Homogenizing media containing a highly conductive honeycomb substructure.

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Contents

Setting of the problem

$$\Omega = I^3,\ I =]-\frac{1}{2}, \frac{1}{2}[\quad \text{and} \quad n \in \mathbb{N}.$$

$$\varepsilon = \frac{1}{2n+1}, \quad \mathbf{Z}_\varepsilon = \{k \in \mathbf{Z}, \quad |k| \leq n\}$$

$$I_\varepsilon^k = \varepsilon k + \varepsilon I, \quad k \in \mathbf{Z}_\varepsilon.$$

Remark

Obviously, $\operatorname{card} \mathbf{Z}_\varepsilon = 1/\varepsilon$ and $x \in \bar{I}$ if and only if there exists $k \in \mathbf{Z}_\varepsilon$ such that $x \in \bar{I}_\varepsilon^k$.

For any $i \in \{1,2,3\}$, we consider $r_\varepsilon^i \geq 0$, $r_\varepsilon^i << \varepsilon$, that is $\frac{r_\varepsilon^i}{\varepsilon} \to 0$ when $\varepsilon \to 0$.

$$T_{\varepsilon,k}^i = \{x = (x_1, x_2, x_3) \in \Omega,\ |x_i - \varepsilon k| < r_\varepsilon^i\},\ \ T_\varepsilon^i = \cup_{k \in \mathbf{Z}_\varepsilon} T_{\varepsilon,k}^i \quad (1)$$

$$T_\varepsilon = \cup_{i=1}^3 T_\varepsilon^i, \quad T_\varepsilon^{ij} = T_\varepsilon^i \cap T_\varepsilon^j, \quad \text{for} \quad 1 \leq i < j \leq 3. \quad (2)$$

is defined for any $k \in \mathbf{Z}_\varepsilon$.

The problem

$$-a\Delta u_\varepsilon = f_\varepsilon \quad \text{in} \quad \Omega \setminus T_\varepsilon \tag{3}$$

$$-\frac{b}{|T_\varepsilon|}\Delta u_\varepsilon = f_\varepsilon \quad \text{in} \quad T_\varepsilon \tag{4}$$

$$u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \tag{5}$$

together with the natural transmission conditions on the interface $\partial T_\varepsilon \setminus \partial\Omega$.

Variational formulation

Assuming that $f_\varepsilon \in H^{-1}(\Omega)$, the variational formulation of our problem is :

To find $u_\varepsilon \in H_0^1(\Omega)$ such that

$$a\int_{\Omega\setminus T_\varepsilon} \nabla u_\varepsilon \nabla v + b \fint_{T_\varepsilon} \nabla u_\varepsilon \nabla v = \langle f_\varepsilon, v\rangle, \quad \forall v \in H_0^1(\Omega), \qquad (6)$$

where we have used the notation

$$\fint_E \cdot = \frac{1}{|E|}\int_E \cdot \quad \text{for any measurable } E \subset \Omega \qquad (7)$$

and $\langle \cdot, \cdot \rangle$ is the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.
Applying the Lax-Milgram theorem we obtain :

Proposition

The variational equation (6) has a unique solution $u_\varepsilon \in H_0^1(\Omega)$.

Aim

The aim of this paper is to describe the asymptotic behaviour of the temperature u_ε as $\varepsilon \to 0$, assuming that the source term is weakly convergent :

$$\exists f \in H^{-1}(\Omega) \quad \text{such that} \quad f_\varepsilon \rightharpoonup f \quad \text{in} \quad H^{-1}(\Omega). \tag{8}$$

We have to remark here that the boundedness of $(f_\varepsilon)_\varepsilon$ implies immediately :

Proposition

$(u_\varepsilon)_\varepsilon$ is bounded in $H_0^1(\Omega)$ and there exists $C > 0$, independent of ε, such that

$$⨍_{T_\varepsilon} |\nabla u_\varepsilon|^2 \leq C. \tag{9}$$

The control-zone homogenization method

In order to obtain further results, specific to the thin substructure considered here, we have to introduce the following operators :

Definition

To any $u \in H^1(\Omega)$ we associate $G^i_\varepsilon(u) \in L^2(\Omega)$ defined by the following :

$$G^i_\varepsilon(u)(x_1, x_2, x_3) = \sum_{k \in \mathbf{Z}_\varepsilon} G^i_{\varepsilon,k}(u)(\overline{x}_i) 1_{I^k_\varepsilon}(x_i),$$

$$\overline{x}_i = (\cdots, \not{x}_i, \cdots) \in I^2$$

$$G^i_{\varepsilon,k}(u) = \frac{1}{2} u \Big|_{x_i = \varepsilon k - r^i_\varepsilon} + \frac{1}{2} u \Big|_{x_i = \varepsilon k + r^i_\varepsilon}, \quad k \in \mathbf{Z}_\varepsilon$$

where $u \big|_{x_i = \varepsilon k \pm r^i_\varepsilon}$ are the traces of u on the corresponding faces of $T^i_{\varepsilon,k}$.

