Analyse d'une classe d'inéquations d'évolution implicites et applications à des problèmes quasi-statiques de contact

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Perspectives

- Nonlocal friction law in the static case - G. Duvaut, J.T. Oden and co-workers (1983)
- J. Martins, J. T. Oden (1985, 1987) proposed and studied normal compliance laws
- Nonlocal friction law in the quasistatic case - M. Cocou, E. Pratt, M. Raous (1995, 1996)
- Numerical analysis of variational inequalities - R. Glowinski, J.L. Lions, R. Trémolières (1976)
- Numerical analysis of static contact problems - A. Capatina, M. Cocou (1991)
- Numerical analysis of quasistatic contact problems - A. Capatina, M. Cocou, M. Raous (2009)
- W. Han and M. Sofonea (2002) studied contact problems for (visco)elastic and elastic-viscoplastic bodies
- Schwarz methods for elliptic variational and quasivariational inequalities - L. Badea and co-workers (2003, 2006, 2008)

1. Analysis of a system of implicit variational inequalities

Let $(V, \langle . , . \rangle, \| \cdot \|)$, and $(H, (\cdot, \cdot)_H, \| \cdot \|_H)$ be two real Hilbert. Let $F : V \times V \to \mathbb{R}$ be a Gâteaux differentiable functional on V and assume that there exist two constants $\alpha,\ \beta > 0$ for which

$$\alpha \|v - u\|^2 \le \langle F'(v) - F'(u), v - u \rangle \tag{1}$$

and

$$\|F'(v) - F'(u)\|_{V'} \le \beta \|v - u\| \tag{2}$$

for all $u, v \in V$, where F' is the Gâteaux derivative of F. Let K be a closed convex cone contained in V with its vertex at 0 and let $(K(g))_{g \in V}$ be a family of nonempty closed convex subsets of K satisfying the following conditions: $0 \in K(0)$ and

$$\text{if } g_n \to g \text{ in } V,\ v_n \in K(g_n) \text{ and } v_n \rightharpoonup v \text{ in } V \text{ then } v \in K(g). \tag{3}$$

We assume that for all $g \in V$ there exists an operator $\gamma(g,\cdot) : K(g) \to H$ such that $\gamma(0,0) = 0$,

$$\begin{aligned} &\text{if } \; g_n \to g \; \text{ in } V, \; v_n \in K(g_n) \text{ and } \; v_n \rightharpoonup v \text{ in } V \\ &\text{then } \; \gamma(g_n, v_n) \to \gamma(g,v) \; \text{ in } H \end{aligned} \tag{4}$$

and for all $g_i \in V$, $v_i \in K(g_i)$, $i = 1, 2$,

$$\|\gamma(g_1,v_1) - \gamma(g_2,v_2)\|_H \leq k_1(\|g_1 - g_2\| + \|v_1 - v_2\|). \tag{5}$$

$$j(g,v,\cdot) \text{ is sequentially weakly continuous on } V \quad \forall\, g \in V,\, v \in K(g), \tag{6}$$

$$j(g,v,\cdot) \; \text{ is sub-additive for all } \; g \in V,\, v \in K(g), \text{ that is} \tag{7}$$

$$j(g,v,w_1 + w_2) \leq j(g,v,w_1) + j(g,v,w_2) \quad \forall\, g,\, w_{1,2} \in V,\, v \in K(g),$$

$$j(g,v,\cdot) \; \text{ is positively homogeneous for all } \; g \in V,\, v \in K(g), \tag{8}$$

that is $j(g,v,\theta w) = \theta j(g,v,w) \quad \forall\, g,\, w \in V,\, v \in K(g),\, \theta \geq 0,$

$$j(0,0,w) = 0 \quad \forall\, w \in V, \tag{9}$$

and there exists $k_2 > 0$ such that

$$
\begin{aligned}
&|j(g_1, v_1, w_2) + j(g_2, v_2, w_1) - j(g_1, v_1, w_1) - j(g_2, v_2, w_2)| \\
&\le\ k_2(\|g_1 - g_2\| + \|\gamma(g_1, v_1) - \gamma(g_2, v_2)\|_H)\|w_1 - w_2\| \\
&\qquad \forall\, g_i,\, w_i \in V,\, v_i \in K(g_i),\, i = 1,\, 2.
\end{aligned}
\tag{10}
$$

We assume that k_1 and k_2 satisfy the following condition:

$$
k_1 k_2 < \alpha. \tag{11}
$$

For all $g \in V$, we consider a functional $b(g, \cdot, \cdot) : K(g) \times V \to \mathbb{R}$ which satisfies the following conditions:

$$
\forall\, g \in V,\, v \in K(g),\ b(g, v, \cdot) \ \text{ is linear and continuous on } V \tag{12}
$$

and

$$
\begin{aligned}
&|b(g_1, v_1, w) - b(g_2, v_2, w)| \le k_b(\|g_1 - g_2\| \\
&\qquad + \|v_1 - v_2\|)\|w\| \quad \forall\, g_i,\, w \in V,\, v_i \in K(g_i),\, i = 1,\, 2,
\end{aligned}
\tag{13}
$$

Let $f \in W^{1,2}(0, T; V)$ be given and $u_0 \in K(f(0))$ be the unique solution of the following implicit elliptic variational inequality:

$$
\langle F'(u_0), w - u_0 \rangle + j(f(0), u_0, w) - j(f(0), u_0, u_0) \ge 0 \quad \forall\, w \in K. \tag{14}
$$

We consider the following evolution system of coupled variational inequalities.

Problem P: Find $u \in W^{1,2}(0,T;V)$ such that

$$(P) \quad \begin{cases} u(0) = u_0,\, u(t) \in K(f(t)) \quad \forall\, t \in]0,T[, \\ \langle F'(u(t)), v - \dot{u}(t)\rangle + j(f(t), u(t), v) - j(f(t), u(t), \dot{u}(t)) \\ \qquad \geq b(f(t), u(t), v - \dot{u}(t)) \quad \forall\, v \in V \text{ a.e. on }]0,T[, \\ b(f(t), u(t), w - u(t)) \geq 0 \quad \forall\, w \in K,\ \forall\, t \in]0,T[. \end{cases}$$

We approximate problem P by using an implicit time discretization scheme. For $\nu \in N^*$, we set $\Delta t := T/\nu$, $t_\iota := \iota\, \Delta t$ and $K^\iota := K(f(t_\iota))$, $\iota = 0, 1, ..., \nu$. If θ is a continuous function of $t \in [0,T]$ valued in some vector space, we use the notations $\theta^\iota := \theta(t_\iota)$ unless $\theta = u$, and if ζ^ι, $\forall\, \iota \in \{0, 1, ..., \nu\}$, are elements of some vector space, then we set

$$\partial\zeta^\iota := \frac{\zeta^{\iota+1} - \zeta^\iota}{\Delta t} \quad \forall\, \iota \in \{0, 1, ..., \nu - 1\}.$$

We denote $u^0 := u_0$ and we approximate (P) using the following sequence of incremental problems $(P^\iota_\nu)_{\iota=0,1,\ldots,\nu-1}$.

