# Homogenization of elliptic problems in perforated domains with general boundary conditions

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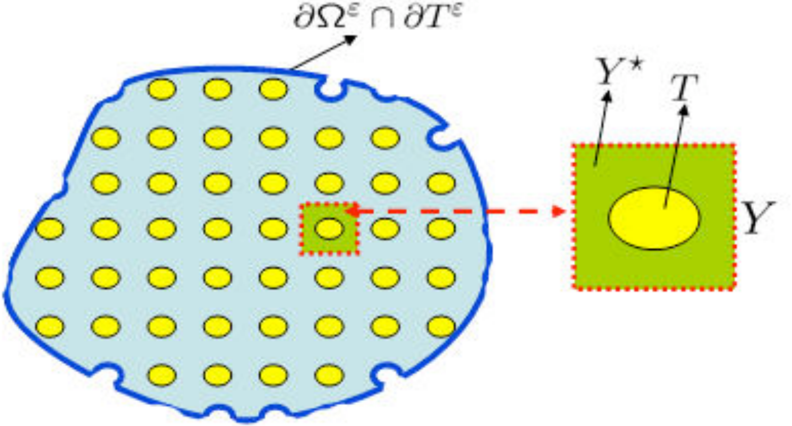
# Introduction




$$x \in \Omega_\varepsilon^*, \quad x = \frac{y}{\varepsilon}, \qquad y \in Y^* \quad y = \varepsilon x.$$

Example 1: Dirichlet-Neumann problem

$$\begin{cases} \displaystyle\int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v \, dx = \int_{\Omega_\varepsilon^*} f \, v \, dx + \int_{\partial S_\varepsilon \cap \Omega} g_\varepsilon v \, d\sigma(x), \\ \forall v \in H_0^1(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*). \end{cases}$$

Example 2: Neumann problem

$$\begin{cases} \displaystyle\int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v \, dx + \int_{\Omega_\varepsilon^*} b_\varepsilon u_\varepsilon \, v dx = \int_{\Omega_\varepsilon^*} f \, v \, dx + \int_{\partial S_\varepsilon \cap \Omega} g_\varepsilon v \, d\sigma(x), \\ \forall v \in H^1(\Omega_\varepsilon^*). \end{cases}$$

- G. Allaire and F. Murat, Homogenization of the homogeneous Neumann problem with nonisolated holes, Asymptotic Analysis 7 (1993).

Example 3: Nonlinear boundary conditions

$$\begin{cases} \displaystyle\int_{\Omega^\varepsilon} A^\varepsilon \nabla u^\varepsilon \nabla v \, dx + a\varepsilon^\gamma \int_{\partial T^\varepsilon} h(u^\varepsilon) v \, d\sigma(x) \\ \qquad \displaystyle = \int_{\Omega^\varepsilon} fv \, dx + \int_{\partial T^\varepsilon} g^\varepsilon v \, d\sigma, \ \forall v \in H_0^1(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*), \end{cases}$$

where $h$ is an increasing nonlinear function satisfying suitable growth conditions.

- C. Conca, J. I. Diaz, A. Linan, C. Timofte, Homogenization in chemical reactive flows, Electronic J. Diff. Eqs., 40 (2004).

Such problems arise in particular, in the modeling of chemical reactive ows.

## **Aim** :To introduce the unfolding method for perforated domains and apply it to some model problems

The results presented here are cointained in joint works with

- Alain Damlamian
- Patrizia Donato
- Georges Griso
- Rachad Zaki

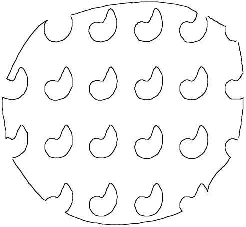
The periodic unfolding method was introduced in order to give an elementary proof for classical periodic homogenization problems, in particular for cases with several micro-scales.

The unfolding method is particularly well adapted for perforated domains defined as follows:
a fixed domain $\Omega$ is given in $\mathbb{R}^n$, together with a reference hole $S$ and a basis of the $\mathbb{R}^n$ whose vectors are macroscopic periods. Then the perforated domain $\Omega^*_\varepsilon$, is obtained by removing from $\Omega$ all the $\varepsilon$-periodic translates of $\varepsilon S$.