Specific operators

These operators have three basic properties :

Proposition

For any $u \in H^1(\Omega)$ and $i \in \{1, 2, 3\}$, we have

$$\fint_{T_\varepsilon^i} |G_\varepsilon^i(u) - u|^2 \le \varepsilon r_\varepsilon^i \left| \frac{\partial u}{\partial x_i} \right|^2_{T_\varepsilon^i} \quad (10)$$

$$\fint_{T_\varepsilon^i} |G_\varepsilon^i(u)|^2 = |G_\varepsilon^i(u)|^2_\Omega \quad (11)$$

$$|G_\varepsilon^i(u) - u|_\Omega \le \varepsilon \left| \frac{\partial u}{\partial x_i} \right|_\Omega \quad (12)$$

where $|\cdot|_\Omega$ is denoting the norm of $L^2(\Omega)$.

The first important consequence is

Theorem

There exists $C > 0$, independent of ε, such that

$$\fint_{T_\varepsilon} |u|^2 \leq C|\nabla u|^2_\Omega, \quad \forall u \in H^1_0(\Omega). \tag{13}$$

$$\fint_{T_\varepsilon} |u|^2 \leq \sum_{i=1}^{3} \fint_{T^i_\varepsilon} |u|^2 \leq 2 \sum_{i=1}^{3} \fint_{T^i_\varepsilon} \left(|G^i_\varepsilon(u) - u|^2 + |G^i_\varepsilon(u)|^2\right).$$

$$\fint_{T_\varepsilon} |u|^2 \leq 2 \left(\max_{i=1,2,3} r^i_\varepsilon\right) |\nabla u|^2_\Omega + 4 \sum_{i=1}^{2} \left(|G^i_\varepsilon(u) - u|^2_\Omega + |u|^2_\Omega\right)$$

As the techniques of [F. BENTALHA, I. GRUAIS, D. POLISEVSKI, Diffusion in a highly rarefied binary structure of general periodic shape, *Applicable Analysis*, **87(6)**, 2008, 635–655.] can be used to the domain T_ε^{ij} ($i < j$), then, according to [M. BELLIEUD, G. BOUCHITTÉ, Homogenization of elliptic problems in a fiber reinforced structure. Non local effects, *Ann. Scuola Norm. Sup. Pis Cl. Sci.(4)*, **26(3)**, 1998, 407–436.], we have :

Theorem

There exists $C > 0$, independent of ε, such that

$$\int_{T_\varepsilon^{ij}} |u|^2 \leq C \max\left(1, \varepsilon^2 \ln \frac{1}{r_\varepsilon^i}, \varepsilon^2 \ln \frac{1}{r_\varepsilon^j}\right) |\nabla u|_\Omega^2, \quad \forall u \in H_0^1(\Omega). \tag{14}$$

Finally, we remind an estimation of the same type which is associated to

$$T_\varepsilon^0 = \cap_{i=1}^3 T_\varepsilon^i. \tag{15}$$

Theorem

There exists $C > 0$, independent of ε, such that

$$\int_{T_\varepsilon^0} |u|^2 \leq C \max\left(1, \frac{\varepsilon^3}{r_\varepsilon}\right) |\nabla u|_\Omega^2, \quad \forall u \in H_0^1(\Omega). \tag{16}$$

See :[F. BENTALHA, I. GRUAIS, D. POLISEVSKI, Diffusion in a highly rarefied binary structure of general periodic shape, *Applicable Analysis*, **87(6)**, 2008, 635–655.]

First, let us introduce the tools of this method. Denoting

$$\mathcal{R} = \{(R^i_\varepsilon)_\varepsilon,\ r^i_\varepsilon << R^i_\varepsilon << \varepsilon,\ i = 1, 2, 3\}, \tag{17}$$

then, for any $(R^i_\varepsilon)_\varepsilon \in \mathcal{R}$ we define the control-zone of the present problem by

$$\mathcal{C}_\varepsilon = \cup_{i=1}^3 \mathcal{C}^i_\varepsilon, \quad \mathcal{C}^i_\varepsilon = \cup_{k \in \mathbf{Z}_\varepsilon} \mathcal{C}^i_{\varepsilon,k}, \quad i \in \{1, 2, 3\}, \tag{18}$$

where for any $k \in \mathbf{Z}_\varepsilon$ and $i \in \{1, 2, 3\}$ we have

$$\mathcal{C}^i_{\varepsilon,k} = \{x = (x_1, x_2, x_3) \in \Omega, \quad |x_i - \varepsilon k| < R^i_\varepsilon\}. \tag{19}$$

