Convergence of invariant measures for nonlinear stochastic equations in variational formulation

Ioana Ciotir

Universitatea "Alexandru Ioan Cuza" Iasi

27 08 2010

Nonlinear stochastic differential equations in Banach spaces

Let H be a real Hilbert separable space with H' the dual and V a reflexive Banach space such that

$$V \subset H = H' \subset V'$$

with dense and compact injections, and ${}_{V'}\langle z, v \rangle_V = \langle z, v \rangle_H$ for all $z \in H$ and $v \in V$.

Let the stochastic differential equation in H of the type

$$dX(t) = A(t, X(t))\, dt + B(t, X(t))\, dW(t) \qquad (1)$$

where $W(t)$, with $t \in [0, T]$ is a Q Wiener process with $Q = I$ on another Hilbert space $(U, \langle , \rangle_U)$ and for $T \in [0, \infty[$ fixed

$$B : [0, T] \times V \rightarrow L_2(U, H),$$

$$A : [0, T] \times V \rightarrow V'$$

progressively measurable.

Definition. An $(\mathcal{F}_t)-$ adapted stochastic process $(X(t))_{t\in[0,T]}$, H - valued continuous, is called solution for the equation (1) if for the equivalence class $\hat{X}$ with respect to $dt \otimes \mathbb{P}$ we have

$$\hat{X} \in L^p([0,T] \times \Omega, dt \otimes \mathbb{P}; V) \cap L^2([0,T] \times \Omega, dt \otimes \mathbb{P}; H)$$

with p from the coercivity and we have

$$X(t) = X(0) + \int_0^t A(s, \bar{X}(s))\, ds + \int_0^t B(s, \bar{X}(s))\, dW(s)$$

$\mathbb{P}-$a.s. where $\bar{X}$ is a $dt \otimes \mathbb{P}-$ version of $\hat{X}$, progressively measurable and $V-$ valued.

Convergence of solutions for nonlinear stochastic differential equation in variational formulation

Consider the stochastic basis $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ and a H - valued cylindrical Wiener process W.
Let the stochastic differential equation

$$\begin{cases} dX(t) + A(X(t))\,dt = \sqrt{Q}dW(t) \\ X(0) = x \end{cases}$$

where the operator $Q \in L(H)$ is symmetric, nonnegative, of trace class and such that Ker $Q = \{0\}$.

Assume that the nonlinear operator $A: V \longrightarrow V'$ satisfy the conditions below

(i) (Hemicontinuity) For all $u, v, x \in V$ the map $\theta \mapsto {}_{V'}\langle A(u+\theta v), x\rangle_V$ is continue from $\mathbb{R}$ to $\mathbb{R}$.

(ii) (Monotonicity) We have

$$_{V'}\langle A(u) - A(v), u - v\rangle_V \geq 0,$$

for all $u, v \in V$.

(iii) (Coercivity) There exist $\gamma > 0$, $\eta \geq 0$ and $p > 2$ such that ${}_{V'}\langle A(u), u\rangle_V \geq \gamma \|u\|_V^p - \eta |u|_H^2$, for all $u \in V$.
If $p = 2$ then there exists $\gamma > 0$ such that

$$_{V'}\langle A(u), u\rangle_V \geq \gamma \|u\|_V^2, \text{ for all } u \in V.$$

(iv) (Boundedness) There exist $\beta_1 > 0$, $\beta_2 \in \mathbb{R}$ such that

$$|A(u)|_{V'} \leq \beta_1 \|u\|_V + \beta_2, \text{ for all } u \in V.$$

(v) $A = \nabla\Phi$ where $\Phi : V \to \mathbb{R}$ is convex and Gateaux differentiable with $\Phi \geq 0$ on V and $\Phi(0) = 0$.

(vi) The operator A is differentiable from V to V' and

$$Tr_H \left[QA'(x) \right] = \sum_{i=1}^{\infty} \left(Qe_i, A'(x) e_i \right) \leq C \left(\|x\|_V^{p-2} + 1 \right),$$

where $\{e_i\} \subset V$ is a complete orthonormal system in H such that $A'(x) e_i \in H$ for all $i \in \mathbb{N}$, $x \in V$ and A' is the Frechet differential of A.

