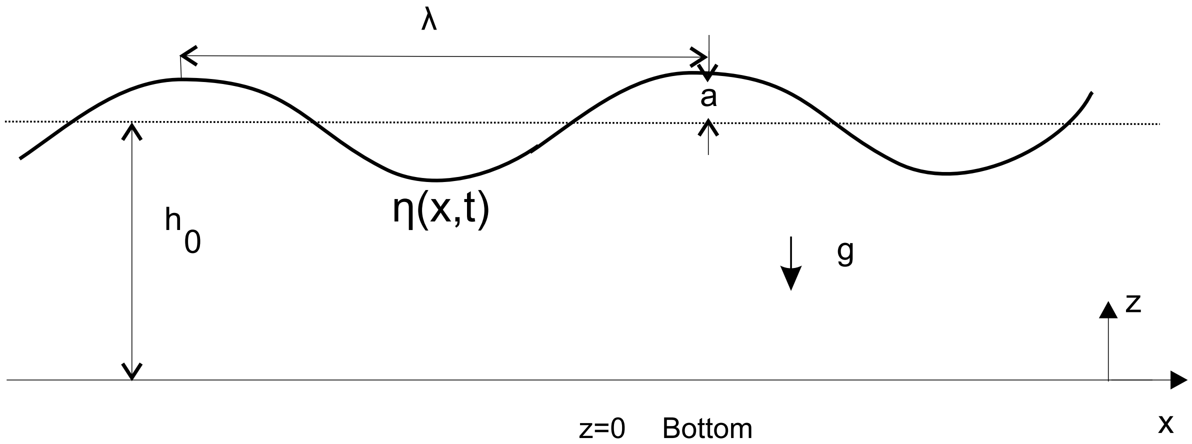
Small-amplitude capillary-gravity water waves: Exact solutions and particle motion beneath such waves

Delia Ionescu-Kruse

Delia.Ionescu@imar.ro

Institute of Mathematics of the Romanian Academy

P.O. Box 1-764, RO-014700, Bucharest, Romania



$$
\left.
\begin{array}{ll}
u_t + uu_x + vu_z = -\frac{1}{\rho}p_x & \text{(EEs)} \\
v_t + uv_x + vv_z = -\frac{1}{\rho}p_z - g & \\
u_x + v_z = 0 & \text{(MC)} \\
v = \eta_t + u\eta_x & \text{on } z = h_0 + \eta(x,t) \text{ (KBCs)} \\
v = 0 & \text{on } z = 0 \\
p = p_0 - \Gamma\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} & \text{on } z = h_0 + \eta(x,t) \text{ (DBC)}
\end{array}
\right\} \quad (1)
$$

There are very few explicit solutions known for the water-wave problems.

- for *pure gravity water waves*: Gerstner's solution([1]; [2]; [3],[4]) and the edge wave solution related to it [5].

Beneath Gerstner's waves it is possible to have a motion of the fluid where all particles describe circles with a depth-dependent radius ([3], [4]).

[1] (1809) GERSTNER F., *Ann. Phys.*

[2] (1863) RANKINE W. J. M., *Phil. Trans. R. Soc. A*.

[3] (2001) CONSTANTIN A., *J. Phys. A*.

[4] (2008) HENRY D., *J. Nonlinear Math. Phys.*

[5] (2001) CONSTANTIN A., *J. Phys. A*.

- for *pure capillary water waves*: Crapper's solution [1] and its generalization in the case of finite depth [2].

In [4], by the use of Longuet-Higgins method [3], the particle trajectories in Crapper's waves are derived. It is found that the orbits of the steeper waves are neither circular nor closed.

[1] (1957) CRAPPER G. D., *J. Fluid Mech.*

[2] (1976) KINNERSLEY W., *J. Fluid Mech.*

[3] (1979) LONGUET-HIGGINS M. S., *J. Fluid Mech.*

[4] (1984) HOGAN S. J., *J. Fluid Mech.*

- for *capillary-gravity water waves* no exact analytic solution has yet been found.

Making use of numerical studies, in [1] the particle trajectories in irrotational nonlinear capillary-gravity waves on ideal fluids of infinite depth are investigated.

