On the relaxed Saint–Venant's problem for transversely isotropic porous elastic circular cylinder[1]

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J.W. Nunziato and S.C. Cowin, A nonlinear theory of elastic materials with voids, Arch. Rational Mech. Anal. (1979) 72, 175–201.

S.C. Cowin and J.W. Nunziato, Linear elastic materials with voids, J. Elasticity (1983) 13, 125–147.

Basic equations of the theory

The theory of elastic materials with voids has been established by Nunziato and Cowin (1979).

In this theory, the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field

$$\rho = \nu \hat{\rho}. \tag{1}$$

This representation has been previously used by Goodman and Cowin (1972) to establish a theory of granular material.

We consider a right circular cylinder of length L and radius a, occupied by a porous elastic material.

We denote by B the interior of cylinder, by ∂B the boundary of B and by $D \subset \mathbb{R}^2$ the interior of the bounded cross section.

We choose a rectangular cartesian system $Ox_1x_2x_3$ so that the Ox_3 axis is parallel with the generator of the cylinder and O is the center of one of ends. The lateral boundary of the cylinder is $\Pi = \partial D \times (0, L)$ and D_0 and D_L are, respectively, the cross sections located at $x_3 = 0$ and $x_3 = L$.

Let **u** be the displacement field over B and φ the volume distribution function. We denote by **U** the four–dimensional vector (u_i, φ).

In the theory of porous materials introduced by Cowin and Nunziato, the equations of equilibrium are

$$\begin{aligned} t_{ji,j} + f_i &= 0, \\ h_{i,i} + g + \ell &= 0. \end{aligned} \tag{2}$$

The components of the stress tensor, the components of the equilibrated stress vector and the intrinsic equilibrated body force for anisotropic porous material [1] are

$$\begin{aligned} t_{ij}(\mathbf{U}) &= C_{ijrs}e_{rs} + B_{ij}\varphi + D_{ijr}\varphi_{,r}, \\ h_i(\mathbf{U}) &= D_{rsi}e_{rs} + d_i\varphi + A_{ij}\varphi_{,j}, \\ g(\mathbf{U}) &= -B_{ij}e_{ij} - \xi\varphi - d_i\varphi_{,i}, \end{aligned} \tag{3}$$

where e_{ij} is the linear strain measure given by

$$e_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{4}$$

C_{ijrs}, B_{ij}, A_{ij}, D_{ijk}, d_i and ξ are the constitutive coefficients which satisfy the symmetry relations

$$C_{ijrs} = C_{rsij} = C_{jirs},\ A_{ij} = A_{ji},\ B_{ij} = B_{ji},\ D_{ijk} = D_{jik}. \tag{5}$$

To these equations we adjoin the following boundary conditions

$$\begin{aligned} u_i &= \widetilde{u}_i \ \text{ on } \ \overline{S}_1, \quad t_{ji} n_j = \widetilde{t}_i \ \text{ on } \ S_2, \\ \varphi &= \widetilde{\varphi} \ \text{ on } \ \overline{S}_3, \quad h_i n_i = \widetilde{h} \ \text{ on } \ S_4, \end{aligned} \tag{6}$$

where $S_r, (r = 1, 2, ..., 4)$, are parts of ∂B such that $\overline{S}_1 \cup S_2 = \overline{S}_3 \cup S_4 = \partial B$, $S_1 \cap S_2 = S_3 \cap S_4 = \emptyset$, $\widetilde{u}_i$, $\widetilde{t}_i$, $\widetilde{\varphi}$ and $\widetilde{h}$ are prescribed functions.

The necessary and sufficient conditions for the existence of a solution of the traction problem ($S_1 = \emptyset$) are given by

$$\begin{aligned} &\int_B f_i dv + \int_{\partial B} \widetilde{t}_i da = 0, \\ &\int_B \varepsilon_{ijr} x_j f_r dv + \int_{\partial B} \varepsilon_{ijr} x_j \widetilde{t}_r da = 0. \end{aligned} \tag{7}$$

The generalized plane strain state

Following Ieşan (1987) and Chiriţă (1979) we define the state of generalized plane strain for the interior of the cross section domain, $D \subset \mathbb{R}^2$, of the considered cylinder to be the state in which the displacement field **w** and the volume distribution ψ depend only on x_1 and x_2

$$w_i = w_i(x_1, x_2), \;\; \psi = \psi(x_1, x_2), \;\; (x_1, x_2) \in D. \tag{8}$$