We consider the nonlinear operators $A : V \longrightarrow V'$ and $A^{\alpha} : V \longrightarrow V'$ that satisfy the conditions above with all constants independents of α and we define the operators

$$A_H(y) = A(y), \quad y \in D(A_H), \quad D(A_H) = \{y \in V : A(y) \in H\}$$

$$A_H^{\alpha}(y) = A^{\alpha}(y), \quad y \in D(A_H^{\alpha}), \quad D(A_H^{\alpha}) = \{y \in V : A^{\alpha}(y) \in H\},$$

and equations

$$\begin{cases} dX^{\alpha}(t) + A_H^{\alpha}(X^{\alpha}(t))\,dt = \sqrt{Q}dW(t), & t \geq 0 \\ X^{\alpha}(t) = 0, \quad \text{on } \partial\mathcal{O} & t \geq 0, \\ X^{\alpha}(0) = x \end{cases} \tag{2}$$

and

$$\begin{cases} dX(t) + A_H(X(t))\,dt = \sqrt{Q}dW(t), & t \geq 0 \\ X(t) = 0, \quad \text{on } \partial\mathcal{O} & t \geq 0, \\ X(0) = x. \end{cases} \tag{3}$$

Theorem. Let A and A^α satisfying Hypotheses above with $\gamma,\ \eta,\ \beta_1,\ \beta_2$ and C independent of α. Assume also that, for all $y \in H$ and all $\varepsilon > 0$ fixed, we have

$$\left(1+\varepsilon A_H^\alpha\right)^{-1} y \longrightarrow \left(1+\varepsilon A_H\right)^{-1} y, \quad \text{strongly in } H, \text{ for } \alpha \longrightarrow 0. \tag{4}$$

Then the following convergence holds

$$\mathbb{E}\left|X^\alpha(t,x) - X(t,x)\right|_H^2 \longrightarrow 0, \text{ for all } x \in V$$

uniformly in t on compact subsets of $[0,\infty)$, as $\alpha \longrightarrow 0$.

Proof (sketch)
Consider the following approximating equations

$$\begin{cases} dX_{\varepsilon}^{\alpha}(t,x) + A_{\varepsilon}^{\alpha}(X_{\varepsilon}^{\alpha}(t,x))\,dt + \varepsilon X_{\varepsilon}^{\alpha}(t,x)\,dt = \sqrt{Q}dW(t), \\ X^{\alpha}(0,x) = x \end{cases}$$

and

$$\begin{cases} dX_{\varepsilon}(t,x) + A_{\varepsilon}(X_{\varepsilon}(t,x))\,dt + \varepsilon X_{\varepsilon}(t,x)\,dt = \sqrt{Q}dW(t), \\ X(0,x) = x. \end{cases}$$

where A_{ε}^{α} and A_{ε} are the Yosida approximations of the operators A_H^{α} and resp. A_H.
We have

$$\mathbb{E}\left|X - X^{\alpha}\right|_H^2 \leq c\left(\mathbb{E}\left|X - X_{\varepsilon}\right|_H^2 + \mathbb{E}\left|X_{\varepsilon} - X_{\varepsilon}^{\alpha}\right|_H^2 + \mathbb{E}\left|X_{\varepsilon}^{\alpha} - X^{\alpha}\right|_H^2\right).$$

By the Ito formula with $\varphi = |.|_H^2$

$$\mathbb{E}\left|X_\varepsilon^\alpha(t,x)\right|_H^2 + 2\gamma\mathbb{E}\int_0^t \left\|\left(1+\varepsilon A_H^\alpha\right)^{-1}\left(X_\varepsilon^\alpha(s,x)\right)\right\|_V^2 ds$$
$$\leq c_t\left(|x|_H^2 + TrQ\right), \quad t \geq 0.$$