Problem $\mathbf{P}^\iota_\nu$: Find $u^{\iota+1} \in K^{\iota+1}$ such that

$$(P^\iota_\nu)\quad \begin{cases} \langle F'(u^{\iota+1}), v - \partial u^\iota\rangle + j(f^{\iota+1}, u^{\iota+1}, v) - j(f^{\iota+1}, u^{\iota+1}, \partial u^\iota) \\ \qquad\qquad \geq b(f^{\iota+1}, u^{\iota+1}, v - \partial u^\iota) \quad \forall\, v \in V, \\ b(f^{\iota+1}, u^{\iota+1}, w - u^{\iota+1}) \geq 0 \quad \forall\, w \in K. \end{cases}$$

It is easily seen that for all $\iota \in \{0, 1, \ldots, \nu - 1\}$ the problem P^ι_ν is equivalent to each of the following variational inequalities: find $u^{\iota+1} \in K^{\iota+1}$ such that

$$(Q^\iota_\nu)\quad \begin{cases} \langle F'(u^{\iota+1}), w - u^{\iota+1}\rangle + j(f^{\iota+1}, u^{\iota+1}, w - u^\iota) \\ \qquad\qquad\qquad -j(f^{\iota+1}, u^{\iota+1}, u^{\iota+1} - u^\iota) \geq 0 \quad \forall\, w \in K, \end{cases}$$

Lemma 1 *Let $u^{\iota+1}$ be the solution of (Q_ν^ι), $\iota \in \{0,1,...,\nu-1\}$. Then*

$$\|u^0\| \leq M_0\|F'(0)\| + M_1\|f^0\|, \;\; \|u^{\iota+1}\| \leq M_0\|F'(0)\| + M_1\|f^{\iota+1}\|, \quad (15)$$

$$\|u^{i+1} - u^i\| \leq M_1\|f^{\iota+1} - f^\iota\|, \quad (16)$$

$$\sum_{\iota=0}^{\nu-1} \|u^{i+1} - u^i\|^2 \leq M_1^2 \Delta t \int_0^T \|\dot{f}(\tau)\|^2 \, d\tau, \quad (17)$$

where

$$M_0 = \frac{1}{\alpha - k_1 k_2}, \quad M_1 = \frac{(k_1+1)k_2}{\alpha - k_1 k_2}. \quad (18)$$

Now, if we define

$$\begin{cases} u_\nu(0) = \widehat{u}_\nu(0) = u^0, \; f_\nu(0) = f^0 \;\; \text{and} \\ \forall\, \iota \in \{0,1,...,\nu-1\}, \; \forall\, t \in]t_\iota, t_{\iota+1}], \\ \; u_\nu(t) = u^{\iota+1}, \; \widehat{u}_\nu(t) = u^\iota + (t - t_\iota)\partial u^\iota, \; f_\nu(t) = f^{\iota+1}, \end{cases}$$

then for all $\nu \in N^*$ the sequence of inequalities $(P_\nu^\iota)_{\iota=0,1,\ldots,\nu-1}$ is equivalent to the following incremental formulation: for almost every $t \in [0,T]$

$$
(P_\nu)\begin{cases} u_\nu(t) \in K(f_\nu(t)),\ \langle F'(u_\nu(t)), v - \dfrac{d}{dt}\widehat{u}_\nu(t)\rangle + j(f_\nu(t), u_\nu(t), v) \\ \quad -j(f_\nu(t), u_\nu(t), \dfrac{d}{dt}\widehat{u}_\nu(t)) \geq b(f_\nu(t), u_\nu(t), v - \dfrac{d}{dt}\widehat{u}_\nu(t)) \quad \forall\, v \in V, \\ b(f_\nu(t), u_\nu(t), w - u_\nu(t)) \geq 0 \quad \forall\, w \in K. \end{cases}
$$

Also, the sequence $(Q_\nu^\iota)_{\iota=0,1,\ldots,\nu-1}$ implies the following inequality: for almost every $t \in [0,T]$

$$
(R_\nu) \quad \langle F'(u_\nu(t)), w - u_\nu(t)\rangle + j(f_\nu(t), u_\nu(t), w - u_\nu(t)) \geq 0 \quad \forall\, w \in K,
$$

which is clearly equivalent to the following inequality: for almost every $t \in [0,T]$

$$
(\widehat{R}_\nu) \;\; F(w) - F(u_\nu(t)) + j(f_\nu(t), u_\nu(t), w - u_\nu(t)) \geq \frac{\alpha}{2}\|w - u_\nu(t)\|^2 \quad \forall\, w \in K.
$$

Lemma 2 *There exist a subsequence of* $(u_\nu, \widehat{u}_\nu)_\nu$, *denoted by* $(u_{\nu_p}, \widehat{u}_{\nu_p})_p$ *, and an element* $u \in W^{1,2}(0,T;V)$ *such that*

$$u_{\nu_p}(t) \rightharpoonup u(t) \quad \text{in} \quad V \quad \forall\, t \in [0,T], \tag{19}$$

$$\widehat{u}_{\nu_p} \rightharpoonup u \quad \text{in} \quad W^{1,2}(0,T;V), \tag{20}$$

$$\frac{d}{dt}\widehat{u}_{\nu_p} \rightharpoonup \dot{u} \quad \text{in} \quad L^2(0,T;V). \tag{21}$$

Also, for all $s \in [\,0,T\,]$, *we have* $u(s) \in K(f(s))$ *and*

$$\liminf_{p\to\infty} \int_0^s j(f_{\nu_p}(t), u_{\nu_p}(t), \frac{d}{dt}\widehat{u}_{\nu_p}(t))\,dt \geq \int_0^s j(f(t), u(t), \dot{u}(t))\,dt. \tag{22}$$

We can prove the following strong convergence and existence result.