In this case, the components of the stress tensor, the components of the equilibrated stress vector and the intrinsic equilibrated body force are function of x_1 and x_2, i.e. $T_{ij} = T_{ij}(x_1, x_2), \;\; H_i = H_i(x_1, x_2)$ and $G = G(x_1, x_2)$. Moreover, we have

$$\begin{aligned} T_{ij}(\mathbf{W}) &= C_{ijk\alpha} w_{k,\alpha} + B_{ij}\psi + D_{ij\alpha}\psi_{,\alpha}, \\ H_i(\mathbf{W}) &= A_{i\alpha}\psi_{,\alpha} + D_{r\alpha i} w_{r,\alpha} + d_i\psi, \\ G(\mathbf{W}) &= -B_{i\alpha} w_{i,\alpha} - \xi\psi - d_\alpha\psi_{,\alpha}, \end{aligned} \tag{9}$$

where $\mathbf{W} = (w_i, \, \psi)$.

Given the body force $f_i(x_1, x_2)$ and the extrinsic equilibrated force (Goodman and Cowin, 1972) $\ell(x_1, x_2)$ on D, the boundary force $\widetilde{T}_i(x_1, x_2)$ and the boundary equilibrated force $\widetilde{H}(x_1, x_2)$ on ∂D the plane strain problem for $D \cup \partial D$ consists in finding a solution $\mathbf{w}, \ \psi$ of the boundary value problem $\mathcal{S}$ defined by the equations

$$\begin{aligned} &[C_{i\alpha k\beta} w_{k,\beta} + B_{i\alpha}\psi + D_{i\alpha\beta}\psi_{,\beta}]_{,\alpha} = -f_i, \\ &[A_{\alpha\beta}\psi_{,\beta} + D_{r\beta\alpha} w_{r,\beta} + d_\alpha \psi]_{,\alpha} - B_{i\alpha} w_{i,\alpha} - \xi\psi - d_\alpha \psi_{,\alpha} = -\ell \text{ in } D, \end{aligned} \tag{10}$$

and the boundary conditions

$$\begin{aligned} \mathcal{P}_i(\mathbf{W}) &\equiv [C_{i\alpha k\beta} w_{k,\beta} + B_{i\alpha}\psi + D_{i\alpha\beta}\psi_{,\beta}] n_\alpha = \widetilde{T}_i, \\ \mathcal{D}(\mathbf{W}) &\equiv [A_{\alpha\beta}\psi_{,\beta} + D_{r\beta\alpha} w_{r,\beta} + d_\alpha\psi] n_\alpha = \widetilde{H} \text{ on } \partial D. \end{aligned} \tag{11}$$

We assume that $f_i, \ell \in C^\infty(D)$ and $\widetilde{T}_i, \widetilde{H} \in C^\infty(\partial D)$. The necessary and sufficient conditions for the existence of a solution of the generalized plane problem (Ieşan and Ciarletta, 1993) are given by

$$\int_D f_i \, da + \int_{\partial D} \widetilde{T}_i \, ds = 0, \;\; \int_D \varepsilon_{3\alpha\beta} x_\alpha f_\beta \, da + \int_{\partial D} \varepsilon_{3\alpha\beta} x_\alpha \widetilde{T}_\beta \, ds = 0, \qquad (12)$$

where ε_{ijk} is the alternating symbol. This implies that, for the existence of solution, the resultant and torque of the supply loads must be zero.

Transversal isotropic elastic materials with voids

We suppose that the axis Ox_3 is an axis of elastic symmetry and the planes normal on this axis are planes of isotropy. In the case of transverse isotropy, the mechanical response of the body remains unaffected due to arbitrary rotations about the direction of Ox_3 and due to reflections about the planes perpendicular to this direction.

For this class of materials we have only ten non-zero independent constitutive coefficients

$$\begin{gathered} c_{ij} \equiv C_{iijj}, \quad i,j \in \{1,2,3\} \text{ (not summed)}, \quad c_{11} = c_{22}, \quad c_{13} = c_{23}, \\ c_{44} \equiv C_{2323} = C_{1313}, \quad b_1 \equiv B_{11} = B_{22}, \quad b_3 \equiv B_{33}, \\ a_1 \equiv A_{11} = A_{22}, \quad a_3 \equiv A_{33} \text{ and } \xi. \end{gathered} \tag{13}$$

We note that in the case of isotropic materials with voids, the number of independent constitutive coefficients is five [1].