Applying again the Ito formula with $\varphi = \Phi_\varepsilon^\alpha$ where

$$\Phi_\varepsilon^\alpha(y) = \inf_{z\in V}\left\{\frac{|y-z|_H^2}{2\varepsilon} + \Phi^\alpha(z)\right\} \quad y \in H$$

and $A_\varepsilon^\alpha = \nabla\Phi_\varepsilon^\alpha$ we obtain

$$\mathbb{E}\left[\Phi_\varepsilon^\alpha\left(X_\varepsilon^\alpha(t,x)\right)\right] + \mathbb{E}\int_0^t \left|A_\varepsilon^\alpha\left(X_\varepsilon^\alpha(s,x)\right)\right|_H^2 ds$$
$$\leq \quad \Phi_\varepsilon^\alpha(x) + c\mathbb{E}\int_0^t\left(\left\|\left(1+\varepsilon A_H^\alpha\right)^{-1}\left(X_\varepsilon^\alpha(s,x)\right)\right\|_V^p + 1\right) ds$$

On the other hand we have

$$\begin{aligned} & \mathbb{E}\left|X_{\varepsilon}^{\alpha}(t,x)-X_{\lambda}^{\alpha}(t,x)\right|_{H}^{2} \\ \leq\ & \mathbb{E}\int_{0}^{t}\left(\varepsilon\left|A_{\varepsilon}^{\alpha}\left(X_{\varepsilon}^{\alpha}(s,x)\right)\right|_{H}^{2}+\lambda\left|A_{\lambda}^{\alpha}\left(X_{\lambda}^{\alpha}(s,x)\right)\right|_{H}^{2}\right)ds \\ & \qquad +\mathbb{E}\int_{0}^{t}\left(\varepsilon\left|X_{\varepsilon}^{\alpha}(t,x)\right|_{H}^{2}+\lambda\left|X_{\lambda}^{\alpha}(t,x)\right|_{H}^{2}\right)ds\leq\bar{C}_{t}(\varepsilon+\lambda) \end{aligned}$$

and consequently

$$\mathbb{E}\left|X_{\varepsilon}^{\alpha}(t,x)-X^{\alpha}(t,x)\right|_{H}^{2}\rightarrow 0$$

for $\varepsilon\rightarrow 0$, uniformly in t on compact sets of $[0,\infty)$ as $\alpha\rightarrow 0$.

For each $\varepsilon > 0$, fixed, we have

$$\mathbb{E}\left|X_\varepsilon^\alpha(t,x) - X_\varepsilon(t,x)\right|_H^2 \longrightarrow 0$$

for $\alpha \longrightarrow 0$, uniformly in t on compact sets of $[0,\infty)$.
By the Ito formula and using the monotonicity of A_ε^α we get that

$$\begin{aligned}
&\mathbb{E}\left|X_\varepsilon^\alpha(t,x) - X_\varepsilon(t,x)\right|_H^2 + \varepsilon\mathbb{E}\int_0^t \left|X_\varepsilon^\alpha(s,x) - X_\varepsilon(s,x)\right|_H^2 ds \\
\leq\ &\left|\mathbb{E}\int_0^t \left\langle A_\varepsilon^\alpha(X_\varepsilon(s,x)) - A_\varepsilon(X_\varepsilon(s,x)), X_\varepsilon^\alpha(s,x) - X_\varepsilon(s,x)\right\rangle_H ds\right|.
\end{aligned}$$

We have

$$\begin{aligned}
&\mathbb{E}\int_0^t \left|\left\langle A_\varepsilon^\alpha(X_\varepsilon(s,x)) - A_\varepsilon(X_\varepsilon(s,x)), X_\varepsilon^\alpha(s,x) - X_\varepsilon(s,x)\right\rangle_H\right| ds \\
\leq\ &\left(\mathbb{E}\int_0^t \left|A_\varepsilon^\alpha(X_\varepsilon(s,x)) - A_\varepsilon(X_\varepsilon(s,x))\right|_H^2 ds\right)^{1/2} \\
&\qquad\times\left(\mathbb{E}\int_0^t \left|X_\varepsilon^\alpha(s,x) - X_\varepsilon(s,x)\right|_H^2 ds\right)^{1/2}.
\end{aligned}$$