Theorem 1 *Under the assumptions (1)-(14) every convergent subsequence of* $(u_\nu, \widehat{u}_\nu)_\nu$*, still denoted by* $(u_\nu, \widehat{u}_\nu)_\nu$*, and its limit* $u \in W^{1,2}(0,T;V)$*, given by lemma 2, satisfy the following properties:*

$$u_\nu(t) \to u(t) \quad \textit{in} \quad V \quad \forall\, t \in [0,T], \tag{23}$$

$$\widehat{u}_\nu \to u \quad \textit{in} \quad L^2(0,T;V), \tag{24}$$

and u *is a solution of problem* P*.*

2. Internal approximation and convergence analysis

We prove a convergence result for a method based on an internal approximation and a backward difference scheme.
First, we consider a semi-discrete approximation of (P), which extends some classical internal approximations. Let $(V_h)_h$ be an internal approximation of V, that is a family of finite-dimensional subspaces of V which satisfies:

$$\begin{array}{c} \text{there exist } U \subset V \text{ such that } \overline{U} = V \text{ and} \\ \forall\, v \in U,\, \exists\, v_h \in V_h \ \text{ for each } h, \ \text{ such that } \ v_h \to v \ \text{ in } \ V. \end{array} \tag{25}$$

Let $(K_h)_h$ be a family of closed convex cones with their vertices at 0 such that $K_h \subset V_h$ for all h and $(K_h)_h$ is an internal approximation of K, i.e.

$$\text{if } \ v_h \in K_h \ \text{ for all } h \ \text{ and } v_h \rightharpoonup v \text{ then } v \in K, \tag{26}$$

$$\forall\, v \in K,\, \exists\, v_h \in K_h \ \text{ for each } h, \text{ such that } \ v_h \to v \ \text{ in } \ V. \tag{27}$$

Let $(K_h(g))_{g\in V}$ be a family of nonempty closed convex subsets of K_h such that $0 \in K_h(0)$ for all h, satisfying the following conditions:

$$\text{if } \; g_n \to g \; \text{ in } V, v_{hn} \in K_h(g_n) \; \text{ and } \; v_{hn} \to v_h \text{ in } V_h \; \text{ then } \; v_h \in K_h(g), \tag{28}$$

$$\text{if } \; v_h \in K_h(g) \; \text{ for all } h \; \text{ and } v_h \rightharpoonup v \text{ then } v \in K(g) \quad \forall\, g \in V. \tag{29}$$

We assume that for all $g \in V$ there exists an operator $\gamma_h(g,\cdot) : K_h(g) \to H$ such that $\gamma_h(0,0) = 0$ and for all $g_i \in V$, $v_{hi} \in K_h(g_i)$, $i = 1, 2$,

$$\|\gamma_h(g_1, v_{h1}) - \gamma_h(g_2, v_{h2})\|_H \le k_1(\|g_1 - g_2\| + \|v_{h1} - v_{h2}\|). \tag{30}$$

For all $g \in V$, let $j_h(g,\cdot,\cdot) : K_h(g) \times V_h \to \mathbb{R}$ be a functional satisfying the following conditions for all $g \in V$:

$$\begin{gathered}\text{if } v_h \in K_h(g) \; \text{ for all } h, \; v_h \rightharpoonup v \; \text{ in } V \text{ and } \; w_h \rightharpoonup w \; \text{ in } \; V \\ \text{then } \; \lim_{h\to 0} j_h(g, v_h, w_h) = j(g, v, w),\end{gathered} \tag{31}$$

$$\text{for all } h \text{ and } v_h \in K_h(g) \quad j_h(g, v_h, \cdot) \; \text{ is sub-additive,} \tag{32}$$

for all h and $v_h \in K_h(g)$ $\quad j_h(g, v_h, \cdot)$ is positively homogeneous, (33)

$$j_h(0, 0, w_h) = 0 \quad \forall\, w_h \in V_h, \tag{34}$$

and

if $v_h(t) \in K_h(g(t))$ for all h and $t \in [0, T]$, $v_h \rightharpoonup v$ in $W^{1,2}(0, T; V)$

$$\text{then } \liminf_{h \to 0} \int_0^T j_h(g(t), v_h(t), \dot{v}_h(t))\, dt \geq \int_0^T j(g(t), v(t), \dot{v}(t))\, dt \tag{35}$$

for all $g \in C([0, T]; V)$,

$$|j_h(g_1, v_{h1}, w_{h2}) + j_h(g_2, v_{h2}, w_{h1}) - j_h(g_1, v_{h1}, w_{h1}) - j_h(g_2, v_{h2}, w_{h2})|$$

$$\leq k_2(\|g_1 - g_2\| + \|\gamma_h(g_1, v_{h1}) - \gamma_h(g_2, v_{h2})\|_H)\|w_{h1} - w_{h2}\| \tag{36}$$

$$\forall\, g_i \in V,\, v_{hi} \in K_h(g_i),\, w_{hi} \in V_h,\, i = 1, 2.$$

Now we consider the following semi-discrete problem.

Problem $\mathbf{P_h}$: Find $u_h \in W^{1,2}(0,T;V_h)$ such that

$$(P_h)\quad \begin{cases} u_h(0) = u_{0h},\ u_h(t) \in K_h(f(t)) \quad \forall\, t \in]0,T[, \\ \langle F'(u_h(t)), v_h - \dot u_h(t)\rangle + j_h(f(t), u_h(t), v_h) - j_h(f(t), u_h(t), \dot u_h(t)) \\ \qquad \geq b(f(t), u_h(t), v_h - \dot u_h(t)) \quad \forall\, v_h \in V_h \text{ a.e. on }]0,T[, \\ b(f(t), u_h(t), z_h - u_h(t)) \geq 0 \quad \forall\, z_h \in K_h, \quad \forall\, t \in]0,T[, \end{cases}$$

The full discretization of (P_h) is obtained by using an implicit scheme as in Section 2 for (P). For $u_h^0 := u_{0h}$ and $\iota \in \{0, 1, ..., \nu - 1\}$, we define $u_h^{\iota+1}$ as the solution of the following problem.