[1] (1985) HOGAN S. J., *J. Fluid Mech.*

In what follows, we investigate the capillary-gravity waves and the internal motion of the fluid under the passage of such waves within the framework of **small-amplitude** waves theory.

We simplify the full system of equations by a linearization which is around still water.

We define the set of **non-dimensional** variables:

$$x \mapsto \lambda x, \quad z \mapsto h_0 z, \quad \eta \mapsto a\eta, \quad t \mapsto \frac{\lambda}{\sqrt{gh_0}} t,$$
$$u \mapsto \sqrt{gh_0}\, u, \quad v \mapsto h_0 \frac{\sqrt{gh_0}}{\lambda} v$$

$$p \mapsto p_0 + \rho g h_0 (1 - z) + \rho g h_0 p$$

In non-dimensional scaled variables, the boundary value problem (1) becomes

$$\left\{\begin{array}{ll} u_t + \epsilon(uu_x + vu_z) = -p_x & \\ \delta^2[v_t + \epsilon(uv_x + vv_z)] = -p_z & \\ u_x + v_z = 0 & \\ v = \eta_t + \epsilon u\eta_x & \text{on } z = 1 + \epsilon\eta(x,t) \\ p = \eta - \delta^2 W_e \frac{\eta_{xx}}{(1+\epsilon^2\delta^2\eta_x^2)^{3/2}} & \text{on } z = 1 + \epsilon\eta(x,t) \\ v = 0 & \text{on } z = 0 \end{array}\right. \tag{2}$$

$\epsilon = \frac{a}{h_0}$ is the **amplitude** parameter

$\delta = \frac{h_0}{\lambda}$ is the **shallowness** parameter

$W_e = \frac{\Gamma}{gh_0^2}$ is a **Weber** number

We suppose that the water flow is irrotational, thus, in addition to the system (2) we also have the eq.:

$$u_z - v_x = 0 \tag{3}$$

which writes in non-dimensional variables as:

$$u_z - \delta^2 v_x = 0 \tag{4}$$

By letting $\epsilon \to 0$, δ and W_e being fixed, we obtain a linear approximation of (2)+(4).

The linearized problem:

$$\begin{cases} u_t + p_x = 0 \\ \delta^2 v_t + p_z = 0 \\ u_x + v_z = 0 \\ u_z - \delta^2 v_x = 0 \\ v = \eta_t & \text{on } z = 1 \\ p = \eta - \delta^2 W_e \eta_{xx} & \text{on } z = 1 \\ v = 0 & \text{on } z = 0 \end{cases} \tag{5}$$

Solving this problem, we get a parameter c_0 by which we can *describe different background flows in the irrotational case*.

From the first four eqs. of (5) and applying the method of separation of variables we get:

$$u(x,z,t) = \frac{\delta}{k\sinh(k\delta)}\cosh(k\delta z)\eta_{tx} + \mathcal{F}(t)$$
$$v(x,z,t) = \frac{1}{\sinh(k\delta)}\sinh(k\delta z)\eta_t$$
$$\eta_{txx} + k^2\eta_t = 0$$

$\mathcal{F}(t)$ an arbitrary function, $k \geq 0$ a constant that might depend on time. For periodic travelling wave solutions, with $k = 2\pi$, we choose

$$\eta(x,t) = \cos(2\pi(x - ct))$$

c is to be determined. From the first 2 eqs. of (5) and the boundary conditions we find the expressions of the pressure p, of c and

$$\mathcal{F}(t) = \text{const} = c_0$$

Thus, a periodic solution of the linear system (5) is:

$$
\begin{aligned}
\eta(x,t) &= \cos(2\pi(x-ct))\\
u(x,z,t) &= \frac{2\pi\delta c}{\sinh(2\pi\delta)}\cosh(2\pi\delta z)\cos(2\pi(x-ct))+c_0\\
v(x,z,t) &= \frac{2\pi c}{\sinh(2\pi\delta)}\sinh(2\pi\delta z)\sin(2\pi(x-ct))\\
p(x,z,t) &= \frac{2\pi\delta c^2}{\sinh(2\pi\delta)}\cosh(2\pi\delta z)\cos(2\pi(x-ct))
\end{aligned}
\tag{6}
$$

with the non-dimensional speed of the linear wave

$$
c^2 = \frac{\tanh(2\pi\delta)}{2\pi\delta}(1+4\pi^2\delta^2 W_e) = \frac{\lambda}{2\pi h_0}\left(1+\frac{4\pi^2\Gamma}{g\lambda^2}\right)\tanh\left(\frac{2\pi h_0}{\lambda}\right) \tag{7}
$$