The constitutive equations (3), in the case of transversely isotropic materials are reduced to

$$
\begin{aligned}
t_{11}(\mathbf{U}) &= c_{11}e_{11} + c_{12}e_{22} + c_{13}e_{33} + b_1\varphi, \\
t_{22}(\mathbf{U}) &= c_{12}e_{11} + c_{22}e_{22} + c_{13}e_{33} + b_1\varphi, \\
t_{33}(\mathbf{U}) &= c_{13}e_{11} + c_{13}e_{22} + c_{33}e_{33} + b_3\varphi, \\
t_{12}(\mathbf{U}) &= (c_{11} - c_{12})e_{12}, \\
t_{13}(\mathbf{U}) &= 2c_{44}e_{13}, \\
t_{23}(\mathbf{U}) &= 2c_{44}e_{23}, \\
h_1(\mathbf{U}) &= a_1\varphi_{,1}, \\
h_2(\mathbf{U}) &= a_1\varphi_{,2}, \\
h_3(\mathbf{U}) &= a_3\varphi_{,3}, \\
g(\mathbf{U}) &= -b_1(e_{11} + e_{22}) - b_3e_{33} + \xi\varphi.
\end{aligned}
\tag{14}
$$

Throughout this paper we assume that the internal energy density

$$\mathcal{W}(\mathbf{U}) = \frac{1}{2}c_{11}e_{11}^2 + c_{12}e_{11}e_{22} + c_{13}e_{11}e_{33} + \frac{1}{2}c_{11}e_{22}^2 + c_{13}e_{22}e_{33} + \\ + \frac{1}{2}c_{33}e_{33}^2 + 2c_{44}e_{23}^2 + 2c_{44}e_{13}^2 + (c_{11} - c_{12})e_{12}^2 + b_1 e_{11}\varphi + b_1 e_{22}\varphi + \\ + b_3 e_{33}\varphi + \frac{1}{2}\xi\varphi^2 + \frac{1}{2}a_1\varphi_{,1}\varphi_{,1} + \frac{1}{2}a_1\varphi_{,2}\varphi_{,2} + \frac{1}{2}a_3\varphi_{,3}\varphi_{,3} \tag{15}$$

is a positive defined quadratic in terms of e_{ij}, φ and $\varphi_{,i}$. This is true if and only if

$$c_{11} > 0, \ \ c_{33} > 0, \ \ c_{55} > 0, \ \ \xi > 0, \ \ a_1 > 0, \ \ a_3 > 0, \\ |c_{12}| < c_{11}, \quad |c_{13}| < \sqrt{\tfrac{1}{2}\left(c_{11} + c_{12}\right)c_{33}}, \\ \left(c_{13} - \frac{b_1 b_3}{\xi}\right)^2 < \left(c_{33} - \frac{b_3^2}{\xi}\right)\left[\tfrac{1}{2}(c_{11} + c_{12}) - \frac{b_1^2}{\xi}\right]. \tag{16}$$

For a state of generalized plane strain $\mathbf{W}=(w_i(x_1,x_2),\psi(x_1,x_2))$, $(x_1,x_2)\in D$, we define the operators

$$
\begin{aligned}
\mathcal{S}_1(\mathbf{W}) &= c_{11}w_{1,11}+c_{12}w_{2,21}+\frac{1}{2}(c_{11}-c_{12})(w_{1,22}+w_{2,12})+b_1\psi_{,1},\\
\mathcal{S}_2(\mathbf{W}) &= c_{11}w_{2,22}+c_{12}w_{1,12}+\frac{1}{2}(c_{11}-c_{12})(w_{2,11}+w_{1,21})+b_1\psi_{,2},\\
\mathcal{S}_3(\mathbf{W}) &= w_{3,\alpha\alpha},\\
\mathcal{C}(\mathbf{W}) &= a_1\psi_{,\alpha\alpha}-b_1w_{\alpha,\alpha}-\xi\psi,\\
\mathcal{H}_1(\mathbf{W}) &= (c_{11}w_{1,1}+c_{12}w_{2,2}+b_1\psi)n_1+\frac{1}{2}(c_{11}-c_{12})(w_{1,2}+w_{2,1})n_2,\\
\mathcal{H}_2(\mathbf{W}) &= \frac{1}{2}(c_{11}-c_{12})(w_{2,1}+w_{1,2})n_1+(c_{11}w_{2,2}+c_{12}w_{1,1}+b_1\psi)n_2,\\
\mathcal{H}_3(\mathbf{W}) &= w_{3,\alpha}n_\alpha,\\
\mathcal{D}(\mathbf{W}) &= a_1\psi_{,\alpha}n_\alpha.
\end{aligned}
\tag{17}
$$

