A splitting method for nonlinear Schrödinger equation

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NSE

Let us consider the nonlinear Schrödinger equation (NSE):

$$\begin{cases} \dfrac{du}{dt} = i\Delta u - i|u|^p u, & x \in \mathbb{R}^d, t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^d. \end{cases} \tag{1}$$

- L^2-norm is conserved
- $E(t) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{p+2}|u|^{p+2}\right)$ is conserved (good sign)

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We define the flow $N(t)$ for the differential equation:

$$\begin{cases} \dfrac{du}{dt} = -i|u|^p u, & x \in \mathbb{R}, t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}, \end{cases} \tag{2}$$

i.e.

$$N(t)\varphi = \exp(it|\varphi|^p)\varphi. \tag{3}$$

The idea of splitting methods is to approximate the solutions of (1) by combining the two flows $S(t) = \exp(-it\Delta)$ and $N(t)$.

Fix a time interval $[0, T]$ and a positive time step τ.
Lie approximation

$$Z(n\tau) = (S(\tau)N(\tau))^n\varphi,\ 0 \le n\tau \le T \tag{4}$$

Strang approximation

$$Z(n\tau) = (S(\tau/2)N(\tau)S(\tau/2))^n\varphi,\ 0 \le n\tau \le T. \tag{5}$$

In $d = 3$, Lubich (Math. of Comp., 2008) considered the cubic nonlinearity with $H^4(\mathbb{R}^3)$ initial data:

$$\max_{n\tau \leq T} \|Z(n\tau) - u(n\tau)\|_{L^2(\mathbb{R}^3)} \leq \tau^2$$

• $H^4(\mathbb{R}^3)$-regularity was imposed to guarantee that the approximate solution Z remains bounded in the $H^2(\mathbb{R}^3)$-norm.

• $H^2(\mathbb{R}^3)$ boundedness implies the $L^2(\mathbb{R}^3)$ convergence rate.

- Our idea: introduce a splitting method and estimate the error below the H^4 regularity.
- Use something similar to Strichartz estimates in order to prove that the error committed in the nonlinear problem could be estimated in terms of the error in the linear part (like stability in the PDE context)
- Before all, stability in the L^2-norm and in the auxiliary norms $L^q(L^r)$.

Linear Schrödinger Equation

$$\begin{cases} iu_t + \Delta u = 0,\ x \in \mathbb{R},\ t \neq 0, \\ u(0,x) = \varphi(x),\ x \in \mathbb{R}, \end{cases}$$

Conservation of the L^2-norm

$$\|S(t)\varphi\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}$$

Dispersive estimate

$$|S(t)\varphi(x)| \leq \frac{1}{(4\pi|t|)^{1/2}}\|\varphi\|_{L^1(\mathbb{R})}$$

Space time estimates

The admissible pairs

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{r}$$

Strichartz estimates for admissible pairs (q, r)

$$\|S(\cdot)\varphi\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}))} \leq C(q, r)\|\varphi\|_{L^2(\mathbb{R})}$$

$$\begin{cases} iu_t + \Delta u = |u|^p u, \ x \in \mathbb{R}, \ t \neq 0 \\ u(0,x) = \varphi(x), \ x \in \mathbb{R} \end{cases}$$

For initial data in $L^2(\mathbb{R})$, Tsutsumi '87 proved the global existence and uniqueness for $p < 4$

$$u \in C(\mathbb{R},\, L^2(\mathbb{R})) \cap L^q_{loc}(\mathbb{R},\, L^r(\mathbb{R}))$$

This result can not be proved by methods based purely on energy arguments.

Lie's Method

$$
\begin{aligned}
Z(n\tau) &= (S(\tau)N(\tau))^n\varphi \\
&= S(n\tau)\varphi + \tau\sum_{k=0}^{n-1} S(n\tau - k\tau)\frac{N(\tau)-I}{\tau}Z(k\tau)
\end{aligned}
$$

- L^2-stability follows from the properties of S and N.
- $l^q(0 \le n\tau \le T, L^r)$-stability is not even true for the linear problem
- H^1-stability is very hard- requires H^3-regularity on the continuous solution (with Lubich's method)

Way too many dificulties?

