SPATIAL BEHAVIOUR IN VISCOELASTIC MATERIALS

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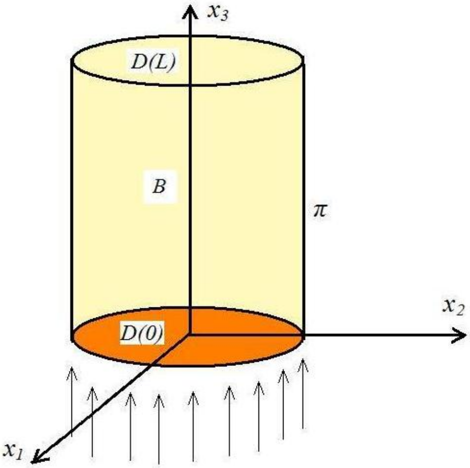
August 29, 2010



x_3
$D(L)$
B
π
x_2
$D(0)$
x_1

Introduction

Flavin and Knops 1987 (Some spatial decay estimates in continuum dynamics, *J. Elasticity*, **17**, 249–264) considered the following initial–boundary value problem

$$\begin{aligned}
&C_{ijrs}u_{r,sj} - \eta\dot{u}_i = \rho\ddot{u}_i, && \text{in } B \times [0,\infty),\\
&u_i = 0, && \text{on } (\pi \cup D(L)) \times [0,\infty),\\
&u_i = \tilde{u}_i(x_1,x_2)\exp(i\omega t), && \text{on } D(0) \times [0,\infty),\\
&u_i = u_i^0, \qquad \dot{u}_i = \dot{u}_i^0, && \text{in } B \ \text{ at } t = 0,
\end{aligned} \tag{2.1}$$

where η is a positive element simulating damping, ω is the frequency of vibrations and $i = \sqrt{-1}$.
It was established that

$$u_i = U_i(x_1,x_2,x_3,t) + v_i(x_1,x_2,x_3)\exp(i\omega t), \tag{2.2}$$

where the amplitude of vibrations v_i satisfies the boundary value problem:

$$\begin{aligned}
&C_{ijrs}v_{r,sj} + i\eta\omega v_i = -\rho\omega^2 v_i, && \text{in } B,\\
&v_i = 0, && \text{on } (\pi \cup D(L)),\\
&v_i = \tilde{u}_i(x_1,x_2), && \text{on } D(0).
\end{aligned} \tag{2.3}$$

In order to investigate the spatial behavior of the amplitude of harmonic vibrations, Flavin and Knops 1987 considered the following measure

$$H(x_3) = \int_{D(x_3)} (T_{3i}\bar{v}_i + \overline{T}_{3i}v_i)dA = \int_{B(x_3)} 2C_{ijrs}v_{i,j}\bar{v}_{r,s}dV \geq 0, \qquad (2.4)$$

where $T_{3i} = C_{3jrs}v_{r,s}$ and $B(l)$ is the cylinder slice bounded by the plane ends $x_3 = l$ and $x_3 = L$.
In the HYPOTHESES

$$\begin{aligned} &\omega \ \text{ is slower than a critical frequency,} \\ &C_{ijrs} \ \text{ is a positive definite tensor,} \end{aligned} \qquad (2.5)$$

the authors proved the following exponential estimate

$$0 \leq H(x_3) \leq H(0)\exp\left(-\frac{x_3 - h}{\nu(\omega, h)}\right), \qquad h \leq x_3 \leq L, \qquad (2.6)$$

where $\nu(\omega, h)$ is a constant.

This result was followed by other works:

1. J.N. Flavin, R.J. Knops and L.E. Payne, Decay Estimates for the Constained Elastic Cylinder of Variable Cross Section, *Quart. Appl. Math.*, 47, 325–350, 1989.
2. R.J. Knops, Spatial Decay Estimates in the Vibrating Anisotropic Elastic Beam, [in:] *Waves and Stability in Continuous Media*, (S. Rionero eds.), World Scientific, Singapore, 192–203, 1991.
3. S. Chiriţă, Spatial Decay Estimates for Solutions Describing Harmonic Vibrations in a Thermoelastic Cylinder, *J. Thermal Stresses*, 18, 421–436, 1995.
4. C. Galeş, Spatial Decay Estimates for Solutions Describing Harmonic Vibrations in the Theory of Swelling Porous Elastic Soils, *Acta Mech.*, 161, 151-164, 2003.
5. F. Passarella and V. Zampoli, Some Exponential Decay Estimates for Thermoelastic Mixtures, *J. Thermal Stresses*, 30, 25–41, 2007.

