Shape optimization in nonlinear elasticity by the level method

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Level set method

• Combine some advantages of the shape sensitivity method and the topological method (homogenization, topological gradient, SIMP)

• Based on the shape derivative (Hadamard, Murat-Simon).
* Easy handling of various objective functions
* Can be adapted to any direct problem (e.g. nonlinear)

• Shape representation by the 0 level set of a scalar field on a fixed mesh (Osher-Sethian).
* Moderate cost
* No numerical instabilities due to remeshing
* Easy topology changes

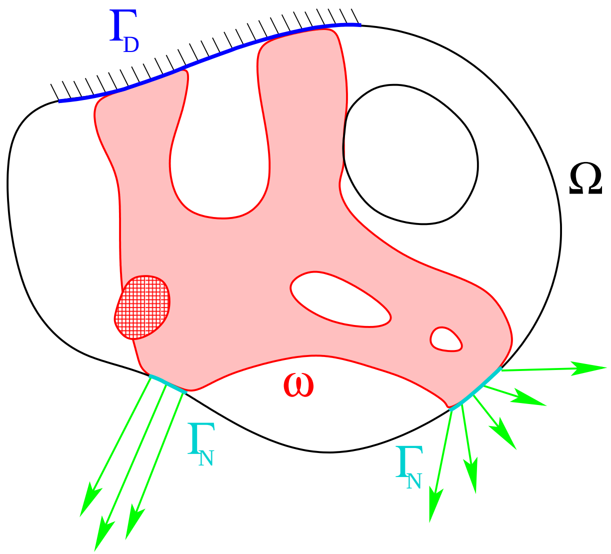
• Remaining drawbacks:
1. reduction of topology (rather than variation) in 2d,
2. local minima.

Hint: coupling with the topological gradient

Setting of the problem

Linearly elastic material. Isotropic Hooke's law A.
u displacement field, $e(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$ deformation tensor.

Linearized elasticity system posed on $\omega \subset \Omega$ (a given open bounded domain):



$$\left\{ \begin{array}{rcl} -\mathrm{div}\big(Ae(u)\big) & = & 0 \quad \text{in } \omega \\ u & = & 0 \quad \text{on } \partial\omega \cap \Gamma_D \\ \big(Ae(u)\big)\cdot n & = & g \quad \text{on } \partial\omega \cup \Gamma_N \end{array} \right.$$

It admits a unique solution.

Objective functions

Most widely used in structural topology optimization : compliance

$$J(\omega) = \int_\omega Ae(u) : e(u)dx = \int_\omega A^{-1}\sigma : \sigma dx = \int_{\partial\omega} g \cdot u\, ds = c(\omega)$$

Global measurement of rigidity.

Many other objective functions are possible and useful (depending on the stress tensor, the displacement field, the eigenvalues etc...)

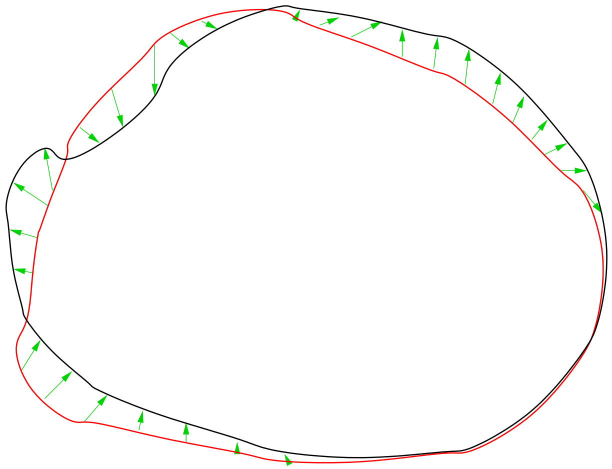
Shape derivative (Murat-Simon, Céa)

ω_0 reference domain. We are interested in variations of the form

$$\omega = \{x + \theta(x) \mid x \in \omega_0\} = (\mathrm{Id} + \theta)\omega_0$$

with $\theta \in W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$.

$\theta(x)$ is a vector field that *deforms* the reference domain ω_0.

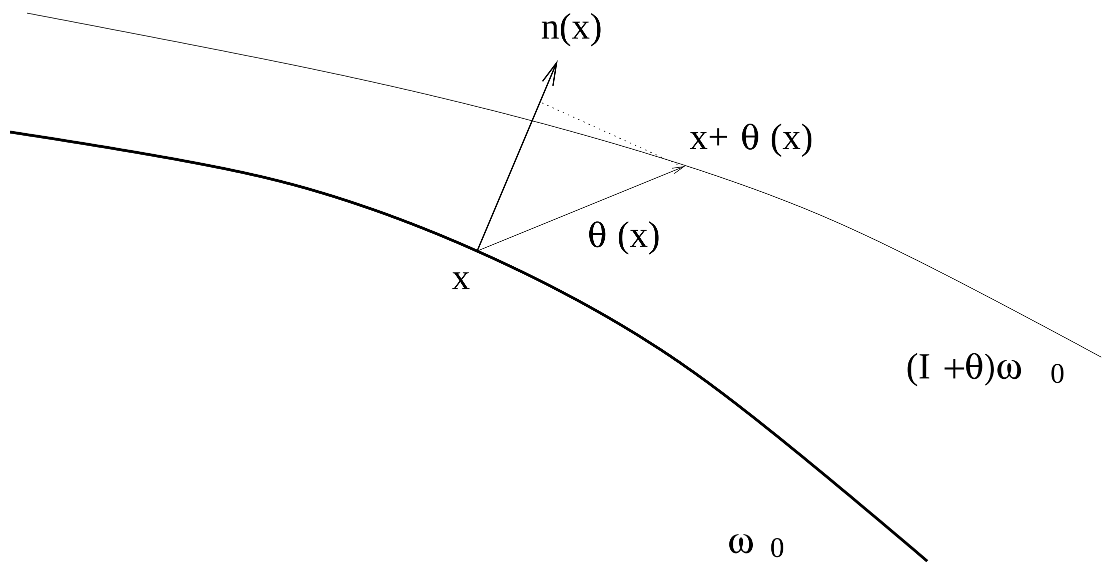


Lemma: For any $\theta \in W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ such that $\|\theta\|_{W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)} < 1$, $(\mathrm{Id} + \theta)$ is a diffeomorphism in $\mathbb{R}^d$.

Definition:
The shape derivative of $\omega \mapsto J(\omega)$ at ω_0 is the Fréchet derivative of $\theta \mapsto J\big((\mathrm{Id} + \theta)\omega_0\big)$ at 0.

Shape derivative (Murat-Simon)

The shape derivative $J'(\omega_0)(\theta)$ depends only of $\theta \cdot n$ on the boundary $\partial\omega_0$.



Lemma: Let ω_0 be a smooth bounded open set and $J(\omega)$ at differentiable function at ω_0. Its derivative satisfies

$$J'(\omega_0)(\theta_1) = J'(\omega_0)(\theta_2)$$

if $\theta_1, \theta_2 \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$ are such that

$$\begin{cases} \theta_2 - \theta_1 \in \mathcal{C}^1(\mathbb{R}^N; \mathbb{R}^N) \\ \theta_1 \cdot n = \theta_2 \cdot n \quad \text{on } \partial\omega_0. \end{cases}$$

Examples of shape derivatives (I)

Objective-function defined in the domain: Let ω_0 be a smooth bounded open set of class $\mathcal{C}^1$ of $\mathbb{R}^N$. Let $f(x) \in W^{1,1}(\mathbb{R}^N)$ and J defined by

$$J(\omega) = \int_\omega f(x)\, dx.$$

Then J is differentiable at ω_0 and

$$J'(\omega_0)(\theta) = \int_{\omega_0} \operatorname{div} \big(\theta(x)\, f(x)\big)\, dx = \int_{\partial\omega_0} \theta(x) \cdot n(x)\, f(x)\, ds$$

for all $\theta \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$.