The test-functions associated to this control-zone are defined by using the following capacitary functions $w^i_\varepsilon \in W^{1,\infty}(\Omega)$ ($i = 1, 2, 3$), given by

$$w^i_\varepsilon(x_1, x_2, x_3) = \begin{cases} 1 - \dfrac{r^i_\varepsilon}{R^i_\varepsilon} & \text{if} \quad x = (x_1, x_2, x_3) \in T^i_\varepsilon \\ 1 - \dfrac{|x_i - \varepsilon k|}{R^i_\varepsilon} & \text{if} \quad x \in (\mathcal{C}^i_{\varepsilon,k} \setminus T^i_{\varepsilon,k}), \quad k \in \mathbf{Z}_\varepsilon \\ 0 & \text{if} \quad x \in \Omega \setminus \mathcal{C}^i_\varepsilon, \end{cases} \tag{20}$$

and the step approximation operators introduced by

Definition

To any $\varphi \in \mathcal{D}(\Omega)$ we associate $\varphi^i_\varepsilon \in L^\infty(\Omega)$ ($i = 1, 2, 3$) :

$$\varphi^i_\varepsilon(x_1, x_2, x_3) = \sum_{k \in \mathbf{Z}_\varepsilon} \varphi\,|_{x_i = \varepsilon k}(\overline{x}_i)\ 1_{I^k_{R^i_\varepsilon}}(x_i), \tag{21}$$

where $I^k_{R^i_\varepsilon} := \varepsilon k + 2R^i_\varepsilon I$.

These operators have the following basic properties :

$$|\nabla w_\varepsilon^i|_{\mathcal{C}_\varepsilon^i} \leq \left(\frac{2}{\varepsilon R_\varepsilon^i}\right)^{1/2} \tag{22}$$

$$|\varphi - \varphi_\varepsilon^i|_{L^\infty(\mathcal{C}_\varepsilon^i)} \leq R_\varepsilon^i |\nabla\varphi|_{L^\infty(\Omega)}. \tag{23}$$

$$|\nabla\varphi_\varepsilon^i|_{\mathcal{C}_\varepsilon^i} \leq \left(\frac{2R_\varepsilon^i}{\varepsilon}\right)^{1/2} |\nabla\varphi|_{L^\infty(\Omega)}. \tag{24}$$

The reticulated case

For any $i \in \{1, 2, 3\}$ let us denote $m_i \geq 0$ as the limit of

$$m_i = \lim_{\varepsilon \to 0} \frac{|T_\varepsilon^i|}{|T_\varepsilon|}. \tag{25}$$

Obviously, we have

$$m_1 + m_2 + m_3 = 1.$$

In this section, we consider the case when

$$m_i > 0, \quad \forall i \in \{1, 2, 3\}, \tag{26}$$

that is the case when all the three parameters r_ε^i have the same order of magnitude with respect to ε. This geometry is called sometimes as the box-structure case.

We can present now the preliminary convergence results :

Proposition

There exists $u \in H_0^1(\Omega)$ such that, on some subsequence, there hold

$$u_\varepsilon \rightharpoonup u \quad \text{in} \quad H_0^1(\Omega) \tag{27}$$

$$G_\varepsilon^i(u_\varepsilon) \to u \quad \text{in} \quad L^2(\Omega), \quad \forall i \in \{1,2,3\} \tag{28}$$

$$\fint_{T_\varepsilon} u_\varepsilon v \to \int_\Omega uv, \quad \forall v \in H_0^1(\Omega). \tag{29}$$

Moreover, for any $i \in \{1,2,3\}$ we have :

$$\fint_{T_\varepsilon^i} \frac{\partial u_\varepsilon}{\partial x_j} v \to \int_\Omega \frac{\partial u}{\partial x_j} v, \quad \forall v \in H_0^1(\Omega), \quad \forall j \in \{1,2,3\}, \quad j \neq i. \tag{30}$$