In all these papers there are used the following assumptions:

$$\begin{aligned} &a)\ \omega \ \text{ is slower than a critical frequency;} \\ &b)\ \text{the internal energy density is a positive definite quadratic form.} \end{aligned} \tag{2.7}$$

Questions:

1. Can we improve the above results?
2. What does viscoelasticity bring new into the problem of spatial behavior of harmonic vibrations?

Formulation of the problem

Let us suppose that a homogeneous and anisotropic linear viscoelastic material fills B. According to the linearized theory of isothermal viscoelasticity, the fundamental system of field equations, in the absence of the body force, consists of the strain–displacement relations

$$e_{rl} = \frac{1}{2}(u_{r,l} + u_{l,r}), \qquad \text{in } B \times (-\infty, \infty), \tag{3.1}$$

the constitutive equations

$$t_{rl} = G_{rlmn}(0)e_{mn} + \int_0^\infty \dot{G}_{rlmn}(s)e^t_{mn}(s)ds, \quad \text{in } B \times [0, \infty), \tag{3.2}$$

and the equations of motion

$$t_{rl,r} = \rho_0 \ddot{u}_l, \qquad \text{in } B \times (0, \infty), \tag{3.3}$$

where u_l are the components of the displacement vector, e_{rl} are the components of the strain tensor, t_{rl} are the components of the stress tensor and e^t_{rl} represent the history up to time t, namely $e^t_{rl}(s) = e_{rl}(t-s)$, $s \geq 0$.

ρ_0 is the constant mass density and $G_{rlmn}(t)$, $0 \leq t < \infty$, are the components of the relaxation tensor satisfying

$$G_{rlmn}(t) = G_{lrmn}(t) = G_{rlnm}(t), \qquad t \geq 0. \tag{3.4}$$

The tensor $G_{rtmn}(0)$ is called the instantaneous elastic modulus. We suppose that the relaxation tensor $G_{rlmn}(\cdot)$ has a continuous derivative $\dot{G}_{rlmn}(\cdot)$ and that the equilibrium elastic modulus

$$G_{rlmn}(\infty) = \lim_{t \to \infty} G_{rlmn}(t) \tag{3.5}$$

exists. We take $G_{rlmn}(\infty)$ to be a positive definite tensor so that the body is a solid. Moreover, we assume that the relaxation tensor is symmetric, that is

$$G_{rlmn}(t) = G_{mnrl}(t), \qquad t \geq 0. \tag{3.6}$$

We suppose that the material is compatible with thermodynamics. This problem has been discussed in various papers (Day 1972, Fabrizio and Morro 1992, Wilkes 1977 and references therein). According to Fabrizio and Morro [1992, pp. 47], it follows that the half–range Fourier sine transform

$$\dot{G}^s_{rlmn}(\omega) = \int_0^\infty \dot{G}_{rlmn}(s) \sin(\omega s) ds, \qquad \omega > 0, \tag{3.7}$$

of the function $\dot{G}_{rlmn} \in L^1([0,\infty))$ is negative definite.

Since the above assumptions upon $\dot{G}^s_{rlmn}(\omega)$ there seem to be not so transparent, we illustrate the relationship between the relaxation tensor $G_{rlmn}(t)$ and the half–range Fourier sine transform $\dot{G}^s_{rlmn}(\omega)$ of $\dot{G}_{rlmn}(t)$ by a simple example. Thus, let us consider the case of decay exponential memory with

$$G_{rlmn}(t) = G^{\infty}_{rlmn} + e^{-\alpha t} H_{rlmn}, \quad \alpha > 0, \quad H_{rlmn} \text{ and } G^{\infty}_{rlmn} \text{independent of } t$$

and G^{∞}_{rlmn} positive definite (the body is a solid). Then, the half–range Fourier sine transform is $\dot{G}^s_{rlmn}(\omega) = -\frac{\alpha\omega}{\alpha^2+\omega^2} H_{rlmn}$. Therefore, $\dot{G}^s_{rlmn}(\omega)$ is negative definite if and only if H_{rlmn} is positive definite.