Examples of shape derivatives (II)

Objective-function defined on the boundary: Let ω_0 be a smooth bounded open set of class $\mathcal{C}^1$ of $\mathbb{R}^N$. Let $f(x) \in W^{2,1}(\mathbb{R}^N)$ and J defined by

$$J(\omega) = \int_{\partial \omega} f(x)\, ds.$$

Then J is differentiable at ω_0 and

$$J'(\omega_0)(\theta) = \int_{\partial \omega_0} \big(\nabla f \cdot \theta + f \big(\mathrm{div}\, \theta - \nabla \theta n \cdot n \big) \big)\, ds$$

for all $\theta \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$. Moreover if ω_0 is smooth of class $\mathcal{C}^2$, then

$$J'(\omega_0)(\theta) = \int_{\partial \omega_0} \theta \cdot n \left(\frac{\partial f}{\partial n} + H f \right) ds,$$

where H is the mean curvature of $\partial \omega_0$ defined by $H = \mathrm{div}\, n$.

Shape derivative of the compliance

$$J(\omega) = \int_{\partial\omega \cup \Gamma_N} g \cdot u \, ds = \int_\omega A\, e(u) : e(u) \, dx,$$

$$J'(\omega_0)(\theta) = \int_{\Gamma_0} \left(\left[\frac{\partial (f \cdot u)}{\partial n} + H f \cdot u \right] - Ae(u) : e(u) \right) \theta \cdot n \, ds,$$

where u is the state (displacement field) in ω_0, and H the mean curvature of $\partial\omega_0$.

No adjoint state involved. The compliance problem is self-adjoint.

Front propagation by level set

Shapes are not meshed, but captured on a fixed mesh of a large box Ω.

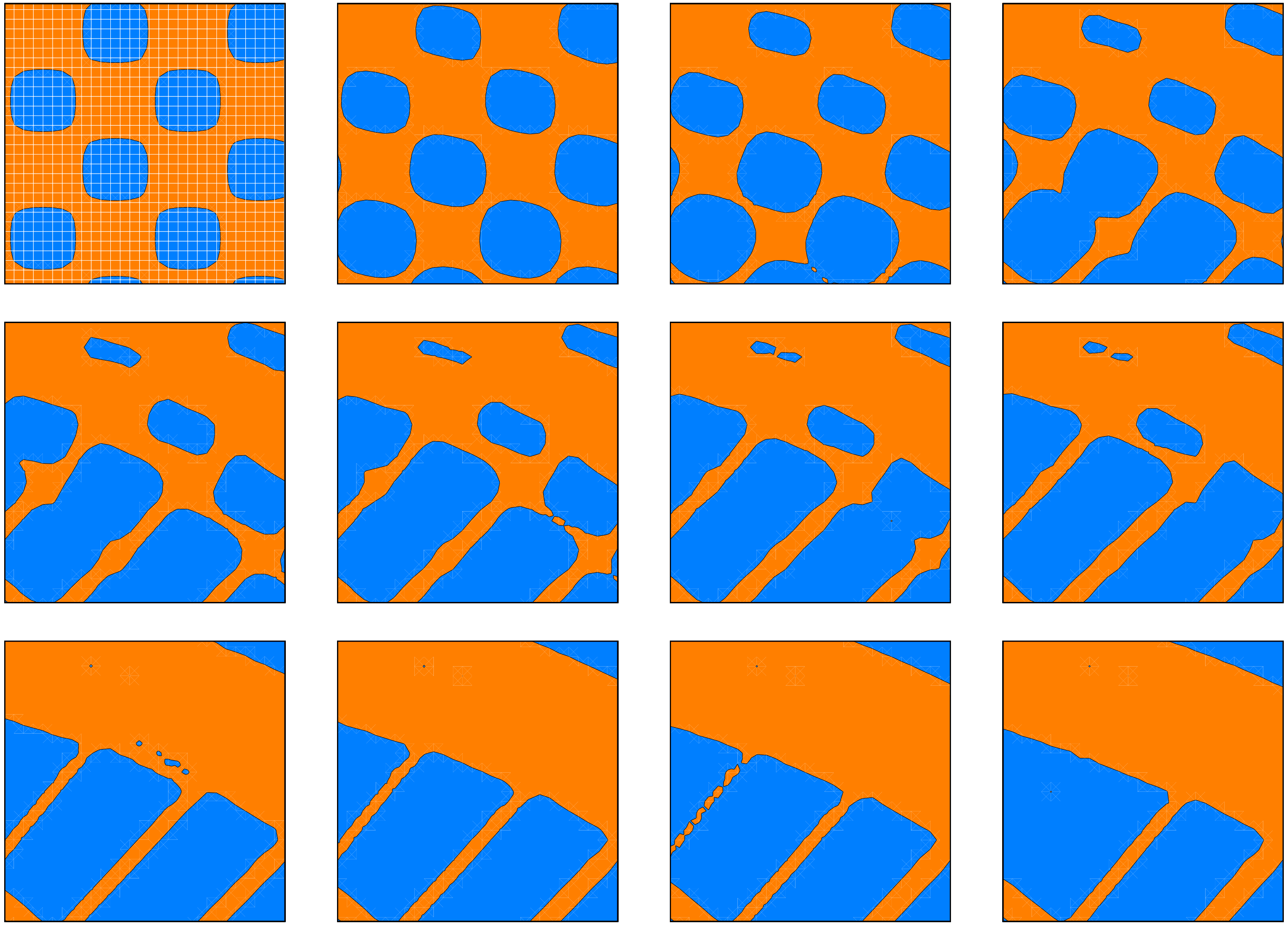
Parameterization of the shape ω by a level set function:

$$\begin{cases} \psi(x) = 0 & \Leftrightarrow x \in \partial\omega \cap \Omega \\ \psi(x) < 0 & \Leftrightarrow x \in \omega \\ \psi(x) > 0 & \Leftrightarrow x \in (\Omega \setminus \omega) \end{cases}$$

- Exterior normal to ω : $n = \nabla\psi / |\nabla\psi|$.
- Mean curvature: $H = \mathrm{div}\ n$.
- These formula make sense everywhere in Ω, not only on the boundary $\partial\omega$.

$\longrightarrow$ *natural extension*

Level set



Hamilton-Jacobi equation

If the shape $\omega(t)$ evolves in pseudo-time t with a normal speed $V(t,x)$, then ψ satisfies a Hamilton-Jacobi equation:

$$\psi\Big(t, x(t)\Big) = 0 \quad \text{for all } x(t) \in \partial\omega(t).$$

deriving in t yields

$$\frac{\partial \psi}{\partial t} + \dot{x}(t) \cdot \nabla\psi = \frac{\partial \psi}{\partial t} + V n \cdot \nabla\psi = 0.$$

As $n = \nabla\psi / |\nabla\psi|$ we obtain

$$\frac{\partial \psi}{\partial t} + V|\nabla\psi| = 0.$$

This equation is valid on the whole domain Ω, not only on the boundary $\partial\omega$, assuming that the velocity is known everywhere.

$\longrightarrow$ the description of the boundary of ω can remain implicit during the algorithm.

Application to shape optimization

If the shape derivative can be expressed as an integral of the form:

$$J'(\omega_0)(\theta) = \int_{\partial\omega_0} v\,\theta \cdot n\,ds,$$

then a valid descent direction, allowing J to decrease at the first order, is

$$\theta = -v\,n$$

Thus, solving

$$\frac{\partial\psi}{\partial t} - v|\nabla\psi| = 0 \quad \text{in } \Omega$$

is equivalent to perform a descent algorithm where t is a descent parameter

Numerical algorithm

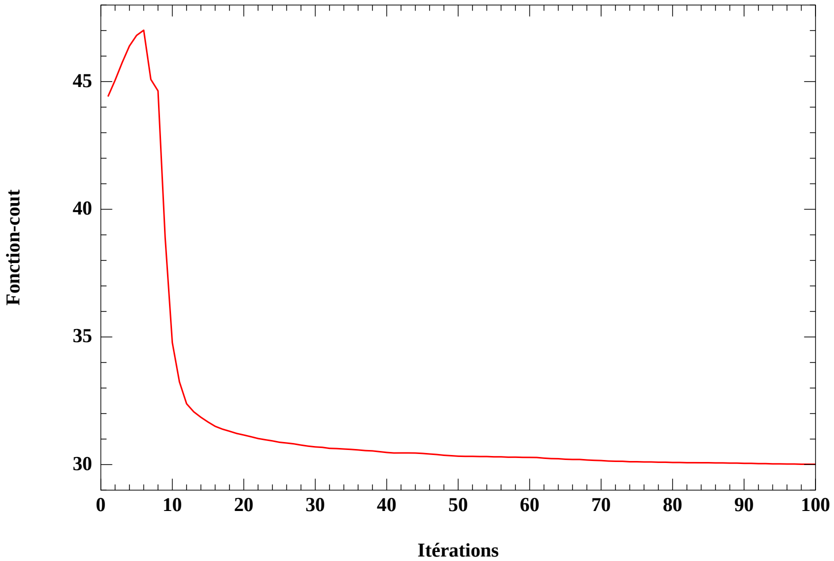
1. Initialization of the level set function ψ_0 (e.g. a product of sinus).

