
x 10^-4
[u2] (mm)
disp.
stress
nodes
c)
[σ22] (MPa)
str.
d–str.
nodes
d)
[σ12] (MPa)
str.
d–str.
nodes

Remarks

Equilibrium Problem at order zero

Equilibrium problem at order one (minimization ?)

Convergence?

$\Rightarrow$ Energy method (Asymptotic expansions by $\Gamma-$convergence)?

Notations

We define

$$F^{\varepsilon}(v) = \begin{cases} \frac{1}{2}(a^{\varepsilon}_{-}(v,v) + a^{\varepsilon}_{+}(v,v) + a^{\varepsilon}_{m}(v,v)) & \text{if } v \in V^{\varepsilon} \\ +\infty & \text{if } v \in (L^2(\Omega))^3 \setminus V^{\varepsilon} \end{cases}$$

where

$$\begin{aligned} a^{\varepsilon}_{\pm}(v,v) &= \int_{\Omega^{\varepsilon}_{\pm}} a^{\pm}[e(v)] \cdot e(v)\, dx \quad \text{(adherents)} \\ a^{\varepsilon}_{m}(v,v) &= \int_{B^{\varepsilon}} a^{m}[e(v)] \cdot e(v)\, dx \quad \text{(adhesive)} \end{aligned}$$

and

$$V^{\varepsilon} = \{v \in (W^{1,2}(\Omega))^3 : v = 0 \text{ on } \Gamma_0\}$$

Assumptions

$$H1)\qquad \begin{cases} a^{\pm}_{ijkl} \in L^{\infty}(\Omega)\ , \\ a^{\pm}_{ijkl} = a^{\pm}_{klij} = a^{\pm}_{jilk} \\ \exists \eta > 0 : a^{\pm}_{ijkl} e_{ij} e_{kl} \geq \eta e_{ij} e_{ij} \quad \forall e_{ij} = e_{ji}\ , \end{cases}$$

$$H2)\qquad \exists \varepsilon_0 : B_\varepsilon \cap (\Gamma_1 \cup \mathrm{supp}(\phi)) = \emptyset\ , \quad \forall\ \ \varepsilon < \varepsilon_0\ .$$

$$H3)\qquad \phi \in (L^2(\Omega))^3\ , g \in (L^2(\Gamma_1))^3\ .$$

Main result at order zero

Theorem

The sequence of functionals F^ε Γ-converges for the strong topology of $X = (L^2(\Omega))^3$ to

$$F^0(v) = \begin{cases} \int_{\Omega_+ \cup \Omega_-} a^\pm e(v) \cdot e(v)\, dx & \text{if } v \in V^0 , \\ +\infty & \text{if } v \in X \setminus V^0 , \end{cases} \tag{1}$$

where

$$V^0 = \{v \in (W^{1,2}(\Omega_0))^3 : v = 0 \text{ on } \Gamma_0 , \ [v] = 0 \text{ on } S\} . \tag{2}$$

Proof

Lemma 4.1. *For all $v \in V^\varepsilon$, there exists a constant $C > 0$ independent of ε such that :*

$$\int_{B^\varepsilon} |\, v(x)\,|^2 \, dx \le C(\varepsilon^2 \int_{B^\varepsilon} |\, e(v(x))\,|^2 \, dx + \varepsilon \int_{\Omega^\varepsilon} |\, e(v(x))\,|^2 \, dx) \, ,$$

$$\int_{\Omega^\varepsilon} |\, v(x)\,|^2 \, dx \le C \int_{\Omega^\varepsilon} |\, e(v(x))\,|^2 \, dx \, ,$$

$$\int_{\Gamma_1} |\, v(x)\,|^2 \, dx \le C \int_{\Omega^\varepsilon} |\, e(v(x))\,|^2 \, dx \, .$$

Lemma 4.2. *There exist constants $C > 0$ such that*

$$\int_{\Omega} |\, u^\varepsilon(x)\,|^2 \, dx \le C \, , \tag{4.1}$$

$$\int_{\Omega} |\, e(u^\varepsilon(x))\,|^2 \, dx \le C \, , \tag{4.2}$$

i. e., there exists a subsequence, not relabeled, such that $u^\varepsilon \rightharpoonup u^0$ in V^ε, and $u^\varepsilon \to u^0$ in X.