The situation is better when $G_{rlmn}(t)$ is a sum of exponential functions, that is

$$G_{rlmn}(t) = G^{\infty}_{rlmn} + \sum_{P=1}^{N} e^{-\alpha_P t} H^P_{rlmn}, \quad \alpha > 0, \quad H^P_{rlmn} \text{ and } G^{\infty}_{rlmn} \text{independent of } t$$

and G^{∞}_{rlmn} positive definite. Clearly, we have $\dot{G}^s_{rlmn}(\omega) = -\sum_{P=1}^{N} \frac{\alpha_P}{\alpha_P^2+\omega^2} H^P_{rlmn}$. So that, it is not necessary that all tensors H^P_{rlmn} to be positive definite such that $-\dot{G}^s_{rlmn}(\omega)$ to be positive definite but just a linear combination of the tensors H^P_{rlmn}.

Let us consider the Cauchy's problem expressed by the relations (3.1), (3.2), (3.3), the lateral boundary conditions

$$u_l = 0\ , \qquad \text{on } \pi \times [0, \infty) \tag{3.8}$$

together with the end boundary conditions

$$u_l = \tilde{u}_l(x_1, x_2)\exp(i\omega t), \qquad \text{on } D(0) \times (0, \infty) \tag{3.9}$$

$$u_l = 0, \qquad \text{on } D(L) \times (0, \infty) \tag{3.10}$$

and the initial history condition of the displacement

$$u_l = a_l(x_1, x_2, x_3, t)\ , \qquad \text{in } B \times (-\infty, 0]. \tag{3.11}$$

In the above relations ω is a positive constant (frequency of vibration), $i = \sqrt{-1}$ is the unit complex and $\tilde{u}_l$, a_l are prescribed functions.
It is easy to see that

$$u_l = U_l(x_1, x_2, x_3, t) + v_l(x_1, x_2, x_3)\exp(i\omega t)\ , \tag{3.12}$$

where U_l (transient solution) absorbs the initial history and satisfies the null boundary conditions and the equations (3.1),(3.2) and (3.3), while v_l (amplitude of steady-state solution) satisfies the boundary value problem consisting of the field equations

$$
\begin{aligned}
&T_{rl,r} = -\rho_0\omega^2 v_l \;, \\
&T_{rl} = G_{rlmn}(0)\varepsilon_{mn} + \left(\int_0^\infty \dot{G}_{rlmn}(s)e^{-i\omega s}ds\right)\varepsilon_{mn}, \\
&\varepsilon_{rl} = \frac{1}{2}(v_{r,l} + v_{l,r}), \qquad \text{in } B,
\end{aligned}
\tag{3.13}
$$

subject to

$$v_l = 0 \;, \qquad \text{on } \pi \tag{3.14}$$

and

$$v_l = \tilde{u}_l(x_1, x_2) \;, \qquad \text{on } D(0) \tag{3.15}$$

$$v_l = 0, \qquad \text{on } D(L). \tag{3.16}$$

If we introduce the notation

$$C_{rlmn}(\omega) = G_{rlmn}(0) + \dot{G}^c_{rlmn}(\omega), \tag{3.17}$$

where $\dot{G}^c_{rlmn}(\omega)$ is the half–range Fourier cosine transform of the function $\dot{G}_{rlmn} \in L^1([0,\infty))$, i.e.

$$\dot{G}^c_{rlmn}(\omega) = \int_0^\infty \dot{G}_{rlmn}(s)\cos(\omega s)ds, \tag{3.18}$$

then from (3.7) and (3.13) we deduce the following system of partial differential equations

$$\left(C_{rlmn}(\omega) - i\dot{G}^{s}_{rlmn}(\omega)\right)v_{n,mr} + \rho_0\omega^2 v_l = 0. \tag{3.19}$$