A main advantage of the method is that by using an unfolding operator, functions defined on (oscillating) perforated domains are transformed into functions defined on a fixed domain. Therefore, no extension operators are required and so it does away with the regularity hypotheses on the boundary of the perforated domain, necessary for the existence of such extensions. For instance, in the two-dimensional case, the reference hole can be of snow-flake type. In this context, when the unit hole is a compact subset of the open reference cell, the condition insuring the existence of extension operators, is replaced by the weaker condition of existence of a Poincaré-Wirtinger inequality in the perforated reference cell.

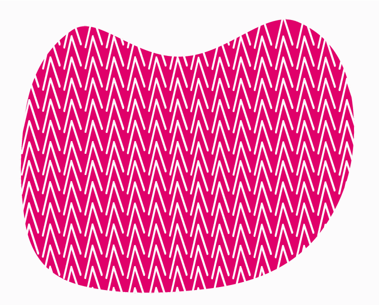
## Consequences

- One can investigate the convergence properties of the unfoldings of functions defined on $\Omega_\varepsilon^*$,
- The use of the periodic unfolding allows to obtain corrector results under minimal regularity assumptions.
- It avoids the hypothesis that the perforated domains have no holes intersecting $\partial\Omega$.

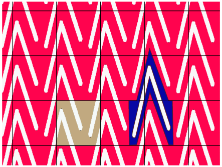


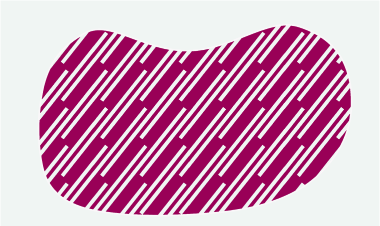
## Remarks

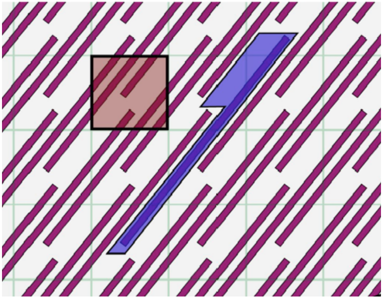
- The unfolding operator is defined in the case when the unit hole is a compact subset of the open unit cell and also in the case when this is impossible to achieve (this can occur in particular in dimensions larger than 2).
- When $S$ is not compact in $Y$, an extra condition in terms of a Poincaré-Wirtinger inequality is required for the union of the unit cell and its translates by a period.
- One can consider also the situation where no choice of the basis of periods gives a parallelotop $Y$ such that $Y^* = Y \setminus S$ is connected (condition necessary for the validity of the Poincaré-Wirtinger inequality). The method applies if there exists a reference cell $Y$ having the paving property with respect to the period basis and such that $Y^*$ is connected.



No choice of parallelotop gives a connected $Y^*$, while there are many possible $Y$'s that give a connected $Y^*$, an example being the following one







**Some applications for which the unfolding method is well-fitted:**

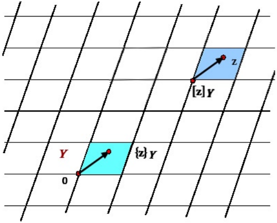
- domains with $\varepsilon-$holes of size of the order of $\varepsilon$ (classical homogenization), or / and of size of the order $r(\varepsilon)\varepsilon$ with $r(\varepsilon) << \varepsilon$ ("small" holes leading to a "strange term"),
- PDE's with complicated conditions on the boundaries of holes, as for example non homogeneous Robin-type conditions or even non linear ones. This is due to the fact that integrals on oscillating boundaries are transformed by a "boundary unfolding operator" into integrals on fixed domains,
- one can consider operators with oscillating coefficients (at a given scale) and perforated domains with holes that are periodic with another (independent) scales,
- multi-scale problems with oscillating coefficients, and any combination of the different situations, for instance, a domain with at the same time $\varepsilon-$size Neumann holes, $r(\varepsilon)\varepsilon-$size Dirichlet and Neumann holes, and also add sieve at some scale.

# Decomposition

Let $Y = \prod_{i=1}^{n} \ell_i$ be a reference cell and $\Omega$ an open subset of $\mathbb{R}^n$.