Construction of solution of the extension, bending and torsion problem

In this section we give the solution of the extension, bending and torsion problem. This problem for the porous elastic cylinder B consists in the determination of a solution $\mathbf{U} = (u_i, \varphi)$ of the equilibrium equations that satisfies the boundary lateral conditions

$$t_i(\mathbf{U}) = 0, \quad h(\mathbf{U}) = 0 \ \text{ on } \ \partial D \times (0, L), \tag{18}$$

and

$$\begin{gathered} \int_D t_{3\alpha}(\mathbf{U})\, da = 0 \quad \int_D t_{33}(\mathbf{U})\, da = -R_3, \\ \int_D \varepsilon_{ijk} x_j t_{3k}(\mathbf{U})\, da = -M_i, \, h(\mathbf{U}) = 0 \ \text{ on } \ x_3 = 0, \end{gathered} \tag{19}$$

where R_3 and M_i are given. We have similar condition on the end $x_3 = L$.

R.C. Batra and J.S. Yang, Saint-Venant's principle for linear elastic porous materials, J. Elasticity, **39**, 265–271, 1995.

Let us consider $\mathbf{W}^{(s)} = (\mathbf{w}^{(s)}, \psi^{(s)}),\ s = 1, 2, 3$ solutions of the problems characterized by the equations

$$\mathcal{S}_i(\mathbf{W}^{(s)}) + f_i^{(s)} = 0,\ \mathcal{C}(\mathbf{W}^{(s)}) + \ell^{(s)} = 0 \text{ in } D, \tag{20}$$

and the boundary conditions

$$\mathcal{H}_i(\mathbf{W}^{(s)}) = \widetilde{T}_i^{(s)},\ \mathcal{D}(\mathbf{W}^{(s)}) = \widetilde{H}^{(s)} \text{ on } \partial D, \tag{21}$$

where

$$\begin{aligned} f_\alpha^{(\gamma)} &= c_{13}\delta_{\alpha\gamma},\ f_3^{(\beta)} = 0,\ f_i^{(3)} = 0, \\ \ell^{(\gamma)} &= -b_3 x_\gamma,\ \ell^{(3)} = -b_3, \\ \widetilde{T}_\alpha^{(\gamma)} &= -c_{13} x_\gamma n_\alpha,\ \widetilde{T}_3^{(i)} = 0, \\ \widetilde{T}_\alpha^{(3)} &= -c_{13} n_\alpha,\ \widetilde{H}^{(i)} = 0, \end{aligned} \tag{22}$$

According to the existence results presented above, these generalized plane strain problems have solutions.

We find a solution of the extension, bending and torsion problem to be

$$\mathbf{U}^{I} = \sum_{s=1}^{4} \alpha_s \mathbf{U}^{(s)} \tag{23}$$

where the vectors $\mathbf{U}^{(s)} = (\mathbf{u}^{(s)}, \psi^{(s)}),\ s = 1, 2, 3, 4$ are defined by

$$\begin{aligned}
&u_\alpha^{(\beta)} = -\frac{1}{2}x_3^2\delta_{\alpha\beta} + w_\alpha^{(\beta)}(x_1, x_2),\\
&u_3^{(\beta)} = x_\beta x_3 + w_3^{(\beta)}(x_1, x_2),\\
&u_\alpha^{(3)} = w_\alpha^{(3)}(x_1, x_2), u_3^{(3)} = x_3 + w_3^{(3)}(x_1, x_2),\\
&u_\alpha^{(4)} = \varepsilon_{3\beta\alpha}x_\beta x_3,\quad u_3^{(4)} = 0,\\
&\varphi^{(i)} = \psi^{(i)},\ i = 1, 2, 3,\ \varphi^{(4)} = 0
\end{aligned} \tag{24}$$

and the unknown constants $\alpha_s,\ s = 1, 2, 3, 4$, are solutions of the following algebraic system

$$\sum_{s=1}^{4} \alpha_s D_{\alpha s} = \varepsilon_{3\alpha\beta} M_\beta, \sum_{s=1}^{4} \alpha_s D_{3s} = -R_3, \qquad (25)$$
$$\sum_{s=1}^{4} \alpha_s D_{4s} = -M_3.$$

with

$$\begin{aligned} D_{3s} &= \int_D t_{33}(\mathbf{U}^{(s)})\, da, \\ D_{\beta s} &= \int_D x_\beta t_{33}(\mathbf{U}^{(s)})\, da, \qquad (26) \\ D_{4s} &= \int_D \varepsilon_{3\alpha\beta} x_\alpha t_{3\beta}(\mathbf{U}^{(s)}) da. \end{aligned}$$

Because the internal energy is positive definite, we can prove [6] that

$$\det(D_{rs}) \neq 0, \qquad (27)$$

so that the system (25) uniquely determines the constants $\alpha_s,\ s = 1, 2, 3, 4$.