Proof

- *step 1* $\exists u^\varepsilon \to u^0$ in X such that $\limsup_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) \leq F^0(u^0)$;
- *step 2* $\forall u^\varepsilon \to u^0$ in X $\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) \geq F^0(u^0)$.

$$F^{\varepsilon,k^m}(v) = \begin{cases} \int_{\Omega^\varepsilon_\pm} a^\pm e(v) : e(v)\, dx + k^m \varepsilon \int_{B^\varepsilon} e(v) : e(v)\, dx & \text{if} \quad v \in V_\varepsilon \ , \\ +\infty & \text{if} \quad v \in X \setminus V_\varepsilon \ . \end{cases}$$

$$R^\varepsilon u(\hat{x}, x_3) = \begin{cases} \frac{1}{2} S^\varepsilon u(\hat{x}) + \frac{x_3}{\varepsilon} [u]_\varepsilon(\hat{x}) & \text{if} \quad | x_3 | \leq \frac{\varepsilon}{2} \\ u^\pm(\hat{x}, x_3) & \text{if} \quad | x_3 | > \frac{\varepsilon}{2} \end{cases}$$

$$\liminf_{\varepsilon\to 0} F^{\varepsilon,k^m}(u^\varepsilon) \geq F^0(u^0) + k^m \int_S | [u^0] \otimes_s e_3 |^2 \, ds \ .$$

Remarks

Using standard arguments, we obtain

$$\begin{cases} \textit{Find } (u^0, \sigma^0) \textit{ such that :} & \\ \mathrm{div}\sigma^0 = 0 & \textit{in } \Omega_+ \cup \Omega_- \\ \sigma^0 = a^{\pm} e(u^0) & \textit{in } \Omega_\pm \\ u^0 = 0 & \textit{on } \Gamma_0 \\ \sigma^0 n = 0 & \textit{on } \Gamma_1 \\ [u^0] = 0 \ , \quad [\sigma^0 n] = 0 & \end{cases}$$

$\Rightarrow$ Model of perfect interface

Main results at order one

$$u^\varepsilon \to u^0 \quad \text{in } L^2(\Omega_0) \; .$$

$$\frac{u^\varepsilon - u^0}{\varepsilon} \rightharpoonup u^1 \quad \text{in } L^2(\Omega_0) \; .$$

Proposition

$$\begin{aligned} \sigma^0_{\alpha 3} &= \mu\left([u^1_\alpha] + u^0_{3,\alpha} \right) , \quad \alpha = 1, 2 \, , \\ \sigma^0_{33} &= \lambda(u^0_{1,1} + u^0_{2,2}) + (\lambda + 2\mu)[u^1_3] \end{aligned}$$

in a distributional sense.

Main results at order one

Proposition

$$[\sigma_{i3}^1] = -\tilde{\sigma}_{i1,1}^0 - \tilde{\sigma}_{i2,2}^0$$

in a distributional sense, and $\tilde{\sigma}_{3\alpha}^0 = \sigma_{3\alpha}^0$.

Proposition

$$\begin{aligned} \tilde{\sigma}_{\alpha\alpha}^0 &= \lambda\left(u_{1,1}^0 + u_{2,2}^0 + [u_3^1]\right) + 2\mu u_{\alpha,\alpha}^0 \,, \quad \alpha = 1,2 \,, \\ \tilde{\sigma}_{12}^0 &= \mu(u_{1,2}^0 + u_{2,1}^0) \,, \end{aligned}$$

in a distributional sense.

Main results at order one

Proposition

$$[\sigma^1_{13}] = -\left(\frac{4\mu(\lambda+\mu)}{\lambda+2\mu} u^0_{1,11} + \mu u^0_{1,22} + \frac{\mu(3\lambda+2\mu)}{\lambda+2\mu} u^0_{2,21} + \frac{\lambda}{\lambda+2\mu} \sigma^0_{33,1} \right) ,$$

$$[\sigma^1_{23}] = -\left(\frac{4\mu(\lambda+\mu)}{\lambda+2\mu} u^0_{2,22} + \mu u^0_{2,11} + \frac{\mu(3\lambda+2\mu)}{\lambda+2\mu} u^0_{1,12} + \frac{\lambda}{\lambda+2\mu} \sigma^0_{33,2} \right) ,$$

in a distributional sense.

Main result at order one

Theorem

$$[u_\alpha^1] = \frac{1}{\mu}\sigma_{\alpha 3}^0(\hat{x},0) - u_{3,\alpha}^0(\hat{x},0) - \frac{1}{2}(u_{\alpha,3}^0(\hat{x},0^+) + u_{\alpha,3}^0(\hat{x},0^-)), \quad \alpha = 1,2,$$

$$[u_3^1] = \frac{1}{\lambda+2\mu}\sigma_{33}^0(\hat{x},0) - \frac{\lambda}{\lambda+2\mu}(u_{1,1}^0(\hat{x},0) + u_{2,2}^0(\hat{x},0)) - \frac{1}{2}(u_{3,3}^0(\hat{x},0^+) + u_{3,3}^0(\hat{x},0^-)),$$