Problem $\mathbf{P^{\iota}_{h\nu}}$: Find $u_h^{\iota+1} \in K_h^{\iota+1}$ such that

$$(P^{\iota}_{h\nu})\quad \begin{cases} \langle F'(u_h^{\iota+1}), v_h - \partial u_h^{\iota}\rangle + j_h(f^{\iota+1}, u_h^{\iota+1}, v_h) - j_h(f^{\iota+1}, u_h^{\iota+1}, \partial u_h^{\iota}) \\ \qquad \geq b(f^{\iota+1}, u_h^{\iota+1}, v_h - \partial u_h^{\iota}) \quad \forall\, v_h \in V_h, \\ b(f^{\iota+1}, u_h^{\iota+1}, z_h - u_h^{\iota+1}) \geq 0 \quad \forall\, z_h \in K_h, \end{cases}$$

where $K_h^{\iota+1} := K_h(f^{\iota+1})$.

If we define the functions

$$\begin{cases} u_{h\nu}(0) = \widehat{u}_{h\nu}(0) = u_{0h} \quad \text{and} \\ \forall\, \iota \in \{0, 1, ..., \nu - 1\}, \quad \forall\, t \in]t_\iota, t_{\iota+1}], \\ \ u_{h\nu}(t) = u_h^{\iota+1}, \\ \ \widehat{u}_{h\nu}(t) = u_h^\iota + (t - t_\iota)\partial u_h^\iota, \end{cases}$$

then for all $\nu \in N^*$ the sequence of inequalities $(P_\nu^{h\iota})_{\iota=0,1,...,\nu-1}$ is equivalent to the following incremental formulation:

for almost every $t \in [0, T]$

$$(P_{h\nu}) \begin{cases} u_{h\nu}(t) \in K_h(f_\nu(t)),\ \langle F'(u_{h\nu}(t)), v_h - \dfrac{d}{dt}\widehat{u}_{h\nu}(t)\rangle + j_h(f_\nu(t), u_{h\nu}(t), v_h) \\ -j_h(f_\nu(t), u_{h\nu}(t), \dfrac{d}{dt}\widehat{u}_{h\nu}(t)) \geq b(f_\nu(t), u_{h\nu}(t), v_h - \dfrac{d}{dt}\widehat{u}_{h\nu}(t))\ \forall\, v_h \in V_h, \\ b(f_\nu(t), u_{h\nu}(t), w_h - u_{h\nu}(t)) \geq 0 \quad \forall\, w_h \in K_h. \end{cases}$$

We have the analogue to theorem 1 in the finite dimensional case.

Theorem 2 *Assume that (1), (2), (12), (13), (28), (30), (32)-(34), (36) hold. Then there exists a subsequence of* $(u_{h\nu}, \widehat{u}_{h\nu})_\nu$*, still denoted by* $(u_{h\nu}, \widehat{u}_{h\nu})_\nu$*, such that*

$$u_{h\nu}(t) \to u_h(t) \quad \textit{in} \quad V \quad \forall\, t \in [0,T], \tag{37}$$

$$\widehat{u}_{h\nu} \to u_h \quad \textit{in} \quad L^2(0,T;V), \tag{38}$$

where u_h *is a solution of* (P_h).

Theorem 3 *Under the assumptions (1)-(14), (25)-(36) there exists a subsequence of* $(u_h)_h$ *such that*

$$u_h(t) \to u(t) \quad \textit{in} \quad V \quad \forall\, t \in [\,0,T\,], \tag{39}$$

$$u_h \to u \quad \textit{in} \quad L^2(0,T;V), \tag{40}$$

$$\dot{u}_h \rightharpoonup \dot{u} \quad \textit{in} \quad L^2(0,T;V), \tag{41}$$

where u *is a solution of* (P).

Theorem 4 *Under the assumptions of theorem 3, there exists a subsequence of* $(u_{h\nu})_{h\nu}$ *such that*

$$u_{h\nu}(t) \to u(t) \quad \textit{in} \quad V \qquad \forall\, t \in [0,T], \tag{42}$$

$$\dot{u}_{h\nu} \rightharpoonup \dot{u} \quad \textit{in} \quad L^2(0,T;V), \tag{43}$$

where $u \in W^{1,2}(0,T;V)$ *is a solution of* (P).

Furthermore any cluster point of $(u_{h\nu})_{h\nu}$ *is a solution of* (P).

3. Subspace correction approximation

Let $V_1, \cdots, V_m$ be some closed subspaces of V. We consider a convex subset $\mathcal{K} \subset V$ satisfying the following assumption.

Assumption 1 *There exists a constant C_0 such that for any $w, v \in \mathcal{K}$ and $w_i \in V_i$ with $w + \sum_{j=1}^{i} w_j \in \mathcal{K}$, $i = 1, \cdots, m$, there exist $v_i \in V_i$, $i = 1, \cdots, m$, satisfying*

$$w + \sum_{j=1}^{i-1} w_j + v_i \in \mathcal{K} \text{ for } i = 1, \cdots, m, \tag{44}$$

$$v - w = \sum_{i=1}^{m} v_i, \tag{45}$$

and

$$\sum_{i=1}^{m} \|v_i\| \leq C_0 \left(\|v - w\| + \sum_{i=1}^{m} \|w_i\| \right). \tag{46}$$

Let $\varphi : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ be a convex and lower semicontinuous functional with respect to the second variable such that

$$\begin{aligned} &|\varphi(v_1, w_2) + \varphi(v_2, w_1) - \varphi(v_1, w_1) - \varphi(v_2, w_2)| \\ &\quad \leq k_1 k_2 ||v_1 - v_2|| ||w_1 - w_2|| \ \ \forall v_1, v_2, w_1, w_2 \in \mathcal{K} \end{aligned} \tag{47}$$

and suppose that

Assumption 2

$$\begin{aligned} &\sum_{i=1}^{m} [\varphi(u, w + \sum_{j=1}^{i-1} w_j + v_i) - \varphi(u, w + \sum_{j=1}^{i-1} w_j + w_i)] \\ &\quad \leq \varphi(u, v) - \varphi(u, w + \sum_{i=1}^{m} w_i) \end{aligned} \tag{48}$$

for any $u \in \mathcal{K}$, and for $v, w \in \mathcal{K}$ and $v_i, w_i \in V_i$, $i = 1, \ldots, m$, as in Assumption 1.