Now, it suffices to show that

$$\left(\mathbb{E}\int_0^t \left|A_\varepsilon^\alpha\left(X_\varepsilon\left(s,x\right)\right)-A_\varepsilon\left(X_\varepsilon\left(s,x\right)\right)\right|_H^2 ds\right)^{1/2} \longrightarrow 0,$$

for $\alpha \longrightarrow 0$, with ε fixed.
From (4) it follows that

$$\left|A_\varepsilon^\alpha\left(X_\varepsilon\right)-A_\varepsilon\left(X_\varepsilon\right)\right|_H^2 = \frac{1}{\varepsilon^2}\left|\left(1+\varepsilon A_H^\alpha\right)^{-1} X_\varepsilon \longrightarrow \left(1+\varepsilon A_H\right)^{-1} X_\varepsilon\right|_H^2 \longrightarrow 0$$

for $\alpha \longrightarrow 0$, for ε fixed and a.e. $[0,t) \times \Omega$.
On the other hand we have

$$\left|A_\varepsilon^\alpha\left(X_\varepsilon\right)-A_\varepsilon\left(X_\varepsilon\right)\right|_H^2 \leq C\left|X_\varepsilon\right|_H^2,$$

a.e. in $[0,t) \times \Omega$, with C independent of $\alpha,\ t,\ x$.
Now, via the Lebesgue dominated convergence theorem we can conclude the proof.

The convergence of the invariant measures

In this section we shall assume, in addition to Hypotheses 0.1, that there exists a real nonnegative continuous increasing function Ψ such that

- the initial value problem

$$z'(r) = -2\Psi(z(r)), \quad z(0) = z_0,$$

  has a unique solution $z(., z_0)$ on $[0, +\infty)$. Moreover, $c(t) = \sup_{z_0 \geq 0} z(t, z_0) < +\infty$ for all $t > 0$ and $\lim_{t\to\infty} c(t) = 0$.

- we assume that

$$(A(x) - A(y), x - y) \geq \Psi\left(|x-y|_H^2\right), \quad x, y \in V.$$

For the solution X of equation above, we consider the transition semigroup

$$P_t\varphi(x) = \mathbb{E}\left[\varphi(X(t,x))\right], \quad x \in H, \ t \geq 0,$$

for all $\varphi \in C_b(H)$ (where $C_b(H)$ is the space of all continuous and bounded functions on H).

Under our assumptions, P_t has an unique invariant measure μ, i.e. a Borel probability measure on H such that

$$\int_H P_t\varphi(x)\,\mu(dx) = \int_H \varphi(x)\,\mu(dx)$$

for all $\varphi \in C_b(H)$, $t > 0$. We know also that μ is ergotic and strongly mixing.

We denote by Λ the set of all invariant measures of P_t^α.

Theorem. Let A and A^{α} satisfying Hypotheses above with $\gamma,\ \eta,\ \beta_1,\ \beta_2$ and C independent of α.
Then the set Λ is tight and then weakly compact.
If we assume also that, for all $y \in H$ and all $\varepsilon > 0$ fixed, we have

$$\left(1+\varepsilon A_H^{\alpha}\right)^{-1} y \to \left(1+\varepsilon A_H\right)^{-1} y, \quad \text{strongly in } H, \text{ for } \alpha \to 0,$$

then $\{\mu^{\alpha}\}_{\alpha}$ is weakly convergent on a subsequence to μ, the invariant measure of P_t.

Proof (sketch)

The main part of the proof is to show that the family of probability measures $\Lambda$is tight, i.e. for all $\varepsilon > 0$ there exists $K_\varepsilon \subset H$, compact, such that $\mu\left(K_\varepsilon^c\right) \leq \varepsilon, \quad \forall \mu \in \Lambda$.

To this propose we apply the Itô formula for

$$\varphi(x) = \varphi_\delta(x) = \frac{|x|_H^2}{1+\delta|x|_H^2}, \quad \delta > 0.$$

We get

$$\begin{aligned} & \mathbb{E}\left(\frac{|X^\alpha(t,x)|_H^2}{1+\delta|X^\alpha(t,x)|_H^2}\right) + 2\gamma\mathbb{E}\left(\int_0^t \frac{\|X^\alpha(s,x)\|_V^p}{\left(1+\delta|X^\alpha(s,x)|_H^2\right)^2} ds\right) \\ \leq \; & \frac{|x|_H^2}{1+\delta|x|_H^2} + \eta\mathbb{E}\left(\int_0^t \frac{|X^\alpha(s,x)|_H^2}{\left(1+\delta|X^\alpha(s,x)|_H^2\right)^2} ds\right) + tTrQ. \end{aligned}$$