In what follows, we solve the three problems defined by relations (20)–(22). First, it is easy to see that $\mathbf{W}^{(3)}$ defined by

$$w_1^{(3)} = -\nu_1 x_1, \;\; w_2^{(3)} = -\nu_1 x_2, \;\; \psi^{(3)} = -\nu_2, \; w_3^{(3)} = 0, \tag{28}$$

with

$$\nu_1 = \frac{c_{13}\xi - b_1 b_3}{(c_{11} + c_{12})\xi - 2b_1}, \;\; \nu_2 = \frac{(c_{11} + c_{12})b_3 - 2c_{13}b_1}{(c_{11} + c_{12})\xi - 2b_1^2}, \tag{29}$$

is solution of the third problem.

Next, we search a solution $\mathbf{W}^{(1)}$ of the first problem in the form

$$\begin{aligned} w_1^{(1)} &= v_1^{(1)} - \frac{1}{2}\nu_1(x_1^2 - x_2^2), \\ w_2^{(1)} &= v_2^{(1)} - \nu_1 x_1 x_2, \\ w_3^{(1)} &= 0, \\ \psi^{(1)} &= \phi^{(1)} - \nu_2 x_1, \end{aligned} \tag{30}$$

where $\mathbf{V}^{(1)} = (v_1^{(1)}, v_2^{(1)}, \phi^{(1)})$ is solution of the problem defined by the equations

$$\mathcal{S}_\alpha(\mathbf{V}^{(1)}) = 0, \ \mathcal{C}(\mathbf{V}^{(1)}) = 0 \text{ in } D, \tag{31}$$

and the boundary conditions

$$\mathcal{H}_\alpha(\mathbf{V}^{(1)}) = 0, \ \mathcal{D}(\mathbf{V}^{(1)}) = a_1 \nu_2 n_1 \text{ on } \partial D. \tag{32}$$

We rewrite this problem in the polar coordinates (r, θ). We denote by u and v the components of the vector $\mathbf{v}^{(1)} = (v_1^{(1)}, v_2^{(1)})$ in polar coordinates.
The constitutive equations in polar coordinates are

$$\begin{aligned}
& t_{rr}(\mathbf{V}^{(1)}) = c_{11} e_{rr}(\mathbf{v}^{(1)}) + c_{12} e_{\theta\theta}(\mathbf{v}^{(1)}) + b_1 \phi^{(1)}, \\
& t_{\theta\theta}(\mathbf{V}^{(1)}) = c_{12} e_{rr}(\mathbf{v}^{(1)}) + c_{11} e_{\theta\theta}(\mathbf{v}^{(1)}) + b_1 \phi^{(1)}, \\
& t_{r\theta}(\mathbf{V}^{(1)}) = (c_{11} - c_{12}) e_{r\theta}(\mathbf{v}^{(1)}), \\
& h_r = a_1 \frac{\partial \phi^{(1)}}{\partial r}, \ h_r = a_1 \frac{\partial \phi^{(1)}}{\partial r}, \\
& h_\theta = a_1 \frac{1}{r} \frac{\partial \phi^{(1)}}{\partial \theta}, \\
& g = -b_1 \left[\frac{1}{r} \frac{\partial}{\partial r}(ru) + \frac{1}{r} \frac{\partial v}{\partial \theta} \right] - \xi \phi^{(1)},
\end{aligned} \tag{33}$$

where

$$\varepsilon_{rr} = \frac{\partial u}{\partial r}, \ \varepsilon_{\theta\theta} = \frac{1}{r} \left(\frac{\partial v}{\partial \theta} + u \right), \ \varepsilon_{r\theta} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{1}{r} v \right). \tag{34}$$