In order to prove these inequalities we remark that

$$\fint_{T_\varepsilon} u_\varepsilon v = \frac{|T_\varepsilon^0|}{|T_\varepsilon|}\fint_{T_\varepsilon^0} u_\varepsilon v - \sum_{1\leq i<j\leq 3}\frac{|T_\varepsilon^{ij}|}{|T_\varepsilon|}\fint_{T_\varepsilon^{ij}} u_\varepsilon v + \sum_{i=1}^{3}\frac{|T_\varepsilon^{i}|}{|T_\varepsilon|}\fint_{T_\varepsilon^{i}} u_\varepsilon v. \tag{31}$$

$$\left|\frac{|T_\varepsilon^0|}{|T_\varepsilon|}\fint_{T_\varepsilon^0} u_\varepsilon v\right| \leq C\frac{|T_\varepsilon^0|}{|T_\varepsilon|}\max\left(1,\ \frac{\varepsilon^3}{r_\varepsilon}\right)|\nabla v|_\Omega \to 0,$$

$$\left|\frac{|T_\varepsilon^{ij}|}{|T_\varepsilon|}\fint_{T_\varepsilon^{ij}} u_\varepsilon v\right| \leq C\frac{|T_\varepsilon^{ij}|}{|T_\varepsilon|}\max\left(1,\ \varepsilon^2\ln\frac{1}{r_\varepsilon^i},\ \varepsilon^2\ln\frac{1}{r_\varepsilon^j}\right)|\nabla v|_\Omega \to 0.$$

$$\fint_{T_\varepsilon^i} u_\varepsilon v \to \int_\Omega uv, \quad \forall v \in H_0^1(\Omega). \tag{32}$$

Next, let $\varphi \in \mathcal{D}(\Omega)$ and $i \in \{1, 2, 3\}$. Denoting by $\nu = (\nu_1, \nu_2, \nu_3)$ the outward normal to ∂T_ε we obviously have $\varphi\nu_j = 0$ on ∂T_ε^i ($j \neq i$) and hence :

$$\int_{T_\varepsilon^i} \frac{\partial u_\varepsilon}{\partial x_j}\varphi = -\int_{T_\varepsilon^i} u_\varepsilon \frac{\partial \varphi}{\partial x_j}.$$

It follows

$$\int_{T_\varepsilon^i} \frac{\partial u_\varepsilon}{\partial x_j}\varphi \to -\int_\Omega u \frac{\partial \varphi}{\partial x_j} = \int_\Omega \frac{\partial u}{\partial x_j}\varphi.$$

The proof is completed by continuity

Now we can present our main result.

Theorem

$(u_\varepsilon)_\varepsilon$ is weakly convergent in $H_0^1(\Omega)$. Its limit, $u \in H_0^1(\Omega)$, is the only solution of the equation

$$-\sum_{i=1}^{3}\left(a+\frac{b}{3}(1-m_i)\right)\frac{\partial^2 u}{\partial x_i^2} = f \quad \text{in} \quad \Omega. \tag{33}$$

For any $i \in \{1,2,3\}$, let $(R^i_\varepsilon)_\varepsilon \in \mathcal{R}$ and $\varphi \in \mathcal{D}(\Omega)$.
We denote

$$v_\varepsilon(\varphi) = \sum_{i=1}^{3} \left(\left(1 - \frac{r^i_\varepsilon}{R^i_\varepsilon} \right) \varphi + (\varphi^i_\varepsilon - \varphi) w^i_\varepsilon \right) \tag{34}$$

$$=: \sum_{i=1,2,3} v^i_\varepsilon(\varphi) \in W^{1,\infty}_0(\Omega).$$

Then we set $v = v_\varepsilon(\varphi)$ in (6) and it follows

$$a \int_{\Omega \setminus T_\varepsilon} \nabla u_\varepsilon \nabla v_\varepsilon(\varphi) + b ⨍_{T_\varepsilon} \nabla u_\varepsilon \nabla v_\varepsilon(\varphi) = \langle f_\varepsilon, v_\varepsilon(\varphi) \rangle.$$

$$a\int_{\Omega\setminus T_\varepsilon}\nabla u_\varepsilon\nabla v_\varepsilon(\varphi)=a\sum_{i=1}^{3}\left(1-\frac{r_\varepsilon^i}{R_\varepsilon^i}\right)\int_{\Omega\setminus T_\varepsilon}\nabla u_\varepsilon\nabla\varphi+$$

$$+a\sum_{i=1}^{3}\int_{\mathcal{C}_\varepsilon^i\setminus T_\varepsilon}\nabla u_\varepsilon\nabla w_\varepsilon^i(\varphi_\varepsilon^i-\varphi)+a\sum_{i=1}^{3}\int_{\mathcal{C}_\varepsilon^i\setminus T_\varepsilon}\nabla u_\varepsilon(\nabla\varphi_\varepsilon^i-\nabla\varphi)w_\varepsilon^i$$