2. Iterations until convergence for $k \geq 1$:

   (a) Computation of u_k and eventually p_k by solving a linearized elasticity problem on the shape ψ_k. Computation of the shape gradient $\longrightarrow$ normal velocity V_k

   (b) Transport of the shape by the speed V_k (Hamilton-Jacobi equation) to obtain a new shape ψ_{k+1}. (Several successive time steps can be applied for a same velocity field). The descent step is controlled by the CFL condition on the transport equation and by the decreasing of the objective function.

   (c) Possible reinitialization of the level set function such that ψ_{k+1} is the signed distance to the interface.


Fonction-cout
45
40
35
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0
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80
90
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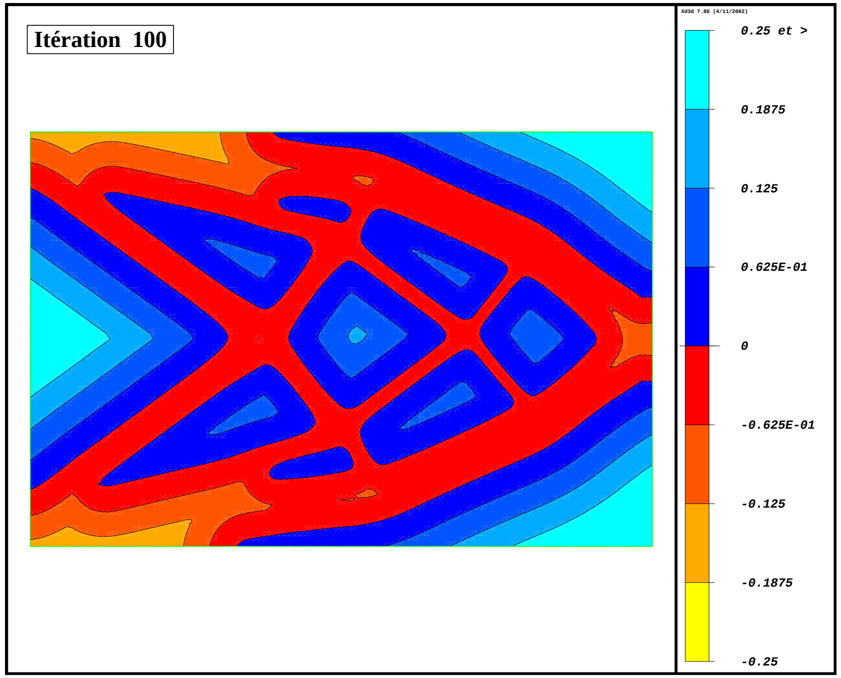
Itérations


Itération 100
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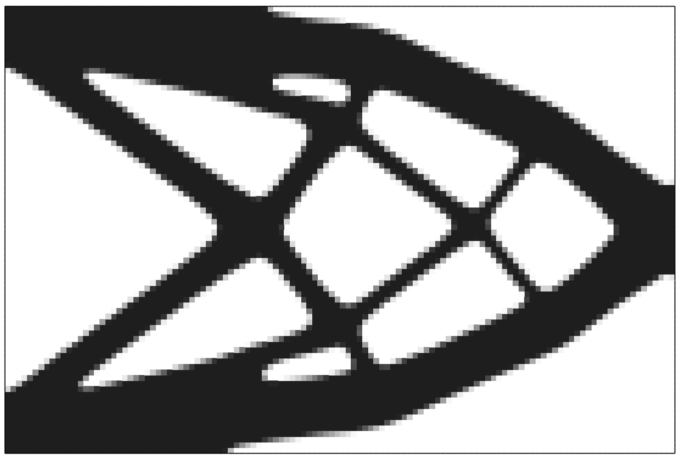

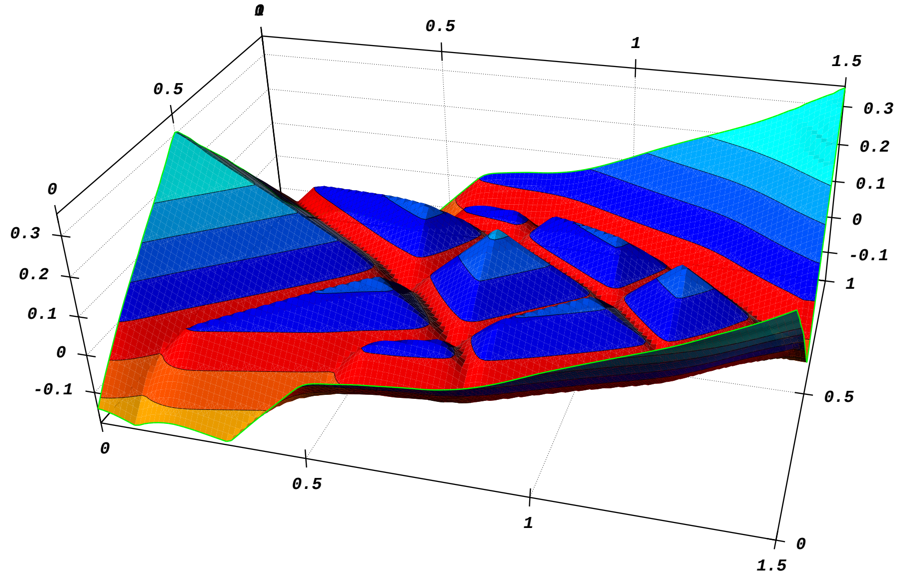
0.5
1
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0
-0.1
0
0.5
1
1.5

Why nonlinear elasticity ?