For each α we integrate with respect to an arbitrary invariant measure μ^α on H and by the invariance property, i.e.

$$\int_H P_t^\alpha \varphi_\delta (x) \, \mu^\alpha (dx) = \int_H \varphi_\delta (x) \, \mu^\alpha (dx)$$

and letting $\delta \to 0$ we get that

$$2\gamma \int_H \|x\|_V^p \, \mu^\alpha (dx) \leq \eta \int_H |x|_H^2 \, \mu^\alpha (dx) + TrQ$$

and consequently that

$$\int_H \|x\|_V^p \, \mu^\alpha (dx) \leq \frac{\eta\theta^2}{2\gamma} + \frac{1}{2\gamma} \left(\frac{\alpha k^p}{\theta^{p-2}} + 1 \right) TrQ,$$

for $\theta > 0$ sufficiently large.

We define

$$B_\theta = \{x \in V \;\; | \|x\|_V \leq \theta\}$$

which is compact in H since $V \subset H$ compactly.Since

$$\begin{aligned}
\mu^\alpha\left(B_\theta^c\right) &= \int_{B_\theta^c} \mu^\alpha(dx) \leq \frac{1}{\theta^p} \int_H \|x\|_V^p \mu^\alpha(dx) \\
&\leq \frac{1}{\theta^p}\left(\frac{\eta^2\theta^2}{2\gamma} + \frac{1}{2\gamma}\left(\alpha k^p \frac{1}{\theta^{p-2}} + 1\right) TrQ\right) \\
&\leq \frac{1}{\theta^{p-2}} c
\end{aligned}$$

where c is independent of α. It follows that Λ is tight and, by Prokhorov's theorem, we get that the set of probability measures Λ is relatively compact (see [4]). Consequently, all sequence from Λ contains a subsequence weakly convergent.

In order to conclude the proof we have to show that, $\{\mu^{\alpha}\}_{\alpha}$ is weakly convergent as $\alpha \to 0$, on a subsequence, to the invariant measure of the transition semigroup P_t .
From the Krylov - Bogoliubov theorem, we have for each α_k and for $\{T_n\} \uparrow +\infty$ that

$$
\begin{aligned}
\int_H \varphi(x)\, \mu^{\alpha_k}(dx) &= \lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} P_t^{\alpha_k} \varphi(x)\, dt \\
&= \lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} \left(P_t^{\alpha_k} \varphi(x) - P_t \varphi(x) \right) dt \\
&\quad + \lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} P_t \varphi(x)\, dt \\
&= \delta_{\alpha_k} + \int_H \varphi(x)\, \mu(dx).
\end{aligned}
$$

Letting $k \to \infty$ we get that $\int_H \varphi(x)\, \overline{\mu}(dx) = \int_H \varphi(x)\, \mu(dx)$ for all $\varphi \in C_b(H)$. Then $\overline{\mu} = \mu$ where μ is the unique invariant measure corresponding to the transition semigroup P_t.

Homogenization

We shall present now an homogenization results for the equation

$$\begin{cases} dX^{\alpha}(t) - \operatorname{div} a\left(\frac{\xi}{\alpha}, \nabla X^{\alpha}\right) dt = \sqrt{Q}dW(t), \text{ on } \mathcal{O} \\ X^{\alpha}(0) = x, \text{ on } \partial\mathcal{O} \end{cases}$$

Let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^d$ and $Y = [0, s]^d$ such that $Y \subset \mathcal{O}$. Consider the following assumptions

(h_1) The function $j : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$, $(\xi, z) \mapsto j(\xi, z)$ is Y- periodic in ξ, convex and twice continuous differentiable with respect to z and there exist $0 < \Lambda_1 \leq \Lambda_2 < \infty$, independent of ξ, such that

$$\Lambda_1 |z|^2 \leq j(\xi, z) \leq \Lambda_2 \left(|z|^2 + 1 \right),$$

for. $\xi \in \mathbb{R}^d$ a.e. for all $z \in \mathbb{R}^d$.