For $z \in \mathbb{R}^n$, $[z]_Y$ denotes the unique integer combination $\sum_{j=1}^{n} k_j \ell_j$ of the periods such that $z - [z]_Y \in Y$, and set

$$\{z\}_Y = z - [z]_Y \in Y \quad \text{a.e. for} \quad z \in \mathbb{R}^n.$$

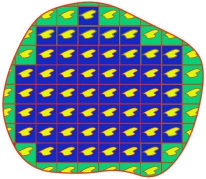



Then for each $x \in \mathbb{R}^n$, one has

$$x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right) \quad \text{a.e. for} \quad x \in \mathbb{R}^n.$$

# The periodic unfolding operator

For $S$, a strict closed subset of $\overline{Y}$ set $Y^* = Y \setminus S$ and $\tau(\varepsilon\overline{S}) = \{\varepsilon\overline{S} \,|\varepsilon(\ell k + \overline{T}),\ k \in \mathbb{Z}^n, k\ell = (k_1\ell_1, ..., k_n\ell_n)\}$.
The perforated domain is defined by $\Omega^*_\varepsilon = \Omega \setminus S_\varepsilon$.



Introduce the sets $\widehat{\Omega}^*_\varepsilon = \widehat{\Omega}_\varepsilon \setminus S_\varepsilon$ and $\Lambda^*_\varepsilon = \Omega^*_\varepsilon \setminus \widehat{\Omega}^*_\varepsilon$.
Definition. *For any function $\phi$ Lebesgue-measurable on $\Omega^*_\varepsilon$, the unfolding operator $\mathcal{T}^*_\varepsilon$ is defined by*

$$\mathcal{T}^*_\varepsilon(\phi)(x,y) = \begin{cases} \phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big) & a.e. \;\; for \;\; (x,y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & a.e. \;\; for \;\; (x,y) \in \Lambda_\varepsilon \times Y^*. \end{cases}$$

Some properties of $\mathcal{T}_\varepsilon^*$.

- For every $\phi$ in $L^1(\Omega_\varepsilon^*)$ and $w$ in $L^p(\Omega_\varepsilon^*)(p \in [1, +\infty[)$,

$$\frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi)(x,y)\, dx\, dy = \int_{\widehat{\Omega}_\varepsilon^*} \phi(x)\, dx = \int_{\Omega_\varepsilon^*} \phi(x)\, dx - \int_{\Lambda_\varepsilon^*} \phi(x)\, dx,$$

$$\|\mathcal{T}_\varepsilon^*(w)\|_{L^p(\Omega \times Y^*)} = |\ Y\ |^{1/p}\ \|w\, 1_{\widehat{\Omega}_\varepsilon^*}\|_{L^p(\Omega_\varepsilon^*)} \leq |\ Y\ |^{1/p}\ \|w\|_{L^p(\Omega_\varepsilon^*)},$$

- Let $\phi_\varepsilon$ be in $L^1(\Omega_\varepsilon^*)$ and satisfying

$$\int_{\Lambda_\varepsilon^*} |\phi_\varepsilon|\, dx \longrightarrow 0.$$

Then

$$\int_{\Omega_\varepsilon^*} \phi_\varepsilon\, dx - \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon)\, dxdy \rightarrow 0.$$

# Macro-micro decomposition

To study the convergence of sequences of gradients, a separation of scales carried out by using a macro-micro decomposition of functions in $W^{1,p}(\Omega_\varepsilon^*)$.
As in the case without holes, the macro approximation $\mathcal{Q}_\varepsilon^*$ is defined by an average at the points of $\Xi_\varepsilon$ and extended to the set $\widehat{\Omega}_\varepsilon^*$ by $Q_1$-interpolation (continuous and piece-wise polynomials of degree $\leq 1$ with respect to each coordinate) as customary in the Finite Element Method.
For $\phi$ in $W^{1,p}(\Omega_\varepsilon^*)$, $p$ in $[1,+\infty]$, set

$$\phi = \mathcal{Q}_\varepsilon^*(\phi) + \mathcal{R}_\varepsilon^*(\phi), \qquad \text{a.e. in } \widehat{\Omega}_\varepsilon^*.$$

By construction, $\mathcal{Q}_\varepsilon^*(\phi)$ is an approximation of $\phi$ while the remainder $\mathcal{R}_\varepsilon^*(\phi)$ of order $\varepsilon$.
As a consequence, for a bounded sequence $\{w_\varepsilon\}$ in $W^{1,p}(\Omega_\varepsilon^*)$, $\{\nabla w_\varepsilon\}$ , $\{\nabla(\mathcal{Q}_\varepsilon(w_\varepsilon))\}$ and $\{\mathcal{T}_\varepsilon^*(\nabla\mathcal{Q}_\varepsilon(w_\varepsilon))\}$ have the "same behavior ". This is not the case for $\mathcal{T}_\varepsilon\big(\nabla(\mathcal{R}_\varepsilon(w_\varepsilon))\big)$ which will give rise to a "correcting" term when passing to the limit.