The equilibrium equations become

$$\begin{aligned}
\frac{\partial t_{rr}}{\partial r} + \frac{1}{r}\frac{\partial t_{r\theta}}{\partial \theta} + \frac{1}{r}(t_{rr} - t_{\theta\theta}) &= 0, \\
\frac{\partial t_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial t_{\theta\theta}}{\partial \theta} + \frac{2}{r} t_{r\theta} &= 0, \\
\frac{1}{r}\frac{\partial}{\partial r}(r h_r) + \frac{1}{r}\frac{\partial h_\theta}{\partial \theta} + g &= 0,
\end{aligned} \tag{35}$$

and the boundary conditions (21) become

$$t_{rr} = 0, \;\; t_{r\theta} = 0, \;\; h_r = \frac{1}{a} a_1 \nu_2 cos\theta. \tag{36}$$

Let us introduce the quantities

$$c_1 = \frac{1}{2}\left(1 - \frac{c_{12}}{c_{11}}\right), \; c_2 = \frac{b_1}{c_{11}}. \tag{37}$$

As in [13], we search a solution of the above problem in the following form

$$u(r,\theta) = U^{(1)}(r)\cos\theta, \; v(r,\theta) = V^{(1)}(r)\sin\theta, \; \phi(r,\theta) = \Psi^{(1)}(r)\cos\theta. \tag{38}$$

where $U^{(1)}$, $V^{(1)}$ and $\Psi^{(1)}$ are solution of the following system of differential equations

$$\begin{aligned}
&r^2\frac{d^2U^{(1)}}{dr^2} + r\frac{dU^{(1)}}{dr} - (1+c_1)U^{(1)} + r(1-c_1)\frac{dV^{(1)}}{dr} - \\
&\qquad\qquad -(1+c_1)V^{(1)} + c_2 r^2\frac{d\Psi^{(1)}}{dr} = 0, \\
&c_1\left(r^2\frac{d^2V^{(1)}}{dr^2} + r\frac{dV^{(1)}}{dr}\right) - (1+c_1)V^{(1)} - r(1-c_1)\frac{dU^{(1)}}{dr} - \\
&\qquad\qquad -(1+c_1)U^{(1)} - c_2 r\Psi^{(1)} = 0, \\
&a_1\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Psi^{(1)}}{dr}\right) - \frac{1}{r^2}\Psi^{(1)}\right] - b_1\left[\frac{1}{r}\frac{d}{dr}(rU^{(1)}) + \frac{1}{r}V^{(1)}\right] - \xi\Psi^{(1)} = 0.
\end{aligned} \tag{39}$$

For this type of system, Ieşan and Nappa (1995) give the following solution

$$
\begin{aligned}
U^{(1)} &= A_1 + Q_1 A_2 r^2 - \frac{c_2}{2p} A_3 [I_0(pr) + I_2(pr)], \\
V^{(1)} &= -A_1 - Q_2 A_2 r^2 + \frac{c_2}{2p} A_3 [I_0(pr) - I_2(pr)], \\
\Psi^{(1)} &= A_3 I_1(pr) + \frac{8 b_1 c_1}{\alpha(1 - 3c_1) p^2} A_2 r,
\end{aligned}
\tag{40}
$$

where I_n is the modified Bessel functions of order n and

$$
\begin{aligned}
p^2 &= \frac{\xi}{a_1} - \frac{b_1^2}{c_{11} a_1}, \\
Q_1 &= \frac{1}{1 - 3c_1} \left(1 - 3c_1 - \frac{3 c_1 c_2 b_1}{a_1 p^2} \right), \\
Q_2 &= \frac{1}{1 - 3c_1} \left(3 - c_1 - \frac{c_1 c_2 b_1}{a_1 p^2} \right).
\end{aligned}
\tag{41}
$$

We note that in view of relation (16) it follows that the real number p is well defined.

From the boundary conditions (36) the unknown constants A_2 and A_1 have the expression

$$\begin{aligned} A_2 &= -\frac{2c_1 b_1 I_2(pa)\nu_2 p^2 a_1(1-3c_1)}{\Gamma a^2 p^4 (1-3c_1) a_1 I_1'(pa) - 16 c_1^2 b_1^2 a I_2(pa)}, \\ A_3 &= \frac{1}{apI_1'(pa)}\left(\nu_2 - \frac{8b_1 c_1 a}{a_1 p^2 (1-3c_1)} A_2\right), \end{aligned} \tag{42}$$

where

$$\Gamma = (2c_{11} + c_{12})Q_1 - c_{12}Q_2 + \frac{8b_1^2 c_1}{a_1(1-3c_1)p^2}. \tag{43}$$