(h_2) Let $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $a(\xi, z) = \nabla_z j(\xi, z)$ satisfying $a(\xi, 0) = 0$ for all $\xi \in \mathbb{R}^d$ and

$$\begin{aligned} \langle a(\xi, z_1) - a(\xi, z_2), z_1 - z_2 \rangle &\geq \Lambda_1 |z_1 - z_2|^2, \\ |a(\xi, z_1) - a(\xi, z_2)| &\leq \Lambda_2 |z_1 - z_2|, \quad \forall\, z_1,\ z_2 \in \mathbb{R}^d. \end{aligned}$$

(h_3) Denote by $a_{i,j}(\xi, z) = \frac{\partial}{\partial z_j} a_i(\xi, z)$. Then there exist $C_1,\ C_2 > 0$, independent of ξ and z, such that

$$C_1 |x|^2 \le \sum_{i,j=1}^{d} a_{ij}(\xi, z)\, x_i x_j \le C_2 |x|^2, \text{ for all } x \in \mathbb{R}^d.$$

(h_4) Consider Q for equation (1) of the form $Q = B^{-\sigma}, \quad \sigma > 2 + \frac{n}{2}$, where

$$\begin{cases} By = -\Delta y, \quad y \in D(B), \\ D(B) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}). \end{cases}$$

Step I

For each $\alpha > 0$ we define

$$a^{\alpha} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, \qquad a^{\alpha}(\xi, z)_i = a\left(\frac{\xi}{\alpha}, z\right)_i,$$

for all $z \in \mathbb{R}^d$ and $a.e.$ $\xi \in \mathbb{R}^d$, $i = \overline{1, d}$.
Consider the operator $A^{\alpha} : H_0^1(\mathcal{O}) \to H^{-1}(\mathcal{O})$ defined by

$$\left(A^{\alpha}(u), v\right) = \int_{\mathcal{O}} \left\langle a^{\alpha}(\xi, \nabla u(\xi)), \nabla v(\xi)\right\rangle_{\mathbb{R}^n} d\xi,$$

for all $u, v \in H_0^1(\mathcal{O})$ and $\Phi^{\alpha} : H_0^1(\mathcal{O}) \to \mathbb{R}_+$ such that $A^{\alpha} = \nabla \Phi^{\alpha}$, i.e.,

$$\Phi^{\alpha}(u) = \int_{\mathcal{O}} j^{\alpha}(\xi, \nabla u(\xi))\, d\xi, \quad \text{for all } u \in H_0^1(\mathcal{O}),$$

where $a^{\alpha}(\xi, z) = \nabla_z j^{\alpha}(\xi, z)$.

We observe that A_H^α satisfies the assumptions of the Trotter type result for $H_0^1(\mathcal{O}) \subset L^2(\mathcal{O}) \subset H^{-1}(\mathcal{O})$ and $p = 2$.
Consider the stochastic differential equation

$$\begin{cases} dX^\alpha(t) + A_H^\alpha(X^\alpha(t))\,dt = \sqrt{Q}dW(t), & t \geq 0, \\ X^\alpha(t) = 0, \quad \text{on } \partial\mathcal{O} & t \geq 0, \\ X^\alpha(0) = x. \end{cases} \tag{5}$$

Consequently equation (5) has a unique solution

$$X^\alpha \in L_W^2\left(\Omega; C\left([0,T]; L^2(\mathcal{O})\right) \cap L^2\left(0,T; H_0^1(\mathcal{O})\right)\right).$$

Step II

We define

$$a^{\text{hom}}(z) = \int_Y a(\xi, z + \operatorname{grad} w_z(\xi))\, d\xi$$

for all $z \in \mathbb{R}^d$ and $w_z \in H^1(Y)$, Y- periodic and satisfying

$$-\operatorname{div} a(\xi, \operatorname{grad} w_z(\xi) + z) = 0 \text{ on } Y.$$

We have the operator $A^{\text{hom}} : H_0^1(\mathcal{O}) \to H^{-1}(\mathcal{O})$

$$\left(A^{\text{hom}}(u), v\right) = \int_{\mathcal{O}} \left\langle a^{\text{hom}}(\nabla u(\xi)), \nabla v(\xi)\right\rangle_{\mathbb{R}^n} d\xi,$$

for all $u, v \in H_0^1(\mathcal{O})$ and $\Phi^{\text{hom}} : H_0^1(\mathcal{O}) \to \mathbb{R}_+$

$$\Phi^{\text{hom}}(u) = \int_{\mathcal{O}} j^{\text{hom}}(\xi, \nabla u(\xi))\, d\xi, \quad \text{for all } u \in H_0^1(\mathcal{O}).$$

