A Linear Programming Approach to Discontinuous Control Problems

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[1](joint work with Oana-Silvia Serea (CMAP))

$$\begin{cases} dX_s^{t,x,u} = b\left(s, X_s^{t,x,u}, u_s\right) ds \left\{+\sigma\left(s, X_s^{t,x,u}, u_s\right) dW_s\right\}, \ t \leq s \leq T, \\ X_t^{t,x,u} = x \in \mathbb{R}^N, \end{cases}$$

h semicontinuous,

$$V(t,x) = \inf \mathbb{E}\left[h\left(X_T^{t,x,u}\right)\right],$$

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- $$\begin{cases} dx_t^{t_0,x_0,u} = b\left(t, x_t^{t_0,x_0,u}, u_t\right) dt, \ t_0 \leq t \leq T, \\ x_{t_0}^{t_0,x_0,u} = x_0 \in \mathbb{R}^N, \end{cases} \tag{1}$$

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- $(0, 0) \in cl\,(R(T, t_0)(0, 0))$ and $(0, 0) \notin R(T, t_0)(0, 0) \Longrightarrow$

$$\inf_{u \in \mathcal{U}} h\left(x_T^{t_0,0,0,u(\cdot)}, y_T^{t_0,0,0,u(\cdot)}\right) = 1 \neq V(t_0, 0, 0).$$

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- $$\begin{cases} \partial_t V(t,x) + \min_{u \in U} \langle \partial_x V(t,x), f(t,x,u) \rangle = 0, \\ \text{if } t \in (0,T), \ x \in \mathbb{R}^N, \end{cases} \qquad \text{(HJ Mayer)}$$

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Lemma

If φ *is a l.s.c. supersolution of (HJ Mayer), s.t.* $\varphi(T,\cdot) \geq h(\cdot)$, *then*

$$\varphi(t_0, x_0) \geq \inf\left\{\varphi(T,x) : x \in cl\left(R(T,t_0)\,x_0\right)\right\},$$

$\forall (t_0, x_0) \in (0, T) \times \mathbb{R}^N$.

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Main assumptions

- $b : \mathbb{R} \times \mathbb{R}^N \times U \longrightarrow \mathbb{R}^N$, $\sigma : \mathbb{R} \times \mathbb{R}^N \times U \longrightarrow \mathbb{R}^{N \times d}$

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- (i) b, σ bounded, uniformly continuous
- (ii) $\exists c > 0$ s.t.
  $|b(t,x,u) - b(t,y,u)| + |\sigma(t,x,u) - \sigma(t,y,u)| \leq c\,|x-y|$
  and
  $|b(t,x,u) - b(s,x,u)| + |\sigma(t,x,u) - \sigma(s,x,u)| \leq$
  $c\,|t-s|^{\frac{\delta_0}{2}}$,
  $\forall (t,s,x,y,u) \in [0,T]^2 \times \mathbb{R}^{2N} \times U$.

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- $$\begin{cases} -\partial_t V_h(t,x) + H\left(x, DV_h(t,x), D^2 V_h(t,x)\right) = 0, \\ \text{for all } (t,x) \in (0,T) \times \mathbb{R}^N, & \text{(HJB)} \\ V_h(T,\cdot) = h(\cdot) \text{ on } \mathbb{R}^N, \end{cases},$$
  $H(t,x,p,A) =$
  $\sup_{u \in U} \left\{-\frac{1}{2} Tr\left(\sigma\sigma^*(t,x,u) A\right) - \langle b(t,x,u), p\rangle\right\}$,

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- $\gamma_{t,x,u}(A \times B \times C \times D) = \frac{1}{T-t}\mathbb{E}\left[\int_t^T 1_{A\times B\times C}\left(\left(s, X_s^{t,x,u}, u_s\right)\right) ds\right] \mathbb{P}\left(X_T^{t,x,u} \in D\right)$,

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- $\int_{[t,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N} \left(|y|^2 + |z|^2\right) \gamma_{t,x,u}(ds, dy, dv, dz) \leq C_0\left(|x|^2 + 1\right)$.

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- $\gamma_{t,x,u} \in \mathcal{P}\left([t,T] \times \mathbb{R}^N \times U \times \mathbb{R}^N\right) : \forall \phi \in C_b^{1,2}\left([0,T] \times \mathbb{R}^N\right)$, $\int_{[t,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N} \left[\begin{array}{c}(T-t)\mathcal{L}^v\phi(s,y) \\ +\phi(t,x) - \phi(T,z)\end{array}\right] \gamma(ds, dy, dv, dz) = 0$ (Itô's formula).