$$[\sigma_{13}^1] = -\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}u_{1,11}^0(\hat{x},0) - \mu u_{1,22}^0(\hat{x},0) - \frac{\mu(3\lambda+2\mu)}{\lambda+2\mu}u_{2,21}^0 - \frac{\lambda}{\lambda+2\mu}\sigma_{33,1}^0(\hat{x},0) - \frac{1}{2}(\sigma_{13,3}^0(\hat{x},0^+) + \sigma_{13,3}^0(\hat{x},0^-)),$$

$$[\sigma_{23}^1] = -\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}u_{2,22}^0(\hat{x},0) - \mu u_{2,11}^0(\hat{x},0) - \frac{\mu(3\lambda+2\mu)}{\lambda+2\mu}u_{1,12}^0(\hat{x},0) - \frac{\lambda}{\lambda+2\mu}\sigma_{33,2}^0(\hat{x},0) - \frac{1}{2}(\sigma_{23,3}^0(\hat{x},0^+) + \sigma_{23,3}^0(\hat{x},0^-))$$

$$[\sigma_{33}^1] = -\sigma_{13,1}^0(\hat{x},0) - \sigma_{23,2}^0(\hat{x},0) - \frac{1}{2}(\sigma_{33,3}^0(\hat{x},0^+) + \sigma_{33,3}^0(\hat{x},0^-)),$$

in $D'(S)$.

Other form

Imperfect interface

$$
\begin{aligned}
[\tilde{u}_1^1] &= \frac{\tilde{\sigma}_{13}^0}{G} - \tilde{u}_{3,1}^0 \\
[\tilde{u}_2^1] &= \frac{\tilde{\sigma}_{23}^0}{G} - \tilde{u}_{3,2}^0 \\
[\tilde{u}_3^1] &= \frac{(-1+\nu+2\nu^2)\tilde{\sigma}_{33}^0 + E\nu(\tilde{u}_{1,1}^0 + \tilde{u}_{2,2}^0)}{E(-1+\nu)}
\end{aligned}
$$

$$
\begin{aligned}
[\tilde{\sigma}_{13}^1] &= \frac{-2E\tilde{u}_{1,11}^0 + E(-1+\nu)\tilde{u}_{1,22}^0 - (1+\nu)(E\tilde{u}_{2,12}^0 + 2\nu\tilde{\sigma}_{33,1}^0)}{2(-1+\nu^2)} \\
[\tilde{\sigma}_{23}^1] &= \frac{-E(1+\nu)\tilde{u}_{1,12}^0 + E(-1+\nu)\tilde{u}_{2,11}^0 - 2(E\tilde{u}_{2,22}^0 + \nu(1+\nu)\tilde{\sigma}_{33,2}^0)}{2(-1+\nu^2)} \\
[\tilde{\sigma}_{33}^1] &= -\tilde{\sigma}_{13,1}^0 - \tilde{\sigma}_{23,2}^0
\end{aligned}
$$

Extension (3D « Orthotropic »)

Imperfect interface

$$
\begin{aligned}
[\tilde{u}_1^1] &= \frac{\tilde{\sigma}_{13}^0}{G_{13}} - \tilde{u}_{3,1}^0 \\
[\tilde{u}_2^1] &= \frac{\tilde{\sigma}_{23}^0}{G_{23}} - \tilde{u}_{3,2}^0 \\
[\tilde{u}_3^1] &= \alpha_o \tilde{\sigma}_{33}^0 - \beta_o \tilde{u}_{1,1}^0 - \gamma_o \tilde{u}_{2,2}^0
\end{aligned}
$$

$$
\begin{aligned}
\alpha_o &= \frac{E_1(E_2 - E_3\nu_{23}^2) - E_2(E_2\nu_{12}^2 + E_3\nu_{13}(\nu_{13} + 2\nu_{12}\nu_{23}))}{E_2E_3(E_1 - E_2\nu_{12}^2)}, \\
\beta_o &= \frac{E_1(\nu_{13} + \nu_{12}\nu_{23})}{E_1 - E_2\nu_{12}^2}, \\
\gamma_o &= \frac{E_2\nu_{12}\nu_{13} + E_1\nu_{23}}{E_1 - E_2\nu_{12}^2}.
\end{aligned}
$$

Extension (3D « Orthotropic »)

Imperfect interface

$$[\tilde{\sigma}^1_{13}] = \bar{\alpha}_o \tilde{u}^0_{1,11} + G_{12}\bar{u}^0_{1,22} + \tilde{\beta}_o \tilde{u}^0_{2,12} - \frac{\nu_{13}+\nu_{12}\nu_{23}}{-1+\nu_{12}\nu_{21}}\sigma^0_{33,1}$$

$$[\tilde{\sigma}^1_{23}] = \bar{\beta}_o \tilde{u}^0_{1,12} + G_{12}\bar{u}^0_{2,11} + \tilde{\gamma}_o \bar{u}^0_{2,22} - \frac{\nu_{23}+\nu_{13}\nu_{21}}{-1+\nu_{12}\nu_{21}}\sigma^0_{33,2}$$