$$\to 3\int_\Omega\nabla u\nabla\varphi+0$$

$$a\sum_{i=1}^{3}\left(1-\frac{r_\varepsilon^i}{R_\varepsilon^i}\right)\int_{\Omega\setminus\mathcal{C}_\varepsilon}\nabla u_\varepsilon\nabla\varphi\to 3a\int_\Omega\nabla u\nabla\varphi.$$

$$b\fint_{T_\varepsilon}\nabla u_\varepsilon\nabla v_\varepsilon(\varphi)=b\sum_{i=1}^{3}\frac{|T_\varepsilon^i|}{|T_\varepsilon|}\fint_{T_\varepsilon^i}\nabla u_\varepsilon\nabla\varphi_\varepsilon^i\left(1-\frac{r_\varepsilon^i}{R_\varepsilon^i}\right)+$$

$$+b\sum_{i=1}^{3}\frac{1}{|T_\varepsilon|}\int_{\mathcal{C}_\varepsilon^i\cap T_\varepsilon}\nabla u_\varepsilon\nabla v_\varepsilon^i(\varphi)$$

$$\to b\sum_{i=1}^{3}m_i\left(\int_\Omega\nabla u\nabla\varphi-\int_\Omega\frac{\partial u}{\partial x_i}\frac{\partial\varphi}{\partial x_i}\right)+0.$$

$$\langle f_\varepsilon, v_\varepsilon(\varphi)\rangle = \sum_{i=1}^{3}\left(1-\frac{r_\varepsilon^i}{R_\varepsilon^i}\right)\langle f_\varepsilon,\varphi\rangle + \sum_{i=1}^{3}\langle f_\varepsilon,(\varphi_\varepsilon^i-\varphi)w_\varepsilon^i\rangle \;\rightarrow\; 3\,\langle f,\varphi\rangle.$$

Resuming, we can say that if we pass to the limit, then we obtain :

$$3a\int_\Omega \nabla u\nabla\varphi + b\sum_{i=1}^{3}(1-m_i)\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i} = 3\,\langle f,\varphi\rangle, \quad \forall\varphi\in\mathcal{D}(\Omega),$$

Remark

In spite of the vanishing volume of the rectangular honeycomb structure the homogenized behaviour is anisotropic, except the case when $m_1 = m_2 = m_3 = \dfrac{1}{3}$.

Remark

The reticulated domain occupied by all the intersections of the layers has no distinct influence upon the asymptotic distribution of the temperature. For $i < j$ and $k \in \{i, j\}$, this follows from

$$\left| \frac{1}{|T_\varepsilon|} \int_{T_\varepsilon^{ij}} \nabla u_\varepsilon \nabla \varphi_\varepsilon^k \right| \leq C |\nabla \varphi|_{L^\infty(\Omega)} \left(\frac{1}{|T_\varepsilon|} \int_{T_\varepsilon^{ij}} |\nabla u_\varepsilon|^2 \right)^{\frac{1}{2}} \left(\frac{|T_\varepsilon^{ij}|}{|T_\varepsilon|} \right)^{\frac{1}{2}} \leq$$
$$\leq C(\varphi) \left(\frac{r_\varepsilon}{\varepsilon} \right)^{\frac{1}{2}} \to 0, \tag{35}$$

The gridwork case

Using the notations and definitions of the previous sections, the gridwork case corresponds to the situation when the horizontal layers are missing and the other two layers have the same order of magnitude with respect to ε.
It follows that

$$T_\varepsilon = T_\varepsilon^1 \cup T_\varepsilon^2 \tag{36}$$

and there exist

$$m_i = \lim_{\varepsilon \to 0} \frac{|T_\varepsilon^i|}{|T_\varepsilon|} \quad (i = 1, 2), \quad \text{such that} \quad m_1 + m_2 = 1. \tag{37}$$

For consistency with the honeycomb case, we also define here

$$m_3 = 0. \tag{38}$$

Obviously, for $i \in \{1, 2\}$ the properties of the corresponding operators are still valid. Then we prove similar results.

The homogenization result in this case is the following :

Theorem

$(u_\varepsilon)_\varepsilon$ is weakly convergent in $H_0^1(\Omega)$. Its limit, $u \in H_0^1(\Omega)$, is the only solution of the equation :

$$-\sum_{i=1}^{3}\left(a + \frac{b}{3}(1 - m_i)\right)\frac{\partial^2 u}{\partial x_i^2} = f \quad \text{in} \quad \Omega. \tag{39}$$

Remark

Assuming that the quantity $|T_\varepsilon|$ is the same in the both cases that we considered, the significant difference between the limit equations shows how important is the internal geometry of the vanishing superconductive material.