# The main convergence results

## Theorem

*Suppose that* $w_\varepsilon$ *in* $W^{1,p}(\Omega_\varepsilon^*)$ *satisfies*

$$\|w_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon^*)} \leq C.$$

*Then there exist a subsequence,* $w$ *in* $W^{1,p}(\Omega)$ *and* $\widehat{w}$ *in* $L^p(\Omega; W^{1,p}_{per}(Y^*))$*, such that*

$$\mathcal{T}_\varepsilon^*(w_\varepsilon) \to w \quad \text{strongly in } L^p_{loc}(\Omega; W^{1,p}(Y^*)),$$

$$\mathcal{T}_\varepsilon^*(w_\varepsilon) \rightharpoonup w \quad \text{weakly in } L^p(\Omega; W^{1,p}(Y^*)),$$

*and*

$$\mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widehat{w} \quad \text{weakly in } L^p(\Omega \times Y^*).$$

## Theorem

*Let $\Omega$ be bounded and with Lipschitz boundary. Suppose that $w_\varepsilon$ belongs to $W_0^{1,p}(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*)$ and satisfies*

$$\|\nabla w_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} \leq C.$$

*Then, there exist a subsequence, $w$ in $W_0^{1,p}(\Omega)$, $\widehat{w}$ in $L^p(\Omega; W^{1,p}_{per}(Y^*))$, such that*

$$\|w_\varepsilon - w\|_{L^p(\Omega_\varepsilon^*)} \to 0,$$

*and*

$$\mathcal{T}_\varepsilon^*(w_\varepsilon) \to w \quad \text{strongly in } L^p(\Omega; W^{1,p}(Y^*)),$$
$$\mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widehat{w}, \quad \text{weakly in } L^p(\Omega \times Y^*).$$

# Model problems

Let $\Omega$ be a bounded domain, $f \in L^2(\Omega)$, $A^\varepsilon(x) = (a^\varepsilon_{ij}(x))_{1\le i,j\le n}$ (uniformly) elliptic and with bounded coefficients, $g_\varepsilon \in L^2(\partial S_\varepsilon \cap \Omega)$.

**1.** Dirichlet-Neumann problem

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u_\varepsilon) = f \quad \text{in } \Omega^*_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega^*_\varepsilon \cap \partial\Omega, \\ A^\varepsilon u_\varepsilon \cdot n_\varepsilon = g_\varepsilon \quad \text{on } \partial S_\varepsilon \cap \Omega. \end{cases} \tag{1}$$

**2.** Neumann problem

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u_\varepsilon) + b_\varepsilon u_\varepsilon = f \quad \text{in } \Omega^*_\varepsilon, \\ A^\varepsilon u_\varepsilon \cdot n_\varepsilon = 0 \quad \text{on } \partial\Omega^*_\varepsilon \cap \partial\Omega, \\ A^\varepsilon u_\varepsilon \cdot n_\varepsilon = g_\varepsilon \quad \text{on } \partial S_\varepsilon \cap \Omega, \end{cases} \tag{2}$$

where $b_\varepsilon$ is measurable, positive a.e. in $\Omega$, essentially bounded as well as its inverse.

## Variational formulation of Dirichlet-Neumann problem

$$\begin{cases} \text{Find } u_\varepsilon \in H_0^1(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*) \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v \, dx = \int_{\Omega_\varepsilon^*} f \, v \, dx + \int_{\partial S_\varepsilon \cap \Omega} g_\varepsilon v \, d\sigma(x), \\ \forall v \in H_0^1(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*). \end{cases}$$

## Variational formulation of Neumann problem

$$\begin{cases} \text{Find } u_\varepsilon \in H^1(\Omega_\varepsilon^*) \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v \, dx + \int_{\Omega_\varepsilon^*} b_\varepsilon u_\varepsilon v dx = \int_{\Omega_\varepsilon^*} f \, v \, dx + \int_{\partial S_\varepsilon \cap \Omega} g_\varepsilon v \, d\sigma(x), \\ \forall v \in H^1(\Omega_\varepsilon^*). \end{cases}$$

- $\displaystyle\int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v \, dx \sim \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(A^\varepsilon) \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \mathcal{T}_\varepsilon^*(\nabla v) \, dxdy.$
- If $A^\varepsilon = A(x/\varepsilon)$ then $\mathcal{T}_\varepsilon^*(A^\varepsilon)(x, y) = A(y)$!!
- To treat nonhomogeneus boundary conditions, we use the boundary unfolding operator.