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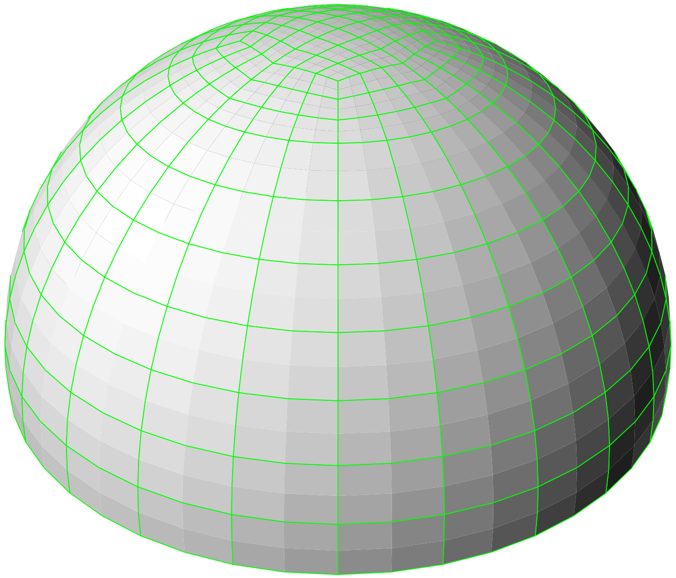
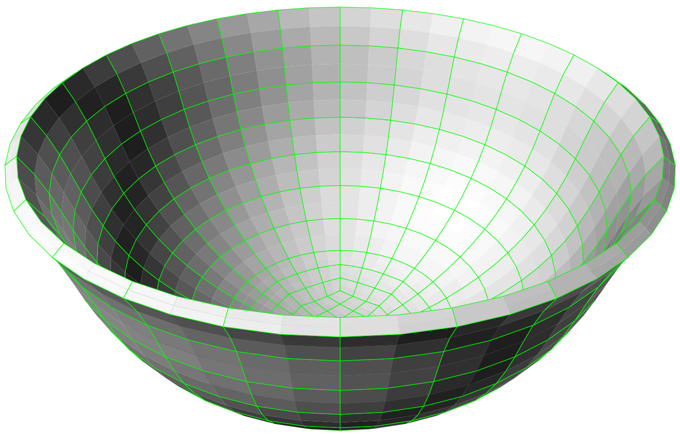
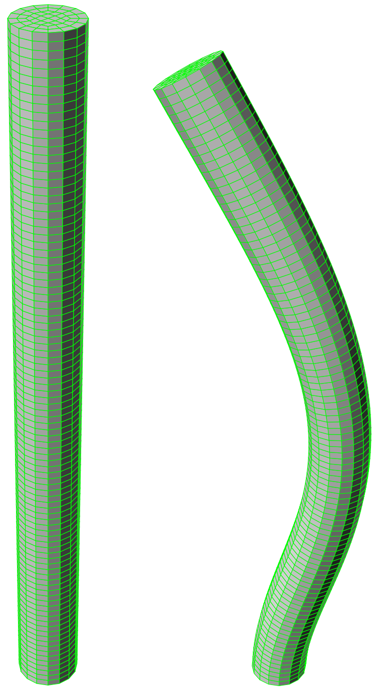
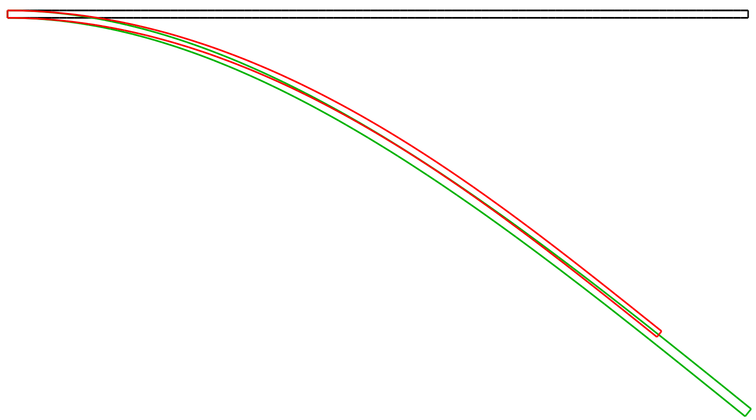








Hyperelasticity

$$\begin{cases} -\text{div }(T(F)) &= f & \text{in } \Omega \\ u &= 0 & \text{on } \Gamma_D \\ T(F)n &= g & \text{on } \Gamma_N. \end{cases}$$

where $F = (I + \nabla u)$ and the first Piola-Kirchhoff stress tensor T derives from a potential $W(F)$ (stored energy function)

$$T_{ij} = \frac{\partial W(F)}{\partial F_{ij}}, \quad i, j \in \{1, \dots, d\},$$

elasticity tensor:

$$\mathcal{A}_{ijkl}(u) = \frac{\partial T_{ij}(F)}{\partial F_{kl}} = \frac{\partial^2 W(F)}{\partial F_{ij} \partial F_{kl}}.$$

Shape derivative

$$\text{If} \qquad J(\Omega) = \int_{\Omega} j\big(x, u(x)\big)\, dx + \int_{\partial\Omega} l\big(x, u(x)\big)\, ds,$$

then

$$\begin{aligned} J'(\omega_0)(\theta) = & \int_{\partial\omega_0} \theta \cdot n \Big(j(u) + T(I + \nabla u) : \nabla p - p \cdot f \Big)\, ds \\ & + \int_{\partial\omega_0} \theta \cdot n \left(\frac{\partial l(u)}{\partial n} + H\, l(u) \right) ds \\ & - \int_{\Gamma_N} \theta \cdot n \left(\frac{\partial (g \cdot p)}{\partial n} + H\, g \cdot p \right) ds \\ & - \int_{\Gamma_D} \theta \cdot n \left(\frac{\partial h}{\partial n} \right) ds, \end{aligned}$$

where $h = u \cdot T(I + \nabla p)n + p \cdot T(I + \nabla u)n$
and p is the adjoint state solution of

$$\left\{ \begin{array}{rcll} -\mathrm{div}\ \left(\mathcal{A}(u)\, \nabla p\right) & = & -j'(u) & \text{in } \omega_0 \\ p & = & 0 & \text{on } \Gamma_D \\ \big(\mathcal{A}(u)\, \nabla p\big) n & = & -l'(u) & \text{on } \Gamma_N. \end{array} \right.$$

Compliance case

$$J(\omega) = \int_{\partial\omega} g \cdot u \, ds \quad \Rightarrow \quad j(u) = 0 \quad \text{and} \quad l(u) = g \cdot u$$

If the load g is constant and applied on a fixed (non optimizable) part of the boundary the shape derivative is

$$J'(\omega_0)(\theta) = \int_{\partial\omega_0} \theta \cdot n \Big(T(I + \nabla u) : \nabla p \Big) \, ds$$

and the adjoint p is solution of

$$\left\{ \begin{array}{rcll} -\mathrm{div} \; (\mathcal{A}(u) \, \nabla p) & = & 0 & \text{in } \omega_0 \\ p & = & 0 & \text{on } \Gamma_D \\ (\mathcal{A}(u) \, \nabla p) n & = & -g & \text{on } \Gamma_N . \end{array} \right.$$

Remarks

- The problem is not self adjoint ($p = -u$) as in the linear case.

- The adjoint problem is linear. It admits a unique solution as soon as $\mathcal{A}(u)$, the linearized operator around the solution u, is coercive.

- The direct problem does not always admits a unique solution...

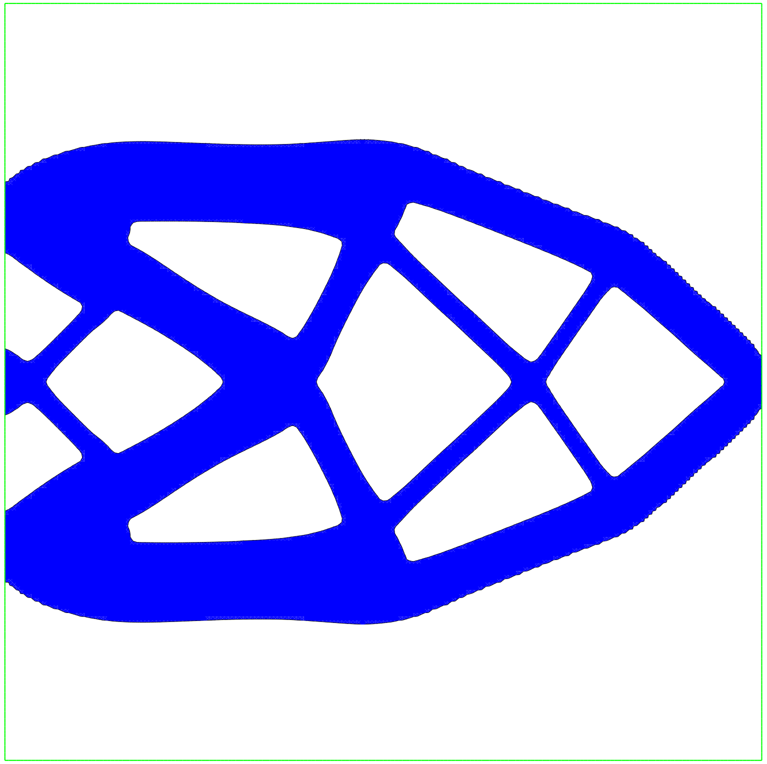
- ... and when a solution exists, it is not always easy to compute it numerically $\rightarrow$ additional restart points in the algorithm.