$$[\tilde{\sigma}^1_{33}] = -\tilde{\sigma}^0_{13,1} - \tilde{\sigma}^0_{23,2}$$

$$\bar{\alpha}_o = \frac{E_3(\nu_{13}+\nu_{12}\nu_{23})^2 - E_1(-1+\nu_{12}\nu_{21})(-1+\nu_{23}\nu_{32})}{(-1+\nu_{12}\nu_{21})(-1+\nu_{12}(\nu_{21}+\nu_{23}\nu_{31})+\nu_{23}\nu_{32}+\nu_{13}(\nu_{31}+\nu_{21}\nu_{32}))}$$

$$\bar{\beta}_o = G_{12} - \frac{E_3(\nu_{13}\nu_{21}+\nu_{23})(\nu_{13}+\nu_{12}\nu_{23})}{(-1+\nu_{12}\nu_{21})(-1+\nu_{12}(\nu_{21}+\nu_{23}\nu_{31})+\nu_{23}\nu_{32}+\nu_{13}(\nu_{31}+\nu_{21}\nu_{32}))} - \frac{E_2(\nu_{12}+\nu_{13}\nu_{32})}{-1+\nu_{12}(\nu_{21}+\nu_{23}\nu_{31})+\nu_{23}\nu_{32}+\nu_{13}(\nu_{31}+\nu_{21}\nu_{32}))}$$

$$\bar{\gamma}_o = \frac{-E_3(\nu_{13}\nu_{21}+\nu_{23})^2 + E_2(-1+\nu_{12}\nu_{21})(-1+\nu_{13}\nu_{31})}{(-1+\nu_{12}\nu_{21})(-1+\nu_{12}(\nu_{21}+\nu_{23}\nu_{31})+\nu_{23}\nu_{32}+\nu_{13}(\nu_{31}+\nu_{21}\nu_{32}))}$$

Extension (3D)

Imperfect interface

$$[\tilde{u}^1] \;=\; A^{an}\tilde{\sigma}_3^0 + B^{an} D\tilde{u}^0$$

$$[\tilde{\sigma}^1]\, e_3 \;=\; C^{an} D^2\tilde{u}^0 + D^{an} D\tilde{\sigma}_3^0$$

Summary

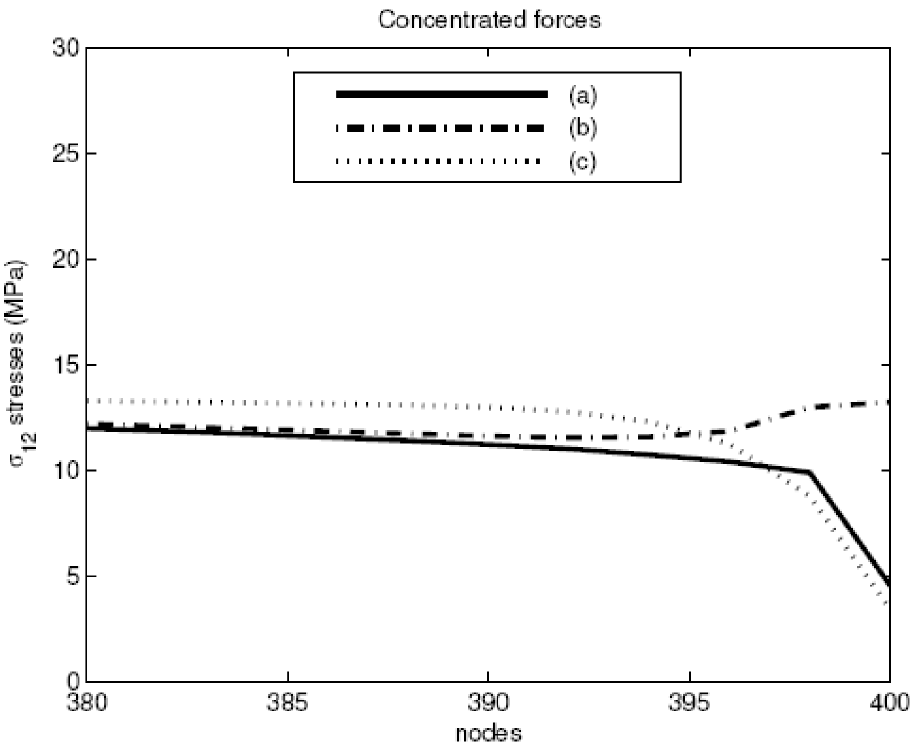
•We have obtained the model of perfect interface via an energy argument (Γ convergence)

•We have obtained the model of imperfect interface via analysis argument

•This derivation extends the existing results (Abdelmoula et al. 1998) to the three-dimensional case

Outlook : We have formally derived a minimum principle for the model of imperfect interface

Concentrated forces



Merci pour votre attention