From the above discussion we can conclude that

$$
\begin{aligned}
w_1^{(1)} &= \left(U^{(1)}(r) - \frac{1}{2}\nu_1 r^2\right)\cos^2\theta - \left(V^{(1)}(r) - \frac{1}{2}\nu_1 r^2\right)\sin^2\theta, \\
w_2^{(1)} &= (U^{(1)}(r) + V^{(1)}(r) - r^2)\sin\theta\cos\theta, \\
w_3^{(1)} &= 0, \\
\psi^{(1)} &= (\Psi^{(1)}(r) - r)\cos\theta.
\end{aligned} \tag{44}
$$

is solution of the first problem defined by (20)–(22).

Similarly, we can find the solution of the second problem to be

$$
\begin{aligned}
w_1^{(2)} &= (U^{(1)}(r) + V^{(1)}(r) - r^2)\sin\theta\cos\theta, \\
w_2^{(2)} &= \left(U^{(1)}(r) - \frac{1}{2}\nu_1 r^2\right)\sin^2\theta - \left(V^{(1)}(r) - \frac{1}{2}\nu_1 r^2\right)\cos^2\theta, \\
w_3^{(2)} &= 0, \\
\psi^{(2)} &= (\Psi^{(1)}(r) - r)\sin\theta.
\end{aligned} \tag{45}
$$

From relations (24), (26), (28), (44) and (45) we obtain the components of the matrix $(D_{ij})_{4\times 4}$

$$\begin{gathered} D_{11}=D_{22}=J,\ D_{33}=E\pi a^2,\ D_{44}=\frac{c_{13}\pi a^4}{2}, \\ D_{12}=D_{21}=D_{\beta 3}=D_{3\beta}=D_{\beta 4}=D_{4\beta}=D_{43}=D_{34}=0, \end{gathered} \tag{46}$$

where

$$\begin{aligned} J &= \frac{\pi}{4}a^4 Q+\frac{\pi a}{p^2}(b_3-c_{13}c_2)A_3\left(apI_0(pa)-2I_1(pa)\right), \\ Q &= E+c_{13}(3Q_1-Q_2)A_2+\frac{8b_1b_3c_1}{a_1p^2(1-3c_1)}A_2, \\ E &= -2\nu_1c_{13}-b_3\nu_2+c_{33}. \end{aligned} \tag{47}$$

From the algebraic systems (25) we find the unknown constants α_s to be

$$\alpha_1=\frac{M_2}{J},\ \alpha_2=-\frac{M_1}{J},\ \alpha_3=-\frac{R_3}{E\pi a^2},\ \ \alpha_4=-\frac{M_3}{c_{13}\pi a^4}. \tag{48}$$

With these, we have the complete expression of solution of the problem $(\mathcal{P}_1)$. In view of the constitutive equations (14) we can observe that the solution constructed in this section corresponds to the null equilibrated stress vector's values on the ends of cylinder.

Solution of the flexure problem

We solve in the following the flexure problem. The flexure problem consist in finding a solution $\mathbf{U} = (u_i, \varphi)$ of the equilibrium equation which satisfy the following boundary conditons

$$t_i(\mathbf{U}) = 0, \quad h(\mathbf{U}) = 0 \ \text{ on } \ \partial D \times (0, L), \tag{49}$$

and

$$\begin{gathered}
\int_D t_{3\alpha}(\mathbf{U})\, da = -R_\alpha, \quad \int_D t_{33}(\mathbf{U})\, da = 0, \\
\int_D \varepsilon_{ijk} x_j t_{3k}(\mathbf{U})\, da = 0, \quad \text{pe} \quad x_3 = 0, L, \\
h(\mathbf{U}) = \begin{cases} \dfrac{\alpha}{J}(\Psi^{(1)}(r) - \nu_2 r)(R_1 \cos\theta + R_2 \sin\theta), & x_3 = 0 \\ -\dfrac{\alpha}{J}(\Psi^{(1)}(r) - \nu_2 r)(R_1 \cos\theta + R_2 \sin\theta), & x_3 = L. \end{cases}
\end{gathered} \tag{50}$$

where R_α are prescribed functions and $J, \Psi^{(1)}(r)$ and ν_2 are quantities defined in the above section.