# The boundary unfolding operator $\mathcal{T}_\varepsilon^b$

Let $p$ be in $(1, +\infty)$. Suppose that $\partial S$ is Lipschitz and has a finite number of connected components. The boundary of the set of holes in $\Omega$ is $\partial S_\varepsilon \cap \Omega$, denote by $\widehat{\partial S}_\varepsilon$ those that are included in $\widehat{\Omega}_\varepsilon$. A well-defined trace operator exists from $W^{1,p}(Y^*)$ to $W^{1-1/p,p}(\partial S)$ (each component of $\partial S$ being with Lipschitz boundary), $\Longrightarrow$ the same is true from $W^{1,p}(\widehat{\Omega}^*_\varepsilon)$ to $W^{1-1/p,p}(\widehat{\partial S}_\varepsilon)$.

Aim now : to give a meaning to the unfolding operator for such traces, to obtain estimates and convergences results for sequences of functions in $W^{1,p}-$type spaces.

Definition *For any function $\varphi$ Lebesgue-measurable on $\partial\widehat{\Omega}^*_\varepsilon \cap \partial S^\varepsilon$, the boundary unfolding operator $\mathcal{T}_\varepsilon^b$ is defined by*

$$\mathcal{T}_\varepsilon^b(\phi)(x,y) = \begin{cases} \phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big) & \text{a.e. for } (x,y) \in \widehat{\Omega}_\varepsilon \times \partial S, \\ 0 & \text{a.e. for } (x,y) \in \Lambda_\varepsilon \times \partial S. \end{cases}$$

If $\varphi \in W^{1,p}(\Omega_\varepsilon^*)$, $\mathcal{T}_\varepsilon^b(\varphi)$ is just the trace on $\partial S$ of $\mathcal{T}_\varepsilon^*(\varphi)$. The integration formula reads

$$\frac{1}{\varepsilon|Y|}\int_{\Omega\times\partial S}\mathcal{T}_\varepsilon^b(\varphi)(x,y)\,dx\,d\sigma(y)=\int_{\widehat{\partial S}_\varepsilon}\varphi(x)\,d\sigma(x),$$

from which it follows that

$$\|\mathcal{T}_\varepsilon^b(\varphi)\|_{L^p(\Omega\times\partial S)}=\varepsilon^{1/p}|Y|^{1/p}\|\varphi\|_{L^p(\widehat{\partial S}_\varepsilon)}.$$

The presence of the power of $\varepsilon$ here is significantly different from the previous estimates concerning the operator $\mathcal{T}_\varepsilon^*$ and induces some interesting effects.

# Convergence results for $\mathcal{T}_\varepsilon^b$

### Theorem

*Suppose $w_\varepsilon$ is in $W^{1,p}(\Omega_\varepsilon^*)$, $g_\varepsilon$ is in $L^{p'}(\widehat{S}_\varepsilon)$ and*

$$\mathcal{T}_\varepsilon^*(w_\varepsilon) \to w \qquad \text{strongly in } L^p(\Omega; W^{1,p}(Y^*)),$$
$$\mathcal{T}_\varepsilon^b(g_\varepsilon) \rightharpoonup g \qquad \text{weakly in } L^{p'}(\Omega \times \partial S).$$

*Then*

$$\varepsilon \int_{\widehat{\partial S}_\varepsilon} g_\varepsilon w_\varepsilon \, d\sigma(x) \to \frac{1}{|Y|} \int_{\Omega \times \partial S} g(x,y)\, w(x,y)\, dx d\sigma(y).$$