For later convenience, we note that T_{rl} may be written in the form

$$T_{rl} = \left(C_{rlmn}(\omega) - i\dot{G}^{s}_{rlmn}(\omega)\right)\varepsilon_{mn}. \tag{3.20}$$

U_l represents essentially the transient and $v_l \exp(i\omega t)$ is the forced oscillation.
It is easy to prove that $U_l \to 0$ when $t \to \infty$, so the cylinder has a vibratory motion.
In what follows we will study the spatial behavior of the amplitude of the steady-state vibration satisfying (3.13) (or (3.19)) under the boundary conditions (3.14), (3.15) and (3.16).

Spatial behaviour

Let us introduce the following cross-sectional functional

$$I(x_3) = -\int_{D(x_3)} \left(T_{r3}\overline{iv}_r + \overline{T}_{r3} i v_r\right) dA$$
$$= \int_{D(x_3)} \left[iC_{r3ml}(\omega)\left(v_{l,m}\overline{v}_r - \overline{v}_{l,m}v_r\right) + \dot{G}^s_{r3ml}(\omega)\left(v_{l,m}\overline{v}_r + \overline{v}_{l,m}v_r\right)\right] dA \,, \tag{4.1}$$

where the superposed bar denotes complex conjugate. By using the symmetry assumption of G_{rlmn} and negative definiteness property of the tensor $\dot{G}^s_{rlmn}(\omega)$ we prove that $I(\cdot)$ is an acceptable measure of the amplitude of steady-state vibrations that decays more rapidly than an exponential of the distance from the excited end of the cylinder.
Thus, from (4.1) by direct differentiation, and by using of the equations (3.19), the boundary conditions (3.14), and the divergence theorem, after a short calculation we obtain

$$\frac{dI}{dx_3}(x_3) = 2\int_{D(x_3)} \dot{G}^s_{rnml}(\omega) v_{l,m}\overline{v}_{r,n} dA \,. \tag{4.2}$$

Now, since the fourth–order tensor $-\dot{\mathbf{G}}^s(\omega)$ is symmetric and positive definite then there exist two positive constants depending on ω, denoted by ν^m and ν^M such that

$$\nu^m(\omega)\zeta_{lr}\overline{\zeta}_{lr} \leq -\dot{G}^s_{lrnp}(\omega)\zeta_{lr}\overline{\zeta}_{np} \leq \nu^M(\omega)\zeta_{lr}\overline{\zeta}_{lr}, \tag{4.3}$$

for every symmetric second order tensor ζ_{lr}. From (4.2) and (4.3) we deduce

$$\frac{dI}{dx_3}(x_3) \leq -2\nu^m(\omega)\int_{D(x_3)} \varepsilon_{lr}\overline{\varepsilon}_{lr}dA \leq 0\ , \tag{4.4}$$

where ε_{lr} is defined by $(3.13)_3$, so that $I(\cdot)$ is a non–increasing function. We also note that $I(L) = 0$, so that $I(x_3) \geq 0$ for all $x_3 \in [0, L]$. Now, integrating the above equation between x_3 and $x_3 + h$, where $x_3 + h \leq L$, we deduce

$$I(x_3 + h) - I(x_3) \leq -2\nu^m(\omega)\int_{B(x_3,h)} \varepsilon_{lr}\overline{\varepsilon}_{lr}dA \leq 0\ , \tag{4.5}$$

where $B(l, h)$ is the cylinder slice bounded by the planes $x_3 = l$ and $x_3 = l + h$. Setting $x_3 + h = L$, we obtain

$$I(x_3) \geq 2\nu^m(\omega)\int_{B(x_3)} \varepsilon_{lr}\overline{\varepsilon}_{lr}dA \geq 0\ . \tag{4.6}$$

On the other hand, letting σ be a positive parameter and using Schwarz and arithmetic-geometric mean inequalities one deduces

$$I(x_3) \leq \int_{D(x_3)} |T_{r3}\overline{iv}_r + \overline{T}_{r3}iv_r|dA \leq \sigma \int_{D(x_3)} T_{rl}\overline{T}_{rl}dA + \frac{1}{\sigma}\int_{D(x_3)} v_r\overline{v}_r dA \ . \tag{4.7}$$