Let $(\mathbf{x}(\mathbf{t}), \mathbf{z}(\mathbf{t}))$ be the path of a particle in the fluid domain, $(x(0), z(0)) := (x_0, z_0)$ at time $t = 0$.
The motion of the particles is described by:

$$\begin{cases} \frac{dx}{dt} = u(x, z, t) = \frac{2\pi\delta c}{\sinh(2\pi\delta)} \cosh(2\pi\delta z) \cos(2\pi(x - ct)) + c_0 \\ \frac{dz}{dt} = v(x, z, t) = \frac{2\pi c}{\sinh(2\pi\delta)} \sinh(2\pi\delta z) \sin(2\pi(x - ct)) \end{cases} \tag{8}$$

Notice that

$$c_0 = \frac{1}{1} \int_x^{x+1} u(s, z, t) ds, \tag{9}$$

representing therefore the strength of the underlying uniform current (see also [1]).

[1] (2010) CONSTANTIN A. AND STRAUSS W., *Comm. Pure Appl. Math.*

Thus,

$\mathbf{c_0 = 0}$ will correspond to a region of still water with no underlying current

$\mathbf{c_0 > 0}$ will characterize a favorable uniform current

$\mathbf{c_0 < 0}$ will characterize an adverse uniform current

Analyzing the ***first-order approximation*** **of the nonlinear ordinary differential equation system** which describes the particle motion below small-amplitude waves, it was obtained that all water particles trace closed, circular or elliptic, orbits (see, for example, [1], [2], [3], [4]).

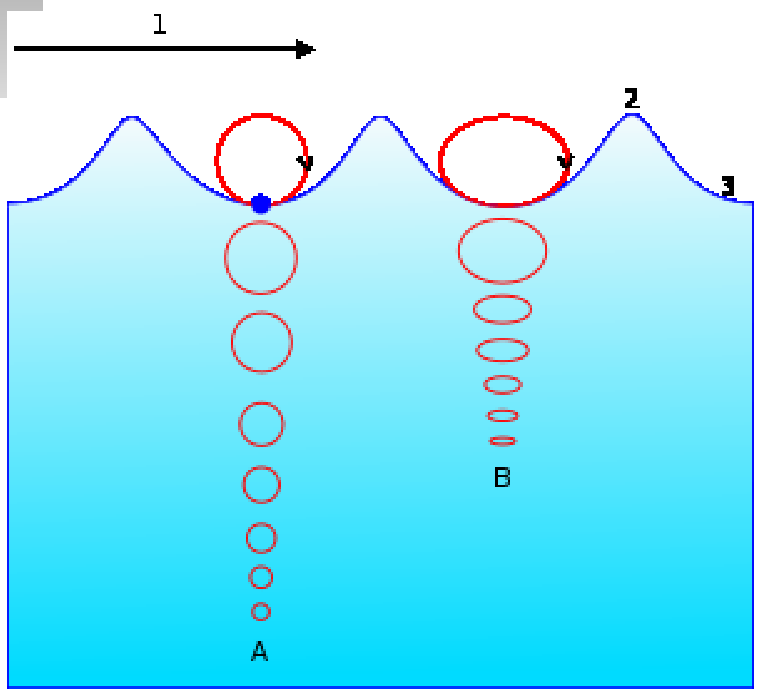
[1] (1953) LAMB H., *Hydrodynamics*.

[2] (1994) DEBNATH L., *Nonlinear Water Waves*.

[3] (1997) JOHNSON R. S., *A Modern Introduction to the Mathematical Theory of Water Waves*.

[4] (2001) LIGHTHILL J., *Waves in Fluids*.

The classical picture:



A = In deep water.
The orbital motion of fluid particles decreases rapidly with increasing depth below the surface.