Consider the equation

$$\begin{cases} dX^{\text{hom}}(t) + A_H^{\text{hom}}\left(X^{\text{hom}}(t)\right) dt = \sqrt{Q}dW(t) \\ X^{\text{hom}}(t) = 0, \quad \text{on} \;\; \partial\mathcal{O} \quad t \geq 0, \\ X^{\text{hom}}(0) = x. \end{cases} \tag{6}$$

The hypotheses from the Trotter result are satisfied for $H_0^1(\mathcal{O}) \subset L^2(\mathcal{O}) \subset H^{-1}(\mathcal{O})$ and $p = 2$, and consequently, equation above has a unique solution

$$X^{\text{hom}} \in L_W^2\left(\Omega; C\left([0, T]; L^2(\mathcal{O})\right) \cap L^2\left(0, T; H_0^1(\mathcal{O})\right)\right).$$

Theorem. Assume that hypotheses (h_i), $i = \overline{1,3}$, and define A_H^{α} and A_H^{hom} as above. Then solution X^{α} to equation (5) is convergent to X^{hom}, the solution of equation (6) as follows

$$\mathbb{E}\left|X^{\alpha}(t,x) - X^{\text{hom}}(t,x)\right|_{L^2(\mathcal{O})}^2 \rightarrow 0,$$

uniformly in t on compact sets of $[0,\infty)$, as $\alpha \rightarrow 0$.
The sequence of invariant measures $\{\mu^{\alpha}\}_{\alpha}$ corresponding to equations (5) is weakly convergent on a subsequence to the invariant measure μ^{hom} corresponding to equation (6).

Proof (sketch)
From [[1], Theorem 1.2 from Chapter 3] we have for all $x \in H_0^1(\mathcal{O})$ and all $\varepsilon > 0$ that

$$(I + \varepsilon \nabla \Phi^{\alpha})^{-1} x \rightarrow \left(I + \varepsilon \nabla \Phi^{\text{hom}}\right)^{-1} x, \quad \text{strongly in } L^2(\mathcal{O}),$$

i.e.

$$(I + \varepsilon A^{\alpha})^{-1} x \rightarrow \left(I + \varepsilon A^{\text{hom}}\right)^{-1} x, \quad \text{strongly in } L^2(\mathcal{O}).$$

Using the Trotter type theorem we get that

$$\mathbb{E} \left| X^{\alpha}(t,x) - X^{\text{hom}}(t,x) \right|^2_{L^2(\mathcal{O})} \rightarrow 0,$$

uniformly in t on compact sets of $[0, \infty)$, as $\alpha \rightarrow 0$.We can now apply the first part and get that

$$\mu^{\alpha} \rightharpoonup \mu^{\text{hom}}$$

weakly on a subsequence, as $\alpha \rightarrow 0$.

- H. Attouch: *Families d'operateurs maximaux monotones et mesurabilite*, Ann. Mat. Pura ed Appl., t. 4, 120, 1979, p. 35-111.
- H. Attouch: *Variational convergence for functions and operators*, Pitman Advanced Publishing Program, 1984.
- V. Barbu, *Analysis and Control of Infinite Dimensional System*, Academic Press, 1993.
- V. Barbu, *Boundary value problems for partial differential equations*, Editura Academiei Romane, 1993.
- V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff International Publishing, Leiden 1976.

- V. Barbu and G. Da Prato, *Ergodicity for Nonlinear Stochastic Equations in Variational Formulation,* Appl.Math Optim, Springer, 53:121-139, 2006.
- H. Brezis *Operateurs Maximaux Monotones et Semigroupes de Contractions dans un Espace de Hilbert,* North Holland 1973.
- E. Pardoux, *Equations aux derivees partielles stochastiques nonlineaires monotones*, These, Universite Paris, 1975.
- G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions,* Cambridge University Press, 1992.
- C. Prevot and M. Röckner, *A concise course on stochastic partial differential equations,* Monogrph , Lecture Notes in Mathematics, Springer, 2006.