Convolution equations with almost periodic solutions

Silvia -Otilia Corduneanu

Gh. Asachi Technical University of Iași
Romania

$$f(x) = M_y[f(x-y)\mu(y)] + \nu * h(x), \quad x \in \mathbb{R} \tag{1}$$

$\mu \in ap(\mathbb{R}), \quad h \in AP(\mathbb{R}), \quad \nu \in m_F(\mathbb{R})$

Consider the case

$$\mu = g\lambda, \quad \nu = \varphi\lambda$$

$$g \in AP(\mathbb{R}), \quad \varphi \in L^1(\mathbb{R})$$

$$f(x) = \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} f(x-y)g(y)\,\mathrm{d}y + \int_{-\infty}^{+\infty} \varphi(y)h(x-y)\,\mathrm{d}y, \quad x \in \mathbb{R} \tag{2}$$

$$\mu \in m_B(G): \qquad (\forall A \subseteq G)\left(\sup_{x\in G} \mid \mu \mid (xA) < \infty\right)$$

DEFINITION 1. The measure $\mu \in m_B(G)$ is said to be an *almost periodic measure*, if for every $f \in K(G)$, $f * \mu \in AP(G)$

$$\begin{cases} \Big(ap(G), \{\| \cdot \|_f\}_{f\in K(G)}\Big) \\ \|\mu\|_f = \|f * \mu\|, \quad f \in K(G), \quad \mu \in m_B(G) \end{cases}$$

$$\begin{cases} \exists M : ap(G) \to \mathbb{C} \\ M(\delta_x * \mu) = M(\mu), \quad x \in G, \quad \mu \in ap(G) \\ M(\lambda) = 1 \end{cases}$$

L.N. Argabright and J. Gil de Lamadrid,
Fourier Analysis of Unbounded Measures on Locally Compact Abelian Groups,
Mem. Amer. Math. Soc 145 (1974)

L.N. Argabright and J. Gil de Lamadrid,
Almost Periodic Measures,
Mem. Amer. Math. Soc. 428 (1990)

J. Gil de Lamadrid,
Sur les Mesures Presque Périodiques,
Astérisque 4, 1973, 61-89

$$f \in AP(G), \quad \mu \in ap(G) \Longrightarrow f\mu \in ap(G)$$

$$\begin{cases} \Phi : G \to \mathbb{C}, \\ \\ \Phi(x) = M_y[f(xy^{-1})g(y)\mu(y)] \end{cases} \tag{3}$$

$$\begin{aligned} &| \, M_y(f(xy^{-1})g(y)\mu(y)) \, |^2 \leq \\ &\leq M_y(| \, f(xy^{-1}) \, |^2 \, \mu(y))M(| \, g \, |^2 \, \mu) \end{aligned} \tag{4}$$

Denote by $\widehat{G}$ the dual of G

$$\widehat{G} \subset AP(G) \tag{5}$$

$$c_\gamma(f) = M(\overline{\gamma} f), \quad \gamma \in \widehat{G}, \quad f \in AP(G) \tag{6}$$

$$\{\gamma \in \widehat{G} | M(\overline{\gamma} f) \neq 0\} = \{\gamma_n \in \widehat{G} | \; n \in \mathbb{N}\} \tag{7}$$

$$f \sim \sum_{n=1}^{\infty} c_{\gamma_n}(f) \gamma_n \tag{8}$$

$$\begin{cases} \mu \in ap(G), \quad \gamma \in \widehat{G}, \\ c_\gamma(\mu) = M(\overline{\gamma} \mu). \end{cases} \tag{9}$$

$$L(y) = \int_A f(x-y)\,\mathsf{d}\lambda(x), \quad y \in \mathbb{R} \tag{10}$$

$$\begin{aligned} &\int_{-t}^{t} M_y[f(x-y)\mu(y)]\,\mathsf{d}x \\ &= M_y\left[\left(\int_{-t}^{t} f(x-y)\,\mathsf{d}x\right)\mu(y)\right] \end{aligned} \tag{11}$$

$$\Phi(x) = M_y[f(x-y)\mu(y)], \quad x \in \mathbb{R} \tag{12}$$

$$M(\Phi) = M(f)M(\mu) \tag{13}$$

$$c_\gamma(\Phi) = c_\gamma(f)c_\gamma(\mu), \quad \gamma \in \widehat{\mathbb{R}} \tag{14}$$

$$f(x) = M_y[f(x-y)\mu(y)]$$
$$+\nu * h(x), \quad x \in \mathbb{R} \tag{15}$$

$$h \sim \sum_{n=1}^{\infty} c_{\gamma_n}(h)\gamma_n \tag{16}$$

$$\nu \in m_F(\mathbb{R}), \quad \sum_{n=1}^{\infty} |\widehat{\nu}(\gamma_n)|^2 < \infty. \tag{17}$$

$$(\exists \delta > 0)(\forall n \in \mathbb{N})(|\, c_{\gamma_n}(\mu) - 1\,| > \delta) \tag{18}$$

$$f \sim \sum_{n=1}^{\infty} \frac{\widehat{\nu}(\gamma_n) c_{\gamma_n}(h)}{1 - c_{\gamma_n}(\mu)} \gamma_n \tag{19}$$

$$\mu = g\lambda, \quad \nu = \varphi\lambda$$

$$g \in AP(\mathbb{R}), \quad \varphi \in L^1(\mathbb{R})$$

$$\begin{aligned} f(x) &= \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} f(x-y)g(y)\,\mathrm{d}y \\ &+ \int_{-\infty}^{\infty} h(x-y)\varphi(y)\,\mathrm{d}y, \quad x \in \mathbb{R} \end{aligned} \tag{20}$$

$$\begin{aligned} f(x) &= \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} f(x-y)\,\mathrm{d}y \\ &+ \int_{0}^{2\pi} h(x-y)\,\mathrm{d}y, \quad x \in \mathbb{R} \end{aligned} \tag{21}$$

$$f(x) = M_y \left[f^p \left(x y^{-1} \right) \mu(y) \right], \quad x \in G \quad (22)$$

$$
\begin{aligned}
&f \in AP_+^\circ(G) \Longleftrightarrow \\
&(\exists \eta > 0)(\forall x \in G)(f(x) \geq \eta)
\end{aligned}
\tag{23}
$$