B = In shallow water.
The elliptical movement of a fluid particle flattens with decreasing depth.

1 = Propagation direction.

2 = Wave crest.

3 = Wave trough.

*The picture is taken from Wikipedia, *Wave-Wikipedia*, the free encyclopedia.

In the moving frame

$$X = 2\pi(x - ct), \qquad Z = 2\pi\delta z \tag{10}$$

the system (8) becomes

$$\begin{cases} \dfrac{dX}{dt} = \dfrac{4\pi^2\delta c}{\sinh(2\pi\delta)} \cosh(Z)\cos(X) + 2\pi(c_0 - c) \\ \dfrac{dZ}{dt} = \dfrac{4\pi^2\delta c}{\sinh(2\pi\delta)} \sinh(Z)\sin(X) \end{cases} \tag{11}$$

I) $\mathbf{c_0 = c}$

In this case, differentiating (11) with respect to t we get

$$\begin{cases} \dfrac{d^2X}{dt^2} = -A^2\sin(2X) \\ \dfrac{d^2Z}{dt^2} = A^2\sinh(2Z) \end{cases} \tag{12}$$

where $A^2 := \dfrac{8\pi^4\delta^2c^2}{\sinh^2(2\pi\delta)}$

The system (12) integrates to:

$$\begin{cases} \left(\frac{dX}{dt}\right)^2 = A^2\cos(2X) + c_1 \\ \left(\frac{dZ}{dt}\right)^2 = A^2\cosh(2Z) + c_2 \end{cases} \tag{13}$$

c_1, c_2 being the integration constants. For the first eq. in (13) we use the substitution

$$\tan(X) = y\,,\ \cos(2X) = \frac{1-y^2}{1+y^2}\,,\ dX = \frac{1}{1+y^2}dy \tag{14}$$

for the second eq. in (13), we use the substitution

$$\tanh(Z) = w\,,\ \cosh(2Z) = \frac{1+w^2}{1-w^2}\,,\ dZ = \frac{1}{1-w^2}dw \tag{15}$$

In the new variables, we obtain:

$$\begin{cases} \left(\frac{dy}{dt}\right)^2 = A^2(1-y^4) + c_1(1+y^2)^2 \\ \left(\frac{dw}{dt}\right)^2 = A^2(1-w^4) + c_2(1-w^2)^2 \end{cases} \tag{16}$$

The solutions involve elliptic integrals of the first kind:

$$\pm \int \frac{dy}{\sqrt{(c_1 - A^2)y^4 + 2c_1 y^2 + c_1 + A^2}} = t \tag{17}$$

$$\pm \int \frac{dw}{\sqrt{(c_2 - A^2)w^4 - 2c_2 w^2 + c_2 + A^2}} = t \tag{18}$$

which may by reduced to their Legendre normal form.

Depending of the sign of $c_1 - A^2$, $c_1 + A^2$, we get:

$$y(t) = \pm \frac{\text{sn}\left(\sqrt{c_1 + A^2}\,t; k_1\right)}{\text{cn}\left(\sqrt{c_1 + A^2}\,t; k_1\right)}, \quad 0 \le k_1^2 := \frac{2A^2}{c_1 + A^2} \le 1 \tag{19}$$

$$y(t) = \pm \sqrt{\frac{A^2 + c_1}{A^2 - c_1}}\,\text{cn}\left(\sqrt{2A^2}\,t; k_2\right), \quad 0 \le k_2^2 := \frac{A^2 + c_1}{2A^2} \le 1 \tag{20}$$

sn is the Jacobian elliptic function *sine amplitude*

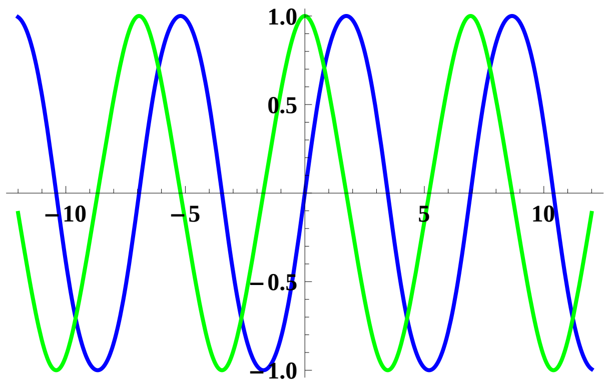
cn is the Jacobian elliptic function *cosine amplitude*