Let us denote by $c^2_{_M}(\omega)$ the largest eigenvalue of the symmetric and positive semi–definite tensor $B_{snpq}(\omega) = C_{rlsn}(\omega)C_{rlpq}(\omega)$, then using the relation

$$T_{rl} = \left(C_{rlmn}(\omega) - i\dot{G}^s_{rlmn}(\omega)\right)\varepsilon_{mn}.$$

it may be established that

$$T_{rl}\overline{T}_{rl} = T'_{rl}T'_{rl} + T''_{rl}T''_{rl} \leq \left(c_{_M}(\omega) + \nu^M(\omega)\right)^2 \varepsilon_{rl}\overline{\varepsilon}_{rl}. \tag{4.8}$$

Further, using the boundary condition $v_l = 0 \text{ on } \pi$ and following a result established by Toupin 1965 concerning the Saint-Venant's principle one deduces the following inequalities

$$\begin{aligned} -\int_{B(x_3,h)} \dot{G}^s_{rnml}(\omega)\varepsilon'_{rn}\varepsilon'_{ml}dV \geq \rho_0 w_0^2(h) \int_{B(x_3,h)} v'_r v'_r dV, \\ -\int_{B(x_3,h)} \dot{G}^s_{rnml}(\omega)\varepsilon''_{rn}\varepsilon''_{ml}dV \geq \rho_0 w_0^2(h) \int_{B(x_3,h)} v''_r v''_r dV, \end{aligned} \tag{4.9}$$

where Υ' and Υ'' the real and imaginary parts of the complex quantity Υ, and $\frac{w_0(h)}{2\pi}$ is the lowest frequency of vibration of the cylinder $B(x_3, h)$ filled by an elastic material whose components of the constant elasticity tensor are $-\dot{G}^s_{rnml}(\omega)$ and whose lateral surface is clamped and plane ends are free. From (4.9) we deduce the following Poincaré type inequality

$$\rho_0 w_0^2(h) \int_{B(x_3,h)} v_r \bar{v}_r dV \leq -\int_{B(x_3,h)} \dot{G}^s_{rnml}(\omega)\varepsilon_{rn}\bar{\varepsilon}_{ml}dV. \tag{4.10}$$

The relations (4.3), (4.7), (4.8), (4.10) assure that the function

$$Q(x_3,h) = \frac{1}{h}\int_{x_3}^{x_3+h} I(y)dy, \qquad (4.11)$$

satisfies the inequality

$$Q(x_3,h) \leq \left(\frac{\sigma\left(c_{_M}(\omega)+\nu^M(\omega)\right)^2}{h} + \frac{\nu^M(\omega)}{\sigma h\rho_0 w_0^2(h)}\right)\int_{B(x_3,h)} \varepsilon_{rl}\overline{\varepsilon}_{rl}dV. \qquad (4.12)$$

On the other hand, from (4.4) and (4.5) we deduce

$$\frac{\partial Q}{\partial x_3}(x_3,h) = \frac{1}{h}\Big(I(x_3+h) - I(x_3)\Big) \leq -\frac{2\nu^m}{h}\int_{B(x_3,h)} \varepsilon_{rl}\overline{\varepsilon}_{rl}dV. \qquad (4.13)$$

Thus, from (4.12) and (4.13) one obtains

$$\gamma(\sigma,h,\omega)\ \frac{\partial Q}{\partial x_3}(x_3,h) + Q(x_3,h) \leq 0, \qquad (4.14)$$

where

$$\gamma(\sigma,h,\omega) = \frac{\sigma\left(c_{_M}(\omega)+\nu^M(\omega)\right)^2}{2\nu^m(\omega)} + \frac{\nu^M(\omega)}{2\sigma\rho_0 w_0^2(h)\nu^m(\omega)}. \qquad (4.15)$$