We consider the problem of finding $u \in \mathcal{K}$, the solution of the following quasi-variational inequality

$$\langle F'(u), v - u\rangle + \varphi(u, v) - \varphi(u, u) \geq 0 \quad \forall\, v \in \mathcal{K}. \tag{49}$$

Algorithm 1 *We start with an arbitrary* $u^0 \in \mathcal{K}$ *and at iteration* $n+1$, *having* $u^n \in \mathcal{K}$, $n \geq 0$, *we compute, for* $i = 1, \cdots, m$, *the local corrections* $w_i^{n+1} \in V_i$, $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in \mathcal{K}$ *satisfying*

$$\begin{aligned}&\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(u^{n+\frac{i-1}{m}} + w_i^{n+1}, u^{n+\frac{i-1}{m}} + v_i) \\ &-\varphi(u^{n+\frac{i-1}{m}} + w_i^{n+1}, u^{n+\frac{i-1}{m}} + w_i^{n+1}) \geq 0, \ \forall\, v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in \mathcal{K},\end{aligned}$$

and then we update

$$u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}.$$

Algorithm 2 *We start with an arbitrary* $u^0 \in \mathcal{K}$ *and at iteration* $n+1$, *having* $u^n \in \mathcal{K}$, $n \geq 0$, *we compute, for* $i = 1, \cdots, m$, *the local corrections* $w_i^{n+1} \in V_i$, $u^{n+\frac{i-1}{m}} + w_i^{n+1} \in \mathcal{K}$ *satisfying*

$$\begin{aligned}&\langle F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(u^{n+\frac{i-1}{m}}, u^{n+\frac{i-1}{m}} + v_i) \\ &-\varphi(u^{n+\frac{i-1}{m}}, u^{n+\frac{i-1}{m}} + w_i^{n+1}) \geq 0, \ \forall\, v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in \mathcal{K},\end{aligned}$$

and then we update

$$u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}.$$

Theorem 5 *Let us assume that Assumptions 1 and 2 are satisfied. Then, if u is the solution of problem (49), $u^{n+\frac{i}{m}}$, $n \geq 0$, $i = 1, \ldots, m$, are its approximations obtained from one of Algorithms 1 or 2 and*

$$\frac{\alpha}{2} \geq mk_1k_2 + \sqrt{2m(25C_0+8)\beta k_1 k_2}, \tag{50}$$

then we have the following error estimations

$$\begin{aligned} &F(u^n) + \varphi(u, u^n) - F(u) - \varphi(u, u) \leq \\ &\left(\tfrac{C_1}{C_1+1}\right)^n \left[F(u^0) + \varphi(u, u^0) - F(u) - \varphi(u, u)\right], \end{aligned} \tag{51}$$

$$\begin{aligned} &\|u^n - u\|^2 \leq \tfrac{2}{\alpha} \left(\tfrac{C_1}{C_1+1}\right)^n \cdot \\ &\left[F(u^0) + \varphi(u, u^0) - F(u) - \varphi(u, u)\right], \end{aligned} \tag{52}$$

where the constant $C_1 > 0$ depends on α, β, k_1, k_2, the number of subspaces m, and on the constant C_0 introduced in Assumption 1.

In the case of Algorithm 1, the constant C_1 can be written as,

$$\begin{aligned}
&C_1 = C_2/C_3\\
&C_2 = \beta m(1+2C_0+\tfrac{C_0}{\varepsilon_1}) + k_1k_2m(1+2C_0+\tfrac{1+3C_0}{\varepsilon_2})\\
&C_3 = \tfrac{\alpha}{2} - k_1k_2(1+\varepsilon_3)m
\end{aligned} \tag{53}$$

where

$$\varepsilon_1 = \varepsilon_2 = \frac{2k_1k_2m}{\frac{\alpha}{2}-k_1k_2m}, \quad \varepsilon_3 = \frac{\frac{\alpha}{2}-k_1k_2m}{2k_1k_2m},$$

Algorithms 1 and 2 can be viewed as multiplicative Schwarz method if the solution space is a Sobolev space and subspaces are associated to the subsets in a domain decomposition $\Omega = \bigcup_{i=1}^m \Omega_i$. If the convex set $\mathcal{K}$ has the property

Property 1 *If $v, w \in \mathcal{K}$, and if $\theta \in C^0(\overline{\Omega})$, $\theta \in C^1(\Omega_i)$, $i = 1, \ldots, m$, with $0 \le \theta \le 1$, then $\theta v + (1-\theta)w \in \mathcal{K}$,*

then Assumption 1 is satisfied with a C_0 depending on $1/\delta$, the overlapping parameter of the domain decomposition. The convex set $K^{\iota+1}$ has the above property.

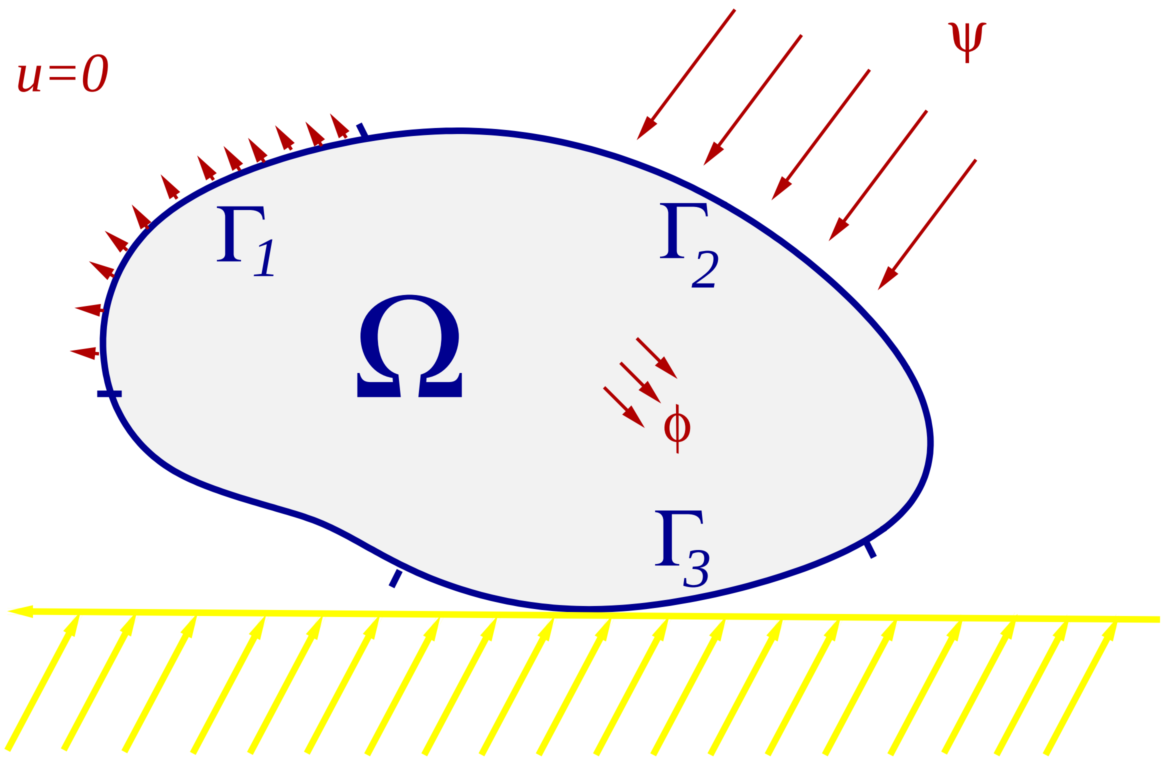
Since $f^{\iota+1}$ and u^ι are fixed in problem (Q_ν^ι), taking