- Other expressions for the compliance are not equivalent:

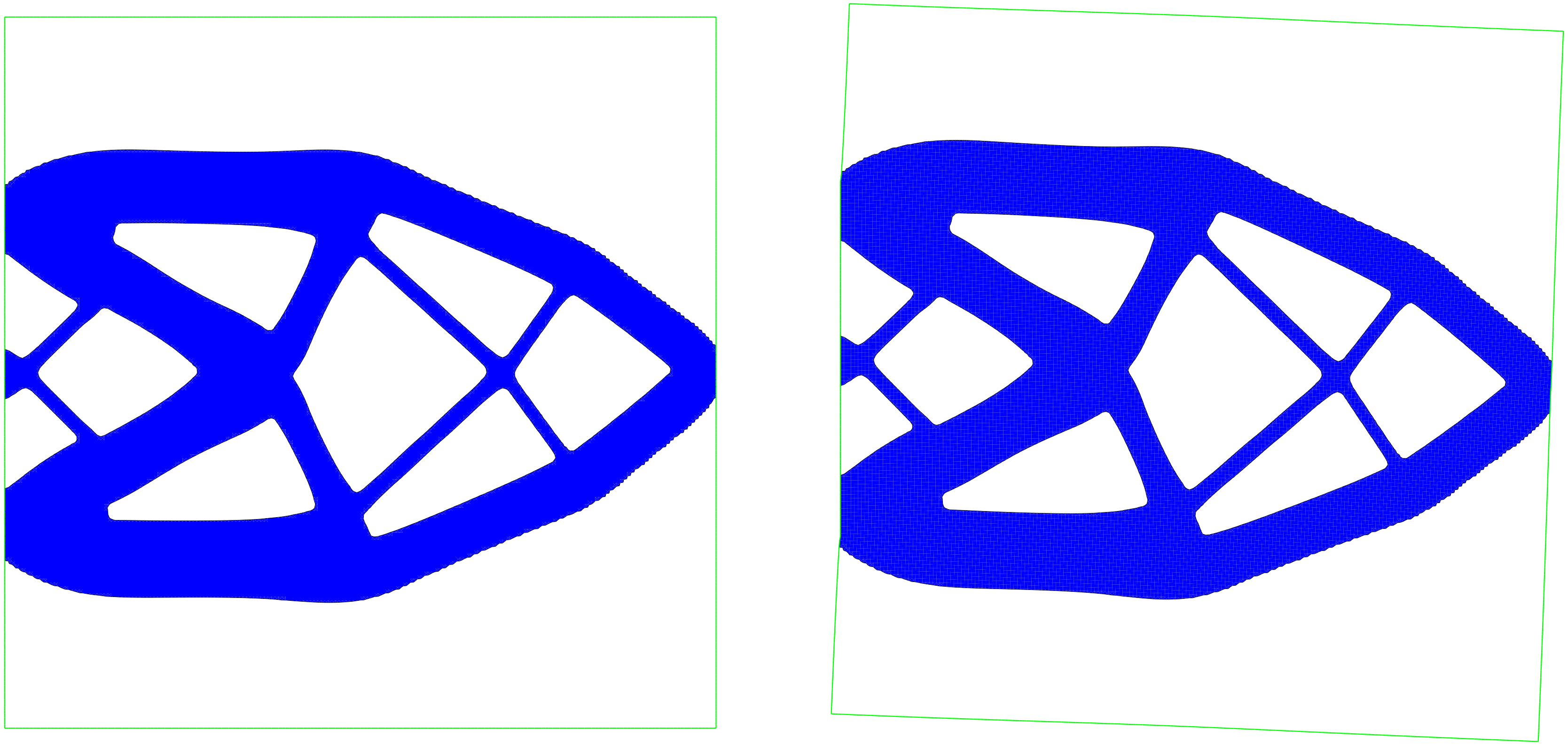
$$\frac{1}{2}\int_\omega T(F) : (FF^T - I)\, dx = \int_\omega W(F)\, dx \neq \int_{\partial\omega} g \cdot u\, ds, \qquad (F = I + \nabla u)$$

and may lead to very different optimal solutions.