They arise from the inversion of the elliptic integral of the first kind

$$t = \int \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \tag{21}$$

$0 \leq k^2 \leq 1$ is the *elliptic modulus* and φ is the *Jacobi amplitude*

$$\text{sn}\,(t;k) := \sin(\varphi), \;\; \text{cn}\,(t;k) := \cos(\varphi) \tag{22}$$



The graphs of sn(t;1/3) and cn(t;1/3).

$$\text{sn}^{\,2}(t;k) + \text{cn}^{\,2}(t;k) = 1$$

$$-1 \leq \text{sn}\,(t;k) \leq 1, \; -1 \leq \text{cn}\,(t;k) \leq 1$$

$$\text{sn}\,(t;0) = \sin(t), \;\; \text{cn}\,(t;0) = \cos(t)$$

$$\text{sn}\,(t;1) = \tanh(t), \;\; \text{cn}\,(t;1) = \text{sech}\,(t)$$

Depending of the sign of $c_2 - A^2$, $c_2 + A^2$, we get:

$$w(t) = \pm\,\mathrm{sn}\,\left(\sqrt{c_2 + A^2}\,t; k_3\right), \quad 0 \leq k_3^2 = \frac{c_2 - A^2}{c_2 + A^2} \leq 1 \qquad (23)$$

$$w(t) = \pm\,\mathrm{cn}\,\left(\sqrt{2A^2}\,t; k_4\right), \quad 0 \leq k_4^2 = \frac{A^2 - c_2}{2A^2} \leq 1 \qquad (24)$$

$$w(t) = \pm\sqrt{1 - \frac{2A^2}{A^2 - c_2}\,\mathrm{sn}^2\left(\sqrt{A^2 - c_2}\,t; k_5\right)},\; 0 \leq k_5^2 = \frac{2A^2}{A^2 - c_2} \leq 1 \qquad (25)$$

Thus, the solution of system (8) has the expression:

$$\begin{cases} x(t) = ct + \frac{1}{2\pi}\text{arctan}\,[y(t)] \\ z(t) = \frac{1}{2\pi\delta}\text{arctanh}\,[w(t)] \end{cases} \tag{26}$$

with $y(t)$ given by (20), (21) and $w(t)$ by (24), (25), (26)(see [1], [2]).

The curves in (26) are not closed curves.

[1] (2009) IONESCU-KRUSE D., *Wave Motion*.

[2] (2010) IONESCU-KRUSE D.,*Nonlinear Anal. Real World Appl.*

This result is in the line with the results obtained in [1]-[11].

[1] (2008) CONSTANTIN A. AND VILLARI G., *J. Math. Fluid Mech.*

[2] (2006) CONSTANTIN A., *Invent. Math.*

[3] (2006) HENRY D., *Int. Math. Res. Not.*

[4] (2007) CONSTANTIN A. AND ESCHER J., *Bull. Amer. Math. Soc.*

[5] (2007) HENRY D., *J. Nonlinear Math. Phys.* .

[6] (2007) HENRY D., *Phil. Trans. R. Soc. A*

[7] (2008) CONSTANTIN A., EHRNSTRÖM M., AND VILLARI G., *Nonlinear Anal. Real World Appl.*

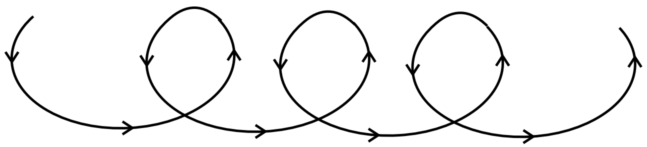
[8] (2008) EHRNSTRÖM M. , *Nonlinearity*.

[9] (2008) EHRNSTRÖM M. AND VILLARI G., *J. Differential Equations*.