$$\psi(u,v) = j(f^{\iota+1}, u, v - u^\iota) \tag{54}$$

this functional has the properties of φ in problem (49), i.e. it is lower semicontinuous and convex in the second variable, and satisfies (47) but does not satisfy Assumption 2.

The one- and two-level methods are directly obtained from Algorithms 1 or 2. We can prove that Assumption 1 holds for closed convex sets K_h satisfying a similar property with that given in Property 1, and also for the discretized form of $K^{\iota+1}$. We are able to explicitly write the dependence of C_0 on the overlapping and mesh parameters. Also, we can give some numerical approximations φ of the functional j for which Assumption 2 holds. Therefore, from Theorem 5, we conclude that these methods globally converge for the discrete form of (Q_ν^ι) if conditions (1) and (2) on F, and condition (10) on j hold. Moreover, from the dependence of C_0 on the mesh and domain decomposition parameters, we conclude that the convergence rate is optimal, i.e. it is similar with that of linear equations.

4. Applications to contact mechanics



We consider a linearly elastic body which occupies the domain Ω of $\mathbb{R}^d$, $d = 2$ or 3, such that the solid is initially in contact with nonlocal bounded friction on Γ_3.
We assume that $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$ and meas$(\Gamma_1) > 0$.

Let
$\boldsymbol{u} = (u_1, ..., u_d)$ be the displacement field,
$\boldsymbol{\varepsilon} = \left(\varepsilon_{ij}(\boldsymbol{u})\right)$ be the infinitesimal strain tensor,
$\boldsymbol{\sigma} = \left(\sigma_{ij}(\boldsymbol{u})\right)$ be the stress tensor,
$\boldsymbol{\mathcal{E}}$ be the elasticity tensor, with the components $\mathcal{E} = (a_{ijkl})$,
$\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ be the given body forces and tractions.
On Γ_1 $\boldsymbol{u} = \boldsymbol{0}$ and in Ω the initial displacements are denoted by $\boldsymbol{u}_0$.
We use the classical decompositions into the normal and tangential components of the displacement vector and stress vector $\boldsymbol{u} = u_N\boldsymbol{n} + \boldsymbol{u}_T$ with $u_N = \boldsymbol{u} \cdot \boldsymbol{n}$, $\boldsymbol{\sigma}\boldsymbol{n} = \sigma_N\boldsymbol{n} + \boldsymbol{\sigma}_T$ with $\sigma_N = (\boldsymbol{\sigma}\boldsymbol{n}) \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is the outward normal unit vector to Γ with the components $n = (n_i)$.

The classical formulation of the quasistatic problem is as follows.

Problem $\mathbf{P_c}$: Find a displacement field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ which satisfies the initial condition $\boldsymbol{u}(0) = \boldsymbol{u}_0$ in Ω and for all $t \in]\,0, T\,[$, the following equations and boundary conditions:

$$
(P_c) \quad \begin{cases}
\text{div } \boldsymbol{\sigma}(\boldsymbol{u}) = -\boldsymbol{\phi} \quad \text{in} \quad \Omega, \\
\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{\mathcal{E}}\, \varepsilon(\boldsymbol{u}) \quad \text{in} \quad \Omega, \\
\boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Gamma_1, \\
\boldsymbol{\sigma n} = \boldsymbol{\psi} \quad \text{on} \quad \Gamma_2, \\
u_N \leq 0, \quad \sigma_N \leq 0, \quad u_N \sigma_N = 0 \quad \text{on} \quad \Gamma_3, \\
|\boldsymbol{\sigma}_T| \leq \mu |\mathcal{R}\sigma_N| \quad \text{on} \quad \Gamma_3 \\
\quad \text{and} \quad \begin{cases} |\boldsymbol{\sigma}_T| < \mu |\mathcal{R}\sigma_N| \Rightarrow \dot{\boldsymbol{u}}_T = \mathbf{0}, \\ |\boldsymbol{\sigma}_T| = \mu |\mathcal{R}\sigma_N| \Rightarrow \exists \lambda \geq 0,\, \dot{\boldsymbol{u}}_T = -\lambda \boldsymbol{\sigma}_T, \end{cases}
\end{cases}
$$

where μ is the coefficient of friction and $\mathcal{R}\sigma_N$ is a regularization of the normal contact force.

In order to obtain a variational formulation for this problem, we adopt the following hypotheses:

$$\boldsymbol{\phi} \in W^{1,2}(0,T;[L^2(\Omega)]^d), \ \boldsymbol{\psi} \in W^{1,2}(0,T;[L^2(\Gamma_2)]^d),$$
$$a_{ijkl} \in L^\infty(\Omega), \ i,j,k,l = 1,...,d, \ \mu \in L^\infty(\Gamma_3), \ \mu \geq 0 \text{ a.e. on } \Gamma_3.$$

We use the following notations:

$$V_0 := \{\boldsymbol{v} \in [H^1(\Omega)]^d; \boldsymbol{v} = \mathbf{0} \text{ a.e. on } \Gamma_1\}, \ \ (\cdot\,,\cdot) = (\cdot\,,\cdot)_{[H^1(\Omega)]^d},$$
$$K_0 := \{\boldsymbol{v} \in V_0\,; v_N \leq 0 \text{ a.e. on } \Gamma_3\},$$
$$H^{\frac{1}{2}}(\Gamma_3) := \{w : \Gamma_3 \to \mathbb{R};\, w \in H^{\frac{1}{2}}(\Gamma),\, w = 0 \text{ a.e. on } \Gamma_1\},$$
$$\forall \boldsymbol{L} \in V_0 \quad S_{\boldsymbol{L}} := \{\boldsymbol{w} \in V_0; \int_\Omega \boldsymbol{\sigma}(\boldsymbol{w}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{\eta}) dx = (\boldsymbol{L}, \boldsymbol{\eta}) \ \ \forall \boldsymbol{\eta} \in V_0 \text{ such that } \boldsymbol{\eta} = \mathbf{0} \text{ a.e. on } \Gamma_3\}.$$