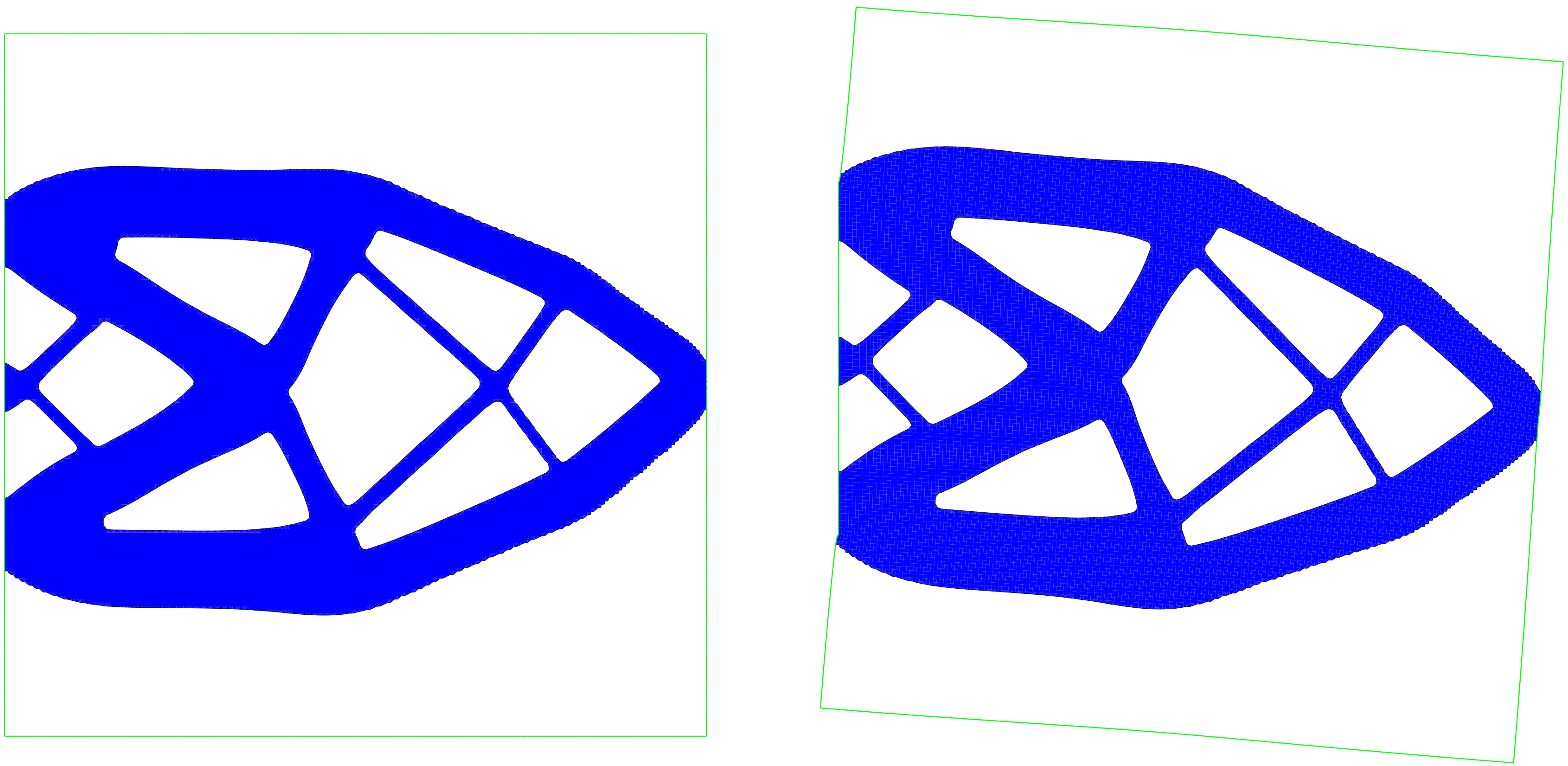
Linear



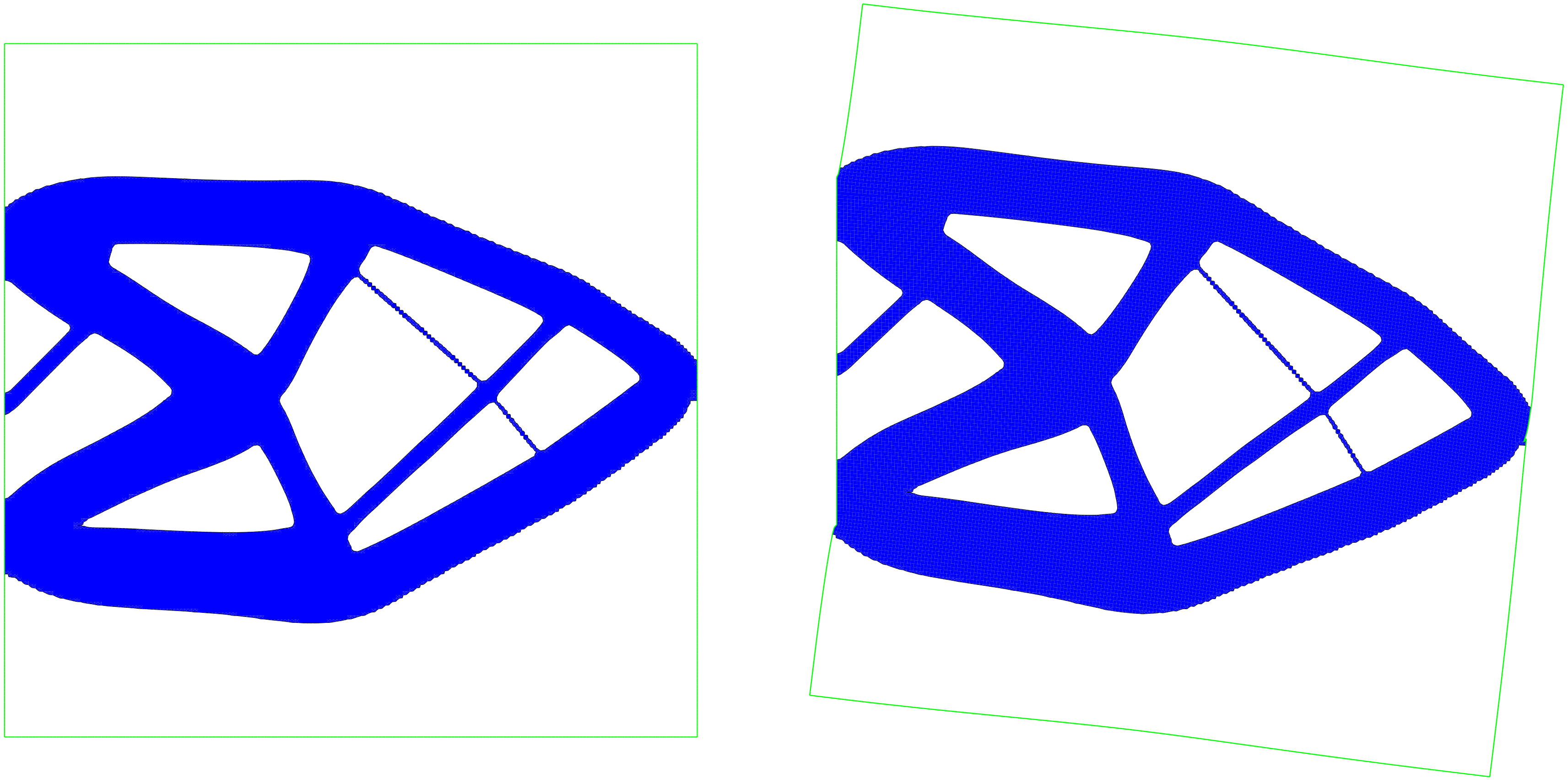
$f = 0.5$



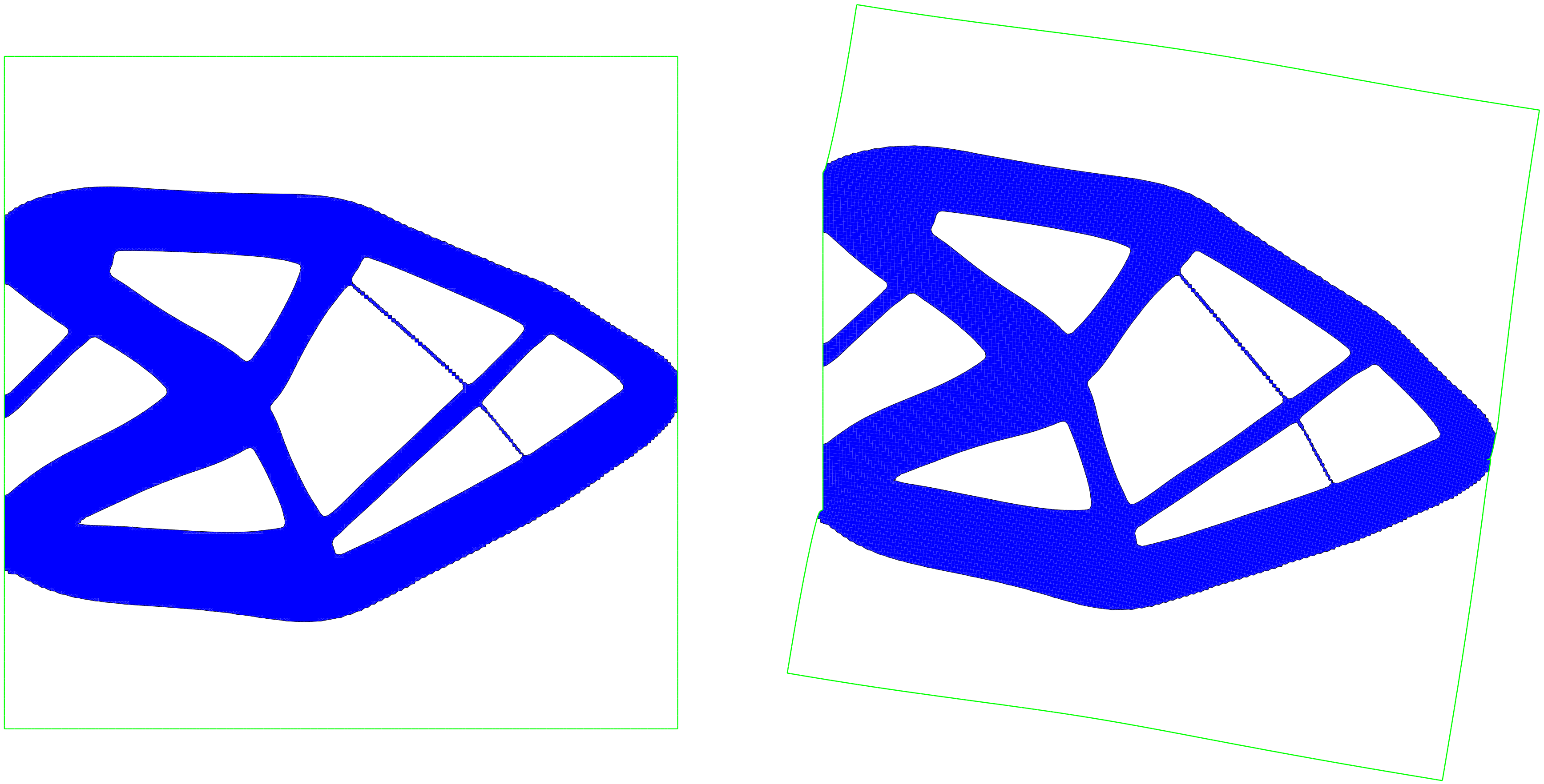
$f = 1$



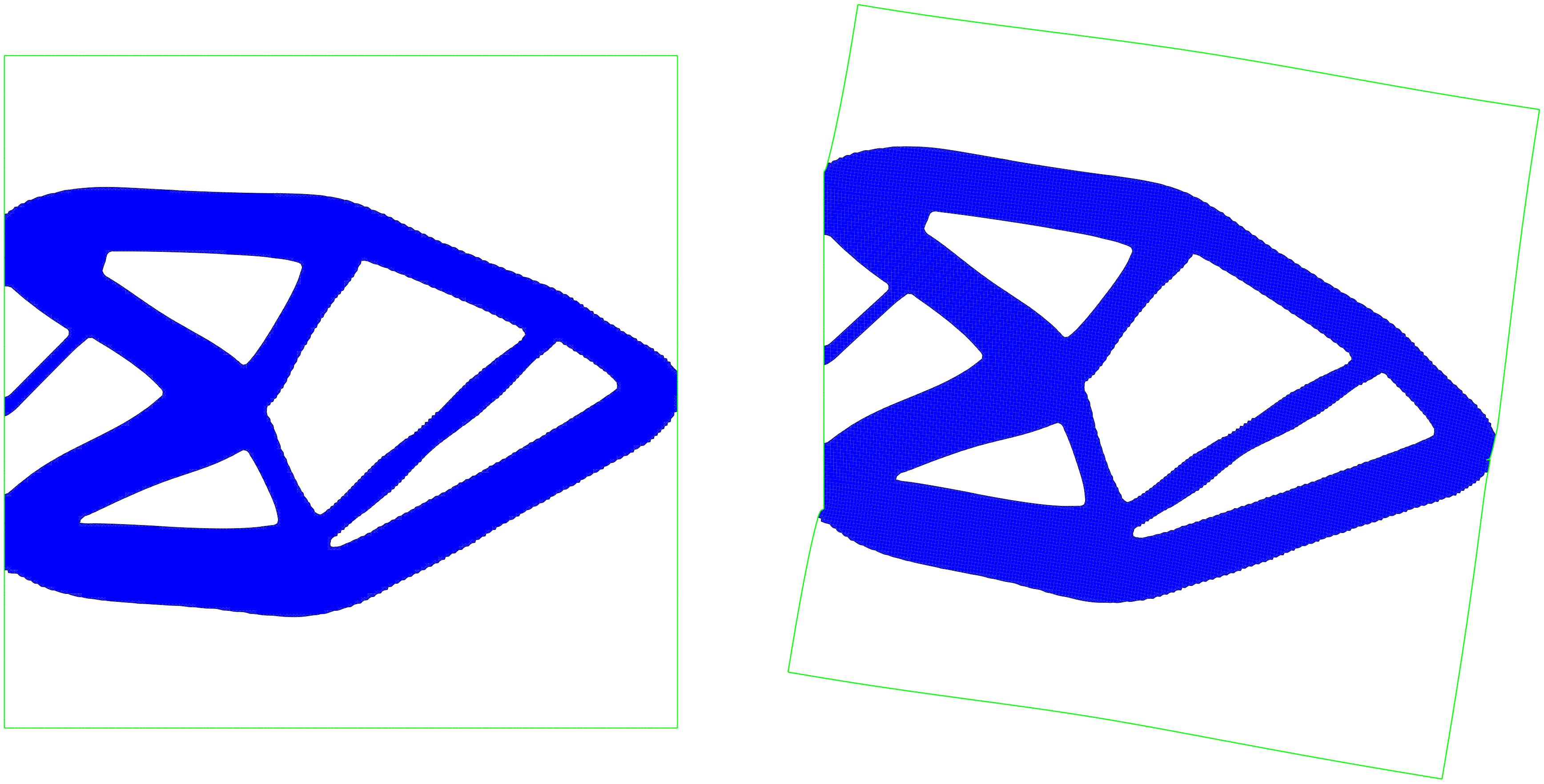
$$f = 1.5$$



$$f = 2$$



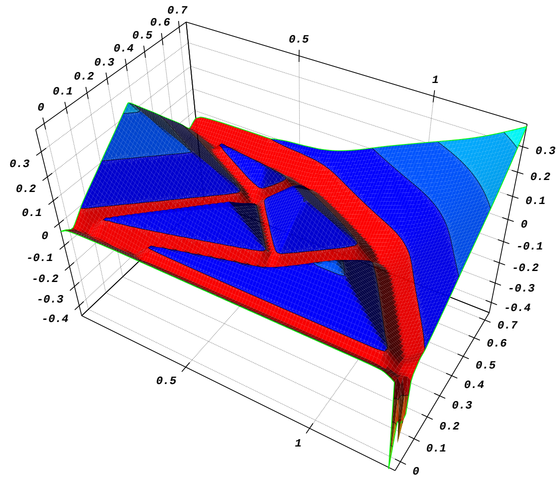
$f = 2.1$



Reinitialization of the level set

• The level set function is periodically reinitialized to avoid it to be *too flat* ($\longrightarrow$ poor precision on ψ) or *too steep* ($\longrightarrow$ poor precision on $\nabla\psi$ i.e. the normal) after some transport steps. It is done by solving

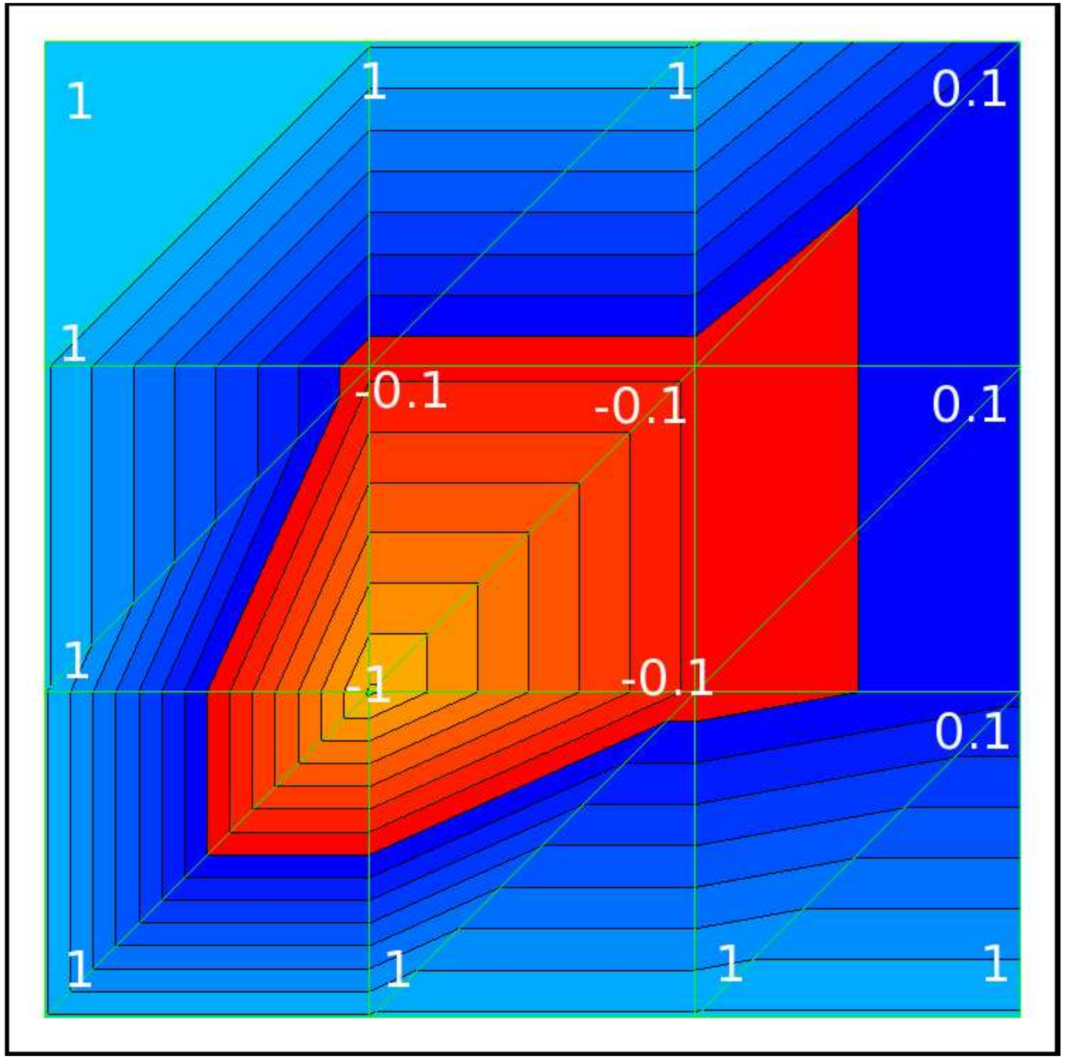
$$\frac{\partial \psi}{\partial t} + \text{sign}(\psi)\Big(|\nabla\psi| - 1\Big) = 0 \quad \text{in } \Omega,$$



whose stationary solution is the signed distance to the interface $\Big\{\psi(t=0,x)=0\Big\}$.