Using the results established in the previous section, we propose the following simplified expression for a solution, $\mathbf{U}^{II} = (\mathbf{u}^{II}, \varphi^{II})$, of the problem $(\mathcal{P}_2)$

$$
\begin{aligned}
u^{II}_\alpha &= -\frac{1}{6}\beta_\alpha x_3^3 + \sum_{\gamma=1,2} x_3 \beta_\gamma w^{(\gamma)}_\alpha, \\
u^{II}_3 &= \frac{1}{2}\beta_\rho x_\rho x_3^2 + w^*_3, \\
\varphi^{II} &= \sum_{\gamma=1,2} x_3 \beta_\gamma \varphi^{(\gamma)},
\end{aligned}
\tag{51}
$$

where $\mathbf{W}^{(s)} = (\mathbf{w}^{(s)}, \psi^{(s)})$ are solutions of the generalized plane strain problem defined in the previous section, β_γ are unknown constants which will be determined, while the function w_3^* is solution of the following problem

$$\begin{aligned} \Delta w_3^* &= -\sum_{\gamma=1,2} \beta_\gamma t_{33}(\mathbf{U}^{(\beta)}) - c_{44} \sum_{\gamma=1,2} \beta_\gamma w_{\alpha,\alpha}^{(\gamma)} \text{ in } D, \\ w_{3,\alpha}^* n_\alpha &= -c_{44} \sum_{\gamma=1,2} \beta_\gamma w_\alpha^{(\gamma)} n_\alpha \text{ on } \partial D \end{aligned} \tag{52}$$

where $\Delta = \dfrac{\partial}{\partial x_1^2} + \dfrac{\partial}{\partial x_2^2}$ is the Laplace operator in two dimensions.
We can observe that, in view of (26), (46), the necessary condition for the existence of solution of the above Neumann type problem holds.

With the help of relations (23), (44) we can rewrite this problem in the polar coordinates (r, θ)

$$
\begin{aligned}
\Delta w_3^* &= (Mr + NI_1(pr))(\beta_1 \cos\theta + \beta_2 \sin\theta) \text{ in } D, \\
\frac{\partial w_3^*}{\partial r}\Big|_{r=a} &= -(U^{(1)}(a) - \frac{1}{2}\nu_1 a^2)(\beta_1 \cos\theta + \beta_2 \sin\theta),
\end{aligned}
\tag{53}
$$

where

$$
\begin{aligned}
M &= -\left\{ \left(1 + \frac{c_{13}}{c_{44}}\right) [(3Q_1 - Q_2)A_2 - 2\nu_1] + \frac{8b_1 c_1}{a_1(1 - 3c_1)c_{44}p^2} A_2 + \frac{c_{13}}{c_{44}} c_{33} + \frac{b_3}{c_{44}} \right\}, \\
N &= \left[\left(1 + \frac{c_{13}}{c_{44}}\right) c_3 - \frac{b_3}{c_{44}} \right] A_3.
\end{aligned}
\tag{54}
$$

In [6], Ghiba gives the expression of solution of this type of problem in the form

$$w_3^* = W(r)(\beta_1 \cos\theta + \beta_2 \sin\theta), \tag{55}$$

with

$$W(r) = M\frac{r^3}{8} + N\frac{I_1(pr)}{p^2} - \left[\frac{3}{4}Ma^2 + 2N\frac{I_1'(pa)}{p^2} + 2(U^{(1)}(a) - \frac{1}{2}\nu_1 a^2)\right]\frac{r}{2}. \tag{56}$$

On the other hand, in view of the end conditions characteristic to the flexure problem (see also Remark 4.2 from the paper [6]), the unknown constants β_α must satisfy the equations

$$\sum_{\gamma=1,2} \beta_\gamma D_{\alpha\gamma} = -R_\alpha. \tag{57}$$

Thus, we obtain

$$\beta_1 = -\frac{R_1}{J}, \quad \beta_2 = -\frac{R_2}{J}. \tag{58}$$

The solution of the problem $(\mathcal{P}_2)$ corresponds to the following equilibrated stress on the ends of cylinder

$$\begin{aligned} h &= \frac{a_1}{J}(\Psi^{(1)}(r) - \nu_2 r)(R_1 \cos\theta + R_2 \sin\theta) \text{ on } D_0, \\ h &= -\frac{a_1}{J}(\Psi^{(1)}(r) - \nu_2 r)(R_1 \cos\theta + R_2 \sin\theta) \text{ on } D_L. \end{aligned} \tag{59}$$

and we can observe that the resultant flux of porosity vanishes on the ends of cylinder.

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