Looking to the semigroup formulation of the two problems the splitting method seems to better approximate the folowing equation

$$\begin{cases} \dfrac{dv}{dt} = i\Delta v + \dfrac{\exp(-i\lambda\tau|v|^p) - 1}{\tau} v, & x \in \mathbb{R}^d, t > 0, \\ v(x,0) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{6}$$

What we can say about the H^1-solutions of this equation? They are local, on $(0, T_0))$ by a fix point argument. Is not very clear what energy is conserved. The initial problem has global solutions and then the stability holds in any time interval $(0, T)$ with $T > T_0$.

The linear semigrup does not satisfy discrete Strichartz estimates

$$\|S(n\tau)\varphi\|_{l^q(\tau\mathbb{Z},L^r(\mathbb{R}^d))} \leq C(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)} \quad \text{for all} \quad \varphi \in L^2(\mathbb{R}^d),$$

since

$$\tau^{1/q}\|S(\tau)\varphi\|_{L^r(\mathbb{R}^d)} \leq C(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)},$$

is not true for all $\varphi \in L^2(\mathbb{R})$.
We have to choose in a convenient way the approximation of the linear semigroup

One of the possible choices is the filtered operator

$$S_\tau(t)\varphi = S(t)\Pi_\tau\varphi$$

where Π_τ filters the high frequencies as follows

$$\widehat{\Pi_\tau\varphi}(\xi) = \hat{\varphi}(\xi)\mathbf{1}_{\{|\xi|\leq\tau^{-1/2}\}}(\xi), \quad \xi\in\mathbb{R}^d. \tag{7}$$

The splitting scheme we propose is the following one:

$$Z_\tau(n\tau) = (S_\tau(\tau)N(\tau))^n\Pi_\tau\varphi. \tag{8}$$

Observe that in this scheme only the linear equation is filtered while the nonlinear one is solved exactly.

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Strichartz estimates for S_τ

Theorem

The semigroup $\{S_\tau(t)\}_{t\in\mathbb{R}}$ *satisfies*

$$\|S_\tau(t)\varphi\|_{L^2(\mathbb{R}^d)} \le \|\varphi\|_{L^2(\mathbb{R}^d)}, \quad \forall\, t \in \mathbb{R}, \tag{9}$$

and

$$\|S_\tau(t)\varphi\|_{L^\infty(\mathbb{R}^d)} \le \frac{C(d)}{\tau^{d/2} + |t|^{d/2}} \|\varphi\|_{L^1(\mathbb{R}^d)}, \quad \forall\, t \in \mathbb{R}. \tag{10}$$

Moreover, for any admissible pairs (q, r) *the following hold*

$$\|S_\tau(\cdot)\varphi\|_{l^q(\tau\mathbb{Z}, L^r(\mathbb{R}^d))} \le C(d,q)\|\varphi\|_{L^2(\mathbb{R}^d)}, \tag{11}$$

Theorem

(Stability) Let $0 < p < 4/d$*. The approximation* Z_τ *introduced in* (8) *satisfies*

$$\max_{n\geq 0} \|Z_\tau(n\tau)\|_{L^2(\mathbb{R}^d)} \leq \|\varphi\|_{L^2(\mathbb{R}^d)}. \tag{12}$$

Moreover, for any $T > 0$ *and* (q, r) *admissible-pair the following*

$$\|Z_\tau(n\tau)\|_{l^q(0\leq n\tau\leq T;\, L^r(\mathbb{R}^d))} \leq C(T, d, p, q)\|\varphi\|_{L^2(\mathbb{R}^d)} \tag{13}$$

holds for some constant $C(T, d, p, q)$ *independent of the time step* τ*.*

Theorem

(Convergence) Let $d \leq 3$*,* $p \in [1, 4/d)$ *and* $\varphi \in H^2(\mathbb{R}^d)$*. The numerical solution* Z_τ *has a first-order error bound in* $L^2(\mathbb{R}^d)$*:*

$$\max_{0\leq n\tau\leq T} \|Z_\tau(n\tau) - u(n\tau)\|_{L^2(\mathbb{R}^d)} \leq \tau C(T, d, p, \|\varphi\|_{H^2(\mathbb{R}^d)}).$$

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