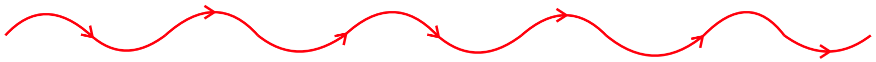
[10] (2008) IONESCU-KRUSE D., *J. Nonlinear Math. Phys.*

[11] (2009) IONESCU-KRUSE, *Nonlinear Anal-Theor*.

Analyzing in more detail the explicit solution (26) we get ([2],[3]) new kind of particle paths (see also [1]):

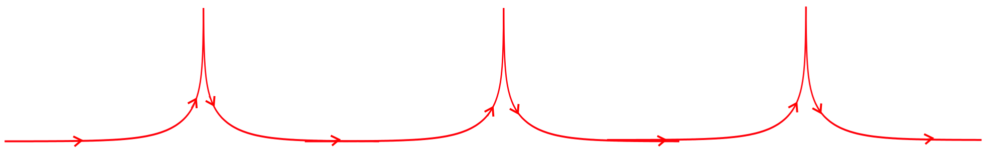


$x'(t) > 0$ $z'(t) > 0$	$x'(t) < 0$ $z'(t) > 0$	$x'(t) < 0$ $z'(t) < 0$	$x'(t) > 0$ $z'(t) < 0$



$x'(t) > 0$ $z'(t) < 0$	$x'(t) > 0$ $z'(t) > 0$

$x'(t) < 0$ $z'(t) < 0$	$x'(t) < 0$ $z'(t) > 0$



$$\lim_{t\to\tilde{t}} x(t) = \text{finite} := \tilde{x}, \quad \lim_{t\to\tilde{t}} z(t) = \infty$$

[1] (2010) CONSTANTIN A. AND STRAUSS W., *Comm. Pure Appl. Math.*

[2] (2009) IONESCU-KRUSE D., *Wave Motion*.

[3] (2010) IONESCU-KRUSE D.,*Nonlinear Anal. Real World Appl.*

II) $\mathbf{c_0 \neq c}$

Differentiating system (11) with respect to t, we get:

$$\frac{d^2X}{dt^2} + b\tan(X)\frac{dX}{dt} + A^2\sin(2X) - b^2\tan(X) = 0 \qquad (27)$$

where $b := 2\pi(c_0 - c)$. By the substitution $\tan(X) = y$, we get

$$\frac{d^2y}{dt^2} - \frac{2y}{1+y^2}\left(\frac{dy}{dt}\right)^2 + by\frac{dy}{dt} + 2A^2y - b^2y(1+y^2) = 0 \qquad (28)$$

This eq. can be written as an Abel differential equation of the second kind. It is solvable and its solution has the parametric form (see [1]):

[1] (2009) IONESCU-KRUSE D., *Wave Motion*.

$$y(\tau) = \pm\sqrt{\frac{\tau^2 - 2A^2}{\left(C - b\ln|\tau + \sqrt{\tau^2 - 2A^2}|\right)^2} - 1}, \qquad (29)$$

C is a constant, and the relation between t and τ is:

$$t = \int \frac{1}{\sqrt{\tau^2 - 2A^2}\sqrt{\tau^2 - 2A^2 - (C - b\ln|\tau + \sqrt{\tau^2 - 2A^2}|)^2}}\, d\tau \qquad (30)$$

The solution of system (8) is written now as

$$\begin{cases} x(\tau) = c\,t(\tau) \pm \frac{1}{2\pi}\text{arctan}\left[\sqrt{\frac{\tau^2 - 2A^2}{\left(C - b\ln|\tau + \sqrt{\tau^2 - 2A^2}|\right)^2} - 1}\right] \\ z(\tau) = \pm\frac{1}{\pi\delta}\text{arctanh}\left[\sqrt{\frac{\tau - \sqrt{2}A}{\tau + \sqrt{2}A}}\right] \end{cases} \qquad (31)$$

Conclusions

- We provide explicit solutions to the nonlinear ODEs which give the particle paths below small-amplitude waves.
  - In the case $c_0 = c$, the solution of the system is represented by Jacobian elliptic functions.
  - In the case $c_0 \neq c$ the system is governed by a solvable Abel differential equation of second kind.
- We give an accurate description of the shapes of the particle paths within the fluid.