(Finite horizon) Occupational measures 2

- $J_h(t,x,u) = \mathbb{E}\left[h\left(X_T^{t,x,u}\right)\right] = \int_{\mathbb{R}^N} h(z)\,\gamma_{t,x,u}\left([t,T] \times \mathbb{R}^N \times U, dz\right).$

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- $\Theta(t,x) =$
$$\left\{\begin{array}{l} \gamma\in\mathcal{P}\left([t,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N\right):\forall\phi\in C_b^{1,2}\left([0,T]\times\mathbb{R}^N\right), \\ \int_{[t,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N}\left[\begin{array}{c}(T-t)\,\mathcal{L}^v\phi(s,y)\\ +\phi(t,x)-\phi(T,z)\end{array}\right]\gamma(ds,dy,dv,dz)=0, \\ \int_{[t,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N}\left(|y|^2+|z|^2\right)\gamma(ds,dy,dv,dz)\le C_0\left(|x|^2+1\right) \end{array}\right.$$

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- $\mathcal{L}^v\phi(s,y) = \frac{1}{2}Tr\left[(\sigma\sigma^*)(s,y,v)\,D^2\phi(s,y)\right] + \langle b(s,y,v), D\phi(s,y)\rangle + \partial_t\phi(s,y),$

Linearized formulation

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- dual formulation:

$$\eta^*(t,x) = \sup\left\{\begin{array}{c} \eta\in\mathbb{R} : \exists\phi\in C_b^{1,2}\left([0,T]\times\mathbb{R}^N\right) \text{ s.t.} \\ \forall (s,y,v,z)\in[t,T]\times\mathbb{R}^N\times V\times\mathbb{R}^N, \\ \eta\le (T-t)\,\mathcal{L}^v\phi(s,y)+h(z)-\phi(T,z)+\phi(t,x), \end{array}\right\},$$

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Theorem

h Lipschitz, bounded $\Longrightarrow V_h(t,x) = h^*(t,x) = \eta^*(t,x),$
$\forall (t,x) \in [0,T) \times \mathbb{R}^N$.

- $V_h(t,x) = \inf_{\gamma \in \Theta(t,x)} \int_{\mathbb{R}^N} h(z)\, \gamma\left([t,T], \mathbb{R}^N, U, dz\right),$
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Theorem

(L) V_h is the smallest lower semicontinuous viscosity supersolution and

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- (L) inf-convolution $h_n(x) = \inf_{y\in\mathbb{R}^N}\left(h(y) \wedge n + n\,|y-x|\right)$ + compactness.

What about the dual formulation? 1

- $\begin{cases} dX_s^{t,x} = 0, \text{ for } 0 \leq t \leq s \leq T = 1, \\ X_t^{t,x} = x \in \mathbb{R}. \end{cases}$, $h(\cdot) = 1_{\{0\}}(\cdot)$.

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- In particular, $V_h\left(\frac{1}{2}, 0\right) = 1$.
- $\eta^*\left(\frac{1}{2}, 0\right)$

$$= \sup \left\{ \begin{array}{c} \eta \in \mathbb{R} : \exists \phi \in C_b^{1,2}([0,1] \times \mathbb{R}) \\ \text{s.t. } \forall (s,y,z) \in \left[\frac{1}{2}, 1\right] \times \mathbb{R}^2, \\ \eta \leq \frac{1}{2}\partial_t \phi(s,y) + h(z) - \phi(1,z) + \phi\left(\frac{1}{2}, 0\right) \end{array} \right\} =$$

0.

Weak control formulation. U.s.c. case

- $\pi = \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, W, u\right)$,

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Proposition

If h is u.s.c., then $V_h(t,x) = V_h^w(t,x)$, $(t,x) \in [0,T]\times\mathbb{R}^N$.

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Proposition

If convexity and h is l.s.c., then $V_h(t,x) = V_h^w(t,x)$.

Weak control formulation. L.s.c. case

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Proposition

If convexity and h is l.s.c., then $V_h(t,x) = V_h^w(t,x)$.

- **Idea of the proof**: use inf-convolution + control rules

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Thank you !