For all $\boldsymbol{L} \in V_0$ and $\boldsymbol{v} \in S_{\boldsymbol{L}}$ we define $\boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{n} \in ([H^{\frac{1}{2}}(\Gamma_3)]^d)'$ by

$$\forall \boldsymbol{w} \in [H^{\frac{1}{2}}(\Gamma_3)]^d \quad \langle \boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{n}, \boldsymbol{w}\rangle = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{v}) \cdot \boldsymbol{\varepsilon}(\bar{\boldsymbol{w}}) dx - (\boldsymbol{L}, \bar{\boldsymbol{w}}), \tag{55}$$

where $\bar{\boldsymbol{w}} \in V_0$ satisfies $\bar{\boldsymbol{w}} = \boldsymbol{w}$ a.e. on Γ_3, and we define the normal component of the stress vector $\sigma_N(\boldsymbol{v}) \in (H^{\frac{1}{2}}(\Gamma_3))'$ by

$$\forall\, w \in H^{\frac{1}{2}}(\Gamma_3) \quad \langle \sigma_N(\boldsymbol{v}), w\rangle = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{v}) \cdot \boldsymbol{\varepsilon}(\bar{\boldsymbol{w}})dx - (\boldsymbol{L}, \bar{\boldsymbol{w}}), \tag{56}$$

where $\bar{\boldsymbol{w}} \in V_0$ satisfies $\bar{\boldsymbol{w}}_T = \boldsymbol{0}$ a.e. on Γ_3, $\bar{\boldsymbol{w}}_N = w$ a.e. on Γ_3.

For all $\boldsymbol{L} \in V_0$ we introduce the functional $J_{\boldsymbol{L}} : S_{\boldsymbol{L}} \times V_0 \to \mathbb{R}$ by

$$J_{\boldsymbol{L}}(\boldsymbol{v}, \boldsymbol{w}) = \int_{\Gamma_3} \mu |\mathcal{R}\sigma_N(\boldsymbol{v})||\boldsymbol{w}_T| ds \quad \forall\, \boldsymbol{v} \in S_{\boldsymbol{L}},\, \boldsymbol{w} \in V_0, \tag{57}$$

where $\mathcal{R} : (H^{\frac{1}{2}}(\Gamma_3))' \to L^2(\Gamma_3)$ is a linear and compact mapping.

Let $\boldsymbol{L} \in V_0$ be given by the relation

$$(\boldsymbol{L}, \boldsymbol{v}) = (\boldsymbol{\phi}, \boldsymbol{v})_{[L^2(\Omega)]^d} + (\boldsymbol{\psi}, \boldsymbol{v})_{[L^2(\Gamma_2)]^d} \quad \forall\, \boldsymbol{v} \in V_0 \tag{58}$$

and let $\boldsymbol{u}_0 \in K_0$ satisfying the following compatibility condition:

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}_0) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w} - \boldsymbol{u}_0)dx + J_{\boldsymbol{L}(0)}(\boldsymbol{u}_0, \boldsymbol{w}) - J_{\boldsymbol{L}(0)}(\boldsymbol{u}_0, \boldsymbol{u}_0) \\ \geq (\boldsymbol{L}(0), \boldsymbol{w} - \boldsymbol{u}_0) \quad \forall\, \boldsymbol{w} \in K_0. \end{aligned} \tag{59}$$

A primal variational formulation of P_c is as follows.

$\mathbf{Problem\ P_0}$: Find $\boldsymbol{u} \in W^{1,2}(0,T;V_0)$ such that

$$(P_0) \begin{cases} \boldsymbol{u}(0) = \boldsymbol{u}_0, \ \boldsymbol{u}(t) \in K_0 \quad \forall\, t \in]\,0,T\,[, \\ \displaystyle\int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}(t)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}(t))dx + J_{\boldsymbol{L}(t)}(\boldsymbol{u}(t),\boldsymbol{v}) - J_{\boldsymbol{L}(t)}(\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) \\ \geq (\boldsymbol{L}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + \langle \sigma_N(\boldsymbol{u}(t)), v_N - \dot{u}_N(t)\rangle \ \forall\, \boldsymbol{v} \in V_0 \text{ a.e. on }]\,0,T\,[, \\ \langle \sigma_N(\boldsymbol{u}(t)), z_N - u_N(t)\rangle \geq 0 \quad \forall\, \boldsymbol{z} \in K_0, \forall\, t \in]\,0,T\,[. \end{cases}$$

Let us define $a : V_0 \times V_0 \to \mathbb{R}$ by

$$a(\boldsymbol{v},\boldsymbol{w}) = \int_\Omega a_{ijkl}\varepsilon_{ij}(\boldsymbol{v})\varepsilon_{kl}(\boldsymbol{w})dx = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{v}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w})dx \quad \forall\, \boldsymbol{v},\boldsymbol{w} \in V_0. \quad (60)$$

The bilinear form $a(\cdot\,,\cdot)$ satisfies

$$\exists\, \beta > 0 \text{ such that } |a(\boldsymbol{v},\boldsymbol{w})| \leq \beta\|\boldsymbol{v}\|\|\boldsymbol{w}\| \quad \forall\, \boldsymbol{v},\boldsymbol{w} \in V_0,$$

$$\exists\, \alpha > 0 \text{ such that } a(\boldsymbol{v},\boldsymbol{v}) \geq \alpha\|\boldsymbol{v}\|^2 \quad \forall\, \boldsymbol{v} \in V_0,$$

where $\|\cdot\| = \|\cdot\|_{[H^1(\Omega)]^d}$.