Proposition 1 *$AP_+^\circ(G)$ is a complete metric space with respect to the metric*

$$d(f,g) = \max\left\{\ln M\left(\frac{f}{g}\right), -\ln m\left(\frac{f}{g}\right)\right\} \tag{24}$$

where

$$M\left(\frac{f}{g}\right) = \inf\{\beta > 0 \mid f \leq \beta g\}. \tag{25}$$

$$m\left(\frac{f}{g}\right) = \sup\{\alpha > 0 \mid \alpha g \leq f\}. \tag{26}$$

A.C.Thompson,
On certain contraction mappings in a partially ordered vector space,
Proc. Amer. Math. Soc.,
14 (1963), p. 438-443

$$T : AP_+^\circ(G) \to AP_+(G)$$

$$Tf(x) = M_y\left[f^p\left(xy^{-1}\right)\mu(y)\right] \tag{27}$$

$$f \in AP_+^\circ(G) \Longrightarrow Tf \in AP_+^\circ(G) \tag{28}$$

$$d\left(Tu, Tv\right) \leq p\, d(u, v). \tag{29}$$

$$f(x) = M_y\left[f^p\left(xy^{-1}\right)\mu(y)\right], \quad x \in G \tag{30}$$

$$f^2(x) = \\ = \lim_{n\to\infty} \frac{1}{\lambda(H_n)} \int_{H_n} f(xy^{-1}) h(y) d\lambda(y) \tag{31}$$

$$\varphi(x) = \\ = \lim_{n\to\infty} \frac{1}{\lambda(H_n)} \int_{H_n} \varphi^{\frac{1}{2}}(xy^{-1}) h(y) d\lambda(y) \tag{32}$$

$$h \in AP_+(G), \quad M(h) > 0 \Longrightarrow \\ \mu = h\lambda \in ap_+(G) \tag{33}$$

$$M_y \left[\varphi^{\frac{1}{2}} \left(xy^{-1}\right) \mu(y)\right] \\ = \lim_{n\to\infty} \frac{1}{\lambda(H_n)} \int_{H_n} \varphi^{\frac{1}{2}}(xy^{-1}) h(y) d\lambda(y) \tag{34}$$

$$G = \bigcup_{n=1}^{\infty} H_n$$

$$\lim_{n\to\infty} \frac{\lambda(xH_n \Delta H_n)}{\lambda(H_n)} = 0, \quad x \in G. \tag{35}$$

$$f(x) = g(x) + \nu * f(x) + \\ + M_y[f(xy^{-1})\mu(y)]. \tag{36}$$

$$\nu(G) + M(\mu) < 1 \tag{37}$$

$U : AP(G) \to AP(G)$,

$$Uf(x) = g(x) + \nu * f(x) + M_y[f(xy^{-1})\mu(y)], \\ f \in AP(G), \quad x \in G. \tag{38}$$

$$\begin{cases} f_0 = g, \\ f_n = Uf_{n-1}, \quad n \in \mathbb{N}^*. \end{cases} \tag{39}$$

$$\inf_{x \in G} g(x) > 0 \Longrightarrow \inf_{x \in G} f(x) > 0 \tag{40}$$

$$M(f) = \frac{M(g)}{1 - \nu(G) - M(\mu)} \tag{41}$$

$$M(\nu * f) = \nu(G)M(f). \tag{42}$$

$$M(\nu * f) =$$

$$= \lim_{n\to\infty} \frac{1}{\lambda(H_n)} \int_{H_n} \int_G f(xy^{-1}) d\nu(y) d\lambda(x). \tag{43}$$

$$\lim_{n\to\infty} \int_G \left[\frac{1}{\lambda(H_n)} \int_{H_n} f(xy^{-1}) d\lambda(x) \right] d\nu(y) =$$

$$= \int_G \left[\lim_{n\to\infty} \frac{1}{\lambda(H_n)} \int_{H_n} f(xy^{-1}) d\lambda(x) \right] d\nu(y) =$$

$$= \nu(G)M(f). \tag{44}$$

$$L(x) = M_y[f(xy^{-1})\mu(y)], \quad x \in G \tag{45}$$

$$M(L) = M(f)M(\mu). \tag{46}$$

$$f \in \widehat{G}, \quad f \in [\widehat{G}], \quad f \in AP(G) \tag{47}$$

$$f(x) = g(x) + \int_0^{\beta} f(x-y)dy + \\ + \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} f(x-y)h(y)dy \tag{48}$$

$$f(x) = g(x) + \nu * f(x) + \\ + M_y[f(xy^{-1})\mu(y)]. \tag{49}$$

$$\int_0^{\beta} f(x-y)dy = \nu * f(x), \quad x \in \mathbb{R} \tag{50}$$

$$\nu = \mathbf{1}_{[0,\beta]}\theta \tag{51}$$

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} f(x-y)h(y)dy = \\ &= M_y[f(xy^{-1})\mu(y)] \end{aligned} \tag{52}$$

$$\mu = h\theta \tag{53}$$

$$\nu(\mathbb{R}) + M(\mu) = \beta + M(h) < 1 \tag{54}$$

$$M(f) = \frac{M(g)}{1-\beta-M(h)} > M(g). \tag{55}$$