Setting the parameter σ such that the above quantity to be minimum, that is

$$\begin{aligned} \sigma_0 &= \frac{1}{c_{_M}(\omega)+\nu^M(\omega)}\sqrt{\frac{\nu^M(\omega)}{\rho_0 w_0^2(h)}}, \\ \gamma_m(h,\omega) &= \gamma(\sigma_0,h,\omega) = \frac{c_{_M}(\omega)+\nu^M(\omega)}{\nu^m(\omega)}\sqrt{\frac{\nu^M(\omega)}{\rho_0 w_0^2(h)}}, \end{aligned} \tag{4.16}$$

from (4.14), we deduce

$$Q(x_3,h) \le Q(0,h)\exp\left(-\frac{x_3}{\gamma_m(h,\omega)}\right). \tag{4.17}$$

Moreover, since $I(\cdot)$ is a non–increasing function on $[0,L]$, we have

$$I(x_3+h) \le Q(x_3,h) \le I(x_3). \tag{4.18}$$

The above relations lead to the following:

Theorem. *The cross–sectional function $I(x_3)$ defined by (4.1) in connection with the boundary value problem (3.13)-(3.16) is an acceptable measure of solution and satisfies the following decay estimate*

$$0 \le I(x_3) \le I(0)\exp\left(-\frac{x_3-h}{\gamma_m(h,\omega)}\right), \qquad h \le x_3 \le L. \tag{4.19}$$

The results may be easily extended to a semi–infinite cylinder, namely the case when $L \to \infty$. Then there are possible the only two possibilities:
a) $I(x_3) \geq 0$ for all $x_3 \in [0, \infty)$; or
b) there exists $x_3^* \in [0, \infty)$ such that $I(x_3^*) < 0$.
In the case a), since $I(\cdot)$ is non–negative, we deduce the same estimate.
Let us consider the case b). Since $I(\cdot)$ is a non–increasing function it follows that

$$I(x_3) < 0, \qquad x_3^* \leq x_3 < \infty, \tag{4.20}$$

so that, on $[x_3^*, \infty)$, we must change the sign of $I(x_3)$ in the relation (4.7).
Repeating the reasoning one deduces the following estimate

$$-I(x_3) \geq -I(x_3^*) \exp\left(\frac{x_3 - h - x_3^*}{\gamma_m(h, \omega)}\right) > 0, \qquad x_3 \in [x_3^* + h, \infty). \tag{4.21}$$

Thus, we have the following Phragmèn–Lindelöf alternative.
Theorem. *In the context of a semi–infinite viscoelastic cylinder for which* $-\dot{G}^s_{rlmn}(\omega)$ *is positive definite the following alternative holds: a) either* $I(\cdot)$ *is a non–negative function on* $[0, \infty)$ *which decays spatially faster than the exponential* $\exp\left(-\frac{x_3 - h}{\gamma_m(h,\omega)}\right)$*; or b) there exists* $x_3^* \in [0, \infty)$ *such that* $I(x_3^*) < 0$*, and then* $-I(x_3)$ *grows spatially faster than the exponential* $\exp\left(\frac{x_3 - h - x_3^*}{\gamma_m(h,\omega)}\right)$.

Concluding remarks

i) Within the framework of linear viscoelasticity, we studied the spatial behavior of solutions describing harmonic vibrations in a right cylinder. Thus, for a cylinder of finite extent, an exponential decay estimate in terms of the distance from the excited end of the cylinder was obtained. In the case of a semi–infinite cylinder, an alternative of Phragmén–Lindelöf type was established.

ii) The dissipative mechanism guarantees the validity of result for every value of the frequency of vibration and for the class of viscoelastic materials compatible with thermodynamics. We utilized the fact that half–range Fourier sine transform $\dot{G}^s_{rlmn}(\omega)$ is negative definite.

iii) By relaxing the above condition, namely assuming that $-\dot{G}^s_{rlmn}(\omega)$ is strongly elliptic, we obtained results addressing to a large class of materials including those new materials with extreme and unusual physical properties like negative Poisson's ratio (that is, so called auxetic materials). These are materials with heterogeneous structure, including natural viscoelastic composites such as bone, ligament, and wood, as well as synthetic composites, biomaterials, and cellular solids with structural hierarchy.

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