Let $\boldsymbol{G}_1, \boldsymbol{G}_2 \in V_0$ and $\boldsymbol{v}_1, \boldsymbol{v}_2$ be such that $\boldsymbol{v}_1 \in S_{\boldsymbol{G}_1}$, $\boldsymbol{v}_2 \in S_{\boldsymbol{G}_2}$. Then from the properties of σ_N, $\mathcal{R}$ and a it follows that the mapping J has the following property: $\exists\, C, C' > 0$ such that

$$
\begin{aligned}
&|J_{\boldsymbol{G}_1}(\boldsymbol{v}_1, \boldsymbol{w}_2) + J_{\boldsymbol{G}_2}(\boldsymbol{v}_2, \boldsymbol{w}_1) - J_{\boldsymbol{G}_1}(\boldsymbol{v}_1, \boldsymbol{w}_1) - J_{\boldsymbol{G}_2}(\boldsymbol{v}_2, \boldsymbol{w}_2)| \\
&\qquad \leq C\bar{\mu} \int_{\Gamma_3} |\mathcal{R}\sigma_N(\boldsymbol{v}_1) - \mathcal{R}\sigma_N(\boldsymbol{v}_2)||\boldsymbol{w}_1 - \boldsymbol{w}_2| ds \qquad (61) \\
&\qquad \leq C'\bar{\mu}(\|\boldsymbol{G}_1 - \boldsymbol{G}_2\| + M\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|)\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|
\end{aligned}
$$

for all $\boldsymbol{G}_i$, $\boldsymbol{w}_i \in V_0$, $\boldsymbol{v}_i \in S_{\boldsymbol{G}_i}$, $i = 1, 2$, where $\bar{\mu} = \|\mu\|_{L^\infty(\Gamma_3)}$.

An incremental formulation can be written by using a time discretization of (P_0) as previously. Therefore we obtain the following sequence of incremental problems $(P_{0,\nu}^\iota)_{\iota=0,1,\ldots,\nu}$.

Problem $\mathbf{P}_{\mathbf{0},\nu}^\iota$: Find $\boldsymbol{u}^{\iota+1} \in K_0$ such that

$$
(P_{0,\nu}^\iota) \quad \begin{cases}
a(\boldsymbol{u}^{\iota+1}, \boldsymbol{v} - \partial\boldsymbol{u}^\iota) + J_{\boldsymbol{L}^{\iota+1}}(\boldsymbol{u}^{\iota+1}, \boldsymbol{v}) - J_{\boldsymbol{L}^{\iota+1}}(\boldsymbol{u}^{\iota+1}, \partial\boldsymbol{u}^\iota) \\
\quad \geq (\boldsymbol{L}^{\iota+1}, \boldsymbol{v} - \partial\boldsymbol{u}^\iota) + \langle \sigma_N(\boldsymbol{u}^{\iota+1}), v_N - \partial u_N^\iota \rangle \quad \forall\, \boldsymbol{v} \in V_0, \\
\langle \sigma_N(\boldsymbol{u}^{\iota+1}), z_N - u_N^{\iota+1} \rangle \geq 0 \quad \forall\, \boldsymbol{z} \in K_0.
\end{cases}
$$

Then $\boldsymbol{u}_\nu \in L^2(0,T;V_0)$ and $\widehat{\boldsymbol{u}}_\nu \in W^{1,2}(0,T;V_0)$ satisfy the following incremental problem:

$$(P_{0,\nu}) \quad \begin{cases} a(\boldsymbol{u}_\nu(t), \boldsymbol{v} - \frac{d}{dt}\widehat{\boldsymbol{u}}_\nu(t)) + J_{\boldsymbol{L}_\nu(t)}(\boldsymbol{u}_\nu(t), \boldsymbol{v}) \\ \quad -J_{\boldsymbol{L}_\nu(t)}(\boldsymbol{u}_\nu(t), \frac{d}{dt}\widehat{\boldsymbol{u}}_\nu(t)) \geq (\boldsymbol{L}_\nu(t), \boldsymbol{v} - \frac{d}{dt}\widehat{\boldsymbol{u}}_\nu(t)) \\ \quad +\langle \sigma_N(\boldsymbol{u}_\nu(t)), v_N - \frac{d}{dt}\widehat{u}_{\nu N}(t)\rangle \quad \forall\, \boldsymbol{v} \in V_0, \forall\, t \in [\,0,T\,], \\ \langle \sigma_N(\boldsymbol{u}_\nu(t)), z_N - u_{\nu N}(t)\rangle \geq 0 \quad \forall\, \boldsymbol{z} \in K_0, \forall\, t \in [\,0,T\,]. \end{cases}$$

We have the following existence and approximation result.

Theorem 6 *Under the above assumptions and if* $\bar{\mu} < \dfrac{\alpha}{C'}$ *there exists a subsequence* $(\boldsymbol{u}_{\nu_p})_p$ *of* $(\boldsymbol{u}_\nu)_\nu$ *such that* $\boldsymbol{u}_{\nu_p}(t) \to \boldsymbol{u}(t)$ *in* V_0 $\forall\, t \in [\,0,T\,]$, $\widehat{\boldsymbol{u}}_{\nu_p} \to \boldsymbol{u}$ *in* $L^2(0,T;V_0)$ *and* $\dfrac{d}{dt}\widehat{\boldsymbol{u}}_{\nu_p} \rightharpoonup \dot{\boldsymbol{u}}$ *in* $L^2(0,T;V_0)$, *as* $p \to \infty$, *where* $\boldsymbol{u}$ *is a solution of* (P_0).

Proof.

Taking $V = V_0$, $K = K_0$, $H = L^2(\Gamma_3)$ and

$$j(\boldsymbol{L}, \boldsymbol{v}, \boldsymbol{w}) = J_{\boldsymbol{L}}(\boldsymbol{v}, \boldsymbol{w}) - (\boldsymbol{L}, \boldsymbol{w}),\ b(\boldsymbol{L}, \boldsymbol{v}, \boldsymbol{w}) = \langle \sigma_N(\boldsymbol{v}), w_N \rangle,$$
$$K(\boldsymbol{L}) = K_0 \cap S_{\boldsymbol{L}},\ \beta(\boldsymbol{L}, \boldsymbol{v}) = \mu |\mathcal{R}\sigma_N(\boldsymbol{v})| \quad \forall\, \boldsymbol{v} \in S_{\boldsymbol{L}},\, \boldsymbol{w} \in V_0,$$

we see that (P_0) can be written in the form (P) with $f = \boldsymbol{L}$, where $\boldsymbol{L}$ is defined by (58). Using the properties of J and Green's formula, it can be easily seen that the hypotheses of theorem 1 are satisfied and the theorem therefore follows. $\square$

- Using a similar approach, one can study the quasistatic unilateral contact problem with nonlocal friction between two linearly elastic bodies.
- The previous abstract results can be equally applied to frictional contact with normal compliance, to bilateral contact problems or, more generally, to contact interface problems.

Perspectives

- Generalizations to: viscoelasticity, monotone operators,...
- Study of the local friction laws
- Study of the corresponding dynamic cases,...