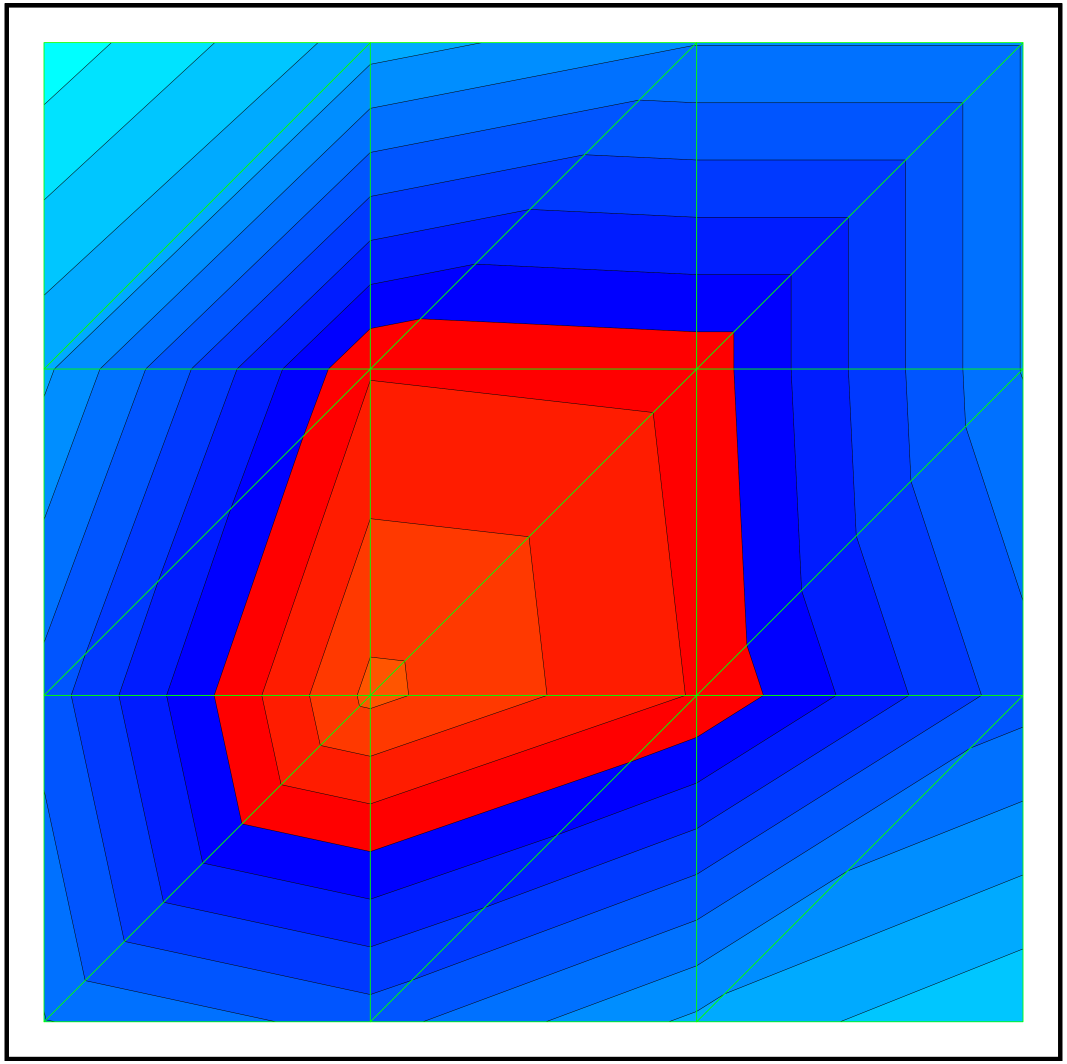
Well well well...
That's what you read in all the papers dealing with level sets
That is certainly true at the continuous level
Let's see what happens for the discrete problem

Reinitialization of the level set



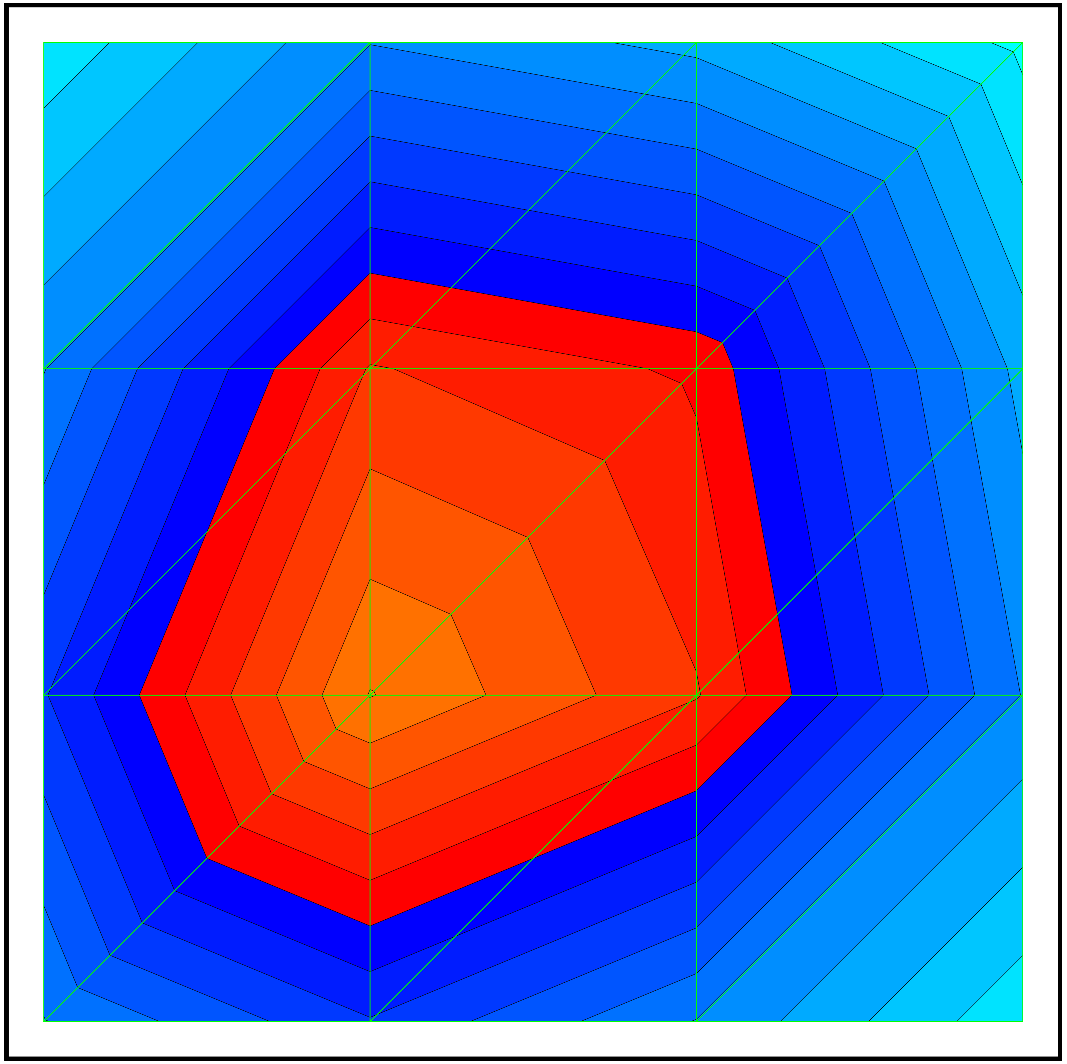
Example of function ψ on a rough mesh

Reinitialization of the level set



Manual and exact computation of the signed distance and plot of the new level sets !

Reinitialization of the level set



Level set after 100 iterations of exact computation of the signed distance !!

No reinitialization !