Several variants, in particular for $w_\varepsilon$ in $W_0^{1,p}(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*)$ satisfying $\|\nabla w_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} \leq C$.
Let $g_\varepsilon \in L^{p'}_{loc}(\partial S_\varepsilon)$ and suppose that

$$\begin{aligned} &\mathcal{T}^b_{\varepsilon,\mathbb{R}^n}(g_\varepsilon) \to g \quad \text{strongly in } L^{p'}_{loc}(\mathbb{R}^n \times \partial S), \\ &\frac{1}{\varepsilon}\mathcal{M}_{\partial S}(\mathcal{T}^b_{\varepsilon,\mathbb{R}^n}(g_\varepsilon)) \rightharpoonup G \quad \text{weakly in } L^{p'}_{loc}(\mathbb{R}^n), \end{aligned}$$

where $\mathcal{T}^b_{\varepsilon,\mathbb{R}^n}$ is the boundary unfolding operator defined in $(\mathbb{R}^n)^*_\varepsilon = \mathbb{R}^n \setminus \overline{S_\varepsilon}$. Then

$$\int_{\partial S_\varepsilon \cap \Omega} g_\varepsilon\, w_\varepsilon\, d\sigma(x) \to \frac{|\partial S|}{|Y|}\int_\Omega \mathcal{M}_{\partial S}(y_{\mathcal{M}}\, g)\cdot\nabla w\, dx + \frac{|\partial S|}{|Y|}\int_\Omega G\, w\, dx + \frac{1}{|Y|}\int_{\Omega\times\partial S} \widehat{w}\; g\; dxdy.$$

# Application to the model problems

## Hypotheses

- There is a matrix $A$ such that

$$\mathcal{T}^*_\varepsilon(A^\varepsilon) \to A \ \text{ a.e. in } \Omega \times Y^* (\text{or in measure in } \Omega \times Y^*).$$

- There exist $g$ in $L^2(\Omega \times \partial S)$ and $G$ in $L^2(\Omega)$ satisfying

$$\begin{aligned} &\mathcal{T}^b_\varepsilon(g_\varepsilon) \rightharpoonup g \quad \text{weakly in } L^2(\Omega \times \partial S), \\ &\frac{1}{\varepsilon}\mathcal{M}_{\partial S}(\mathcal{T}^b_\varepsilon(g_\varepsilon)) \rightharpoonup G \quad \text{weakly in } L^2(\Omega). \end{aligned}$$

Two standard examples of functions $g_\varepsilon$ satisfying these hypotheses

$$\begin{aligned} &g_\varepsilon(x) = \varepsilon\, \mathbf{g}(\{x/\varepsilon\}_Y) \quad \text{if} \quad \mathcal{M}_{\partial S}(\mathbf{g}) \neq 0 \Longrightarrow g = 0,\ G = \mathcal{M}_{\partial S}(\mathbf{g}), \\ &g_\varepsilon(x) = \quad \mathbf{g}(\{x/\varepsilon\}_Y) \quad \text{if} \quad \mathcal{M}_{\partial S}(\mathbf{g}) = 0 \Longrightarrow G = 0. \end{aligned}$$

# Neumann homogenized problem

$$\begin{cases} -\text{div}\,(A^0 \nabla u) + \dfrac{|Y^*|}{|Y|} \mathcal{M}_{Y^*}(b)\, u = \dfrac{|Y^*|}{|Y|} f - div\, \mathcal{G} \quad \text{in } \Omega, \\ A^0 \nabla u \cdot n = \mathcal{G} \cdot n \quad \text{on } \ \partial\Omega. \end{cases}$$

Remark "Strange" phenomenon: the non-homogeneous Neumann condition on the boundary of the holes inside $\Omega$ contribute to a non-homogeneous Neumann condition on $\partial\Omega$ in the limit problem.

$$\mathcal{G}(x) \doteq \frac{|\partial S|}{|Y|} (G + \mathcal{M}_{\partial S}(y_{\mathcal{M}} g)(x)) - \frac{|Y^*|}{|Y|} \mathcal{M}_{Y^*} \left( A(x,\cdot)\, \nabla_y \chi_0(x,\cdot) \right) \ \text{ in } \ \Omega,$$

where $y_{\mathcal{M}} = y - \mathcal{M}_{Y^*}(y)$, $\chi_0$ ($Y$ -periodic) is defined by

$$\begin{cases} -\displaystyle\sum_{i,k=1}^{n} \frac{\partial}{\partial y_i} \left( a_{ik}(x,y) \frac{\partial \chi_0(x,y)}{\partial y_k} \right) = g \quad in \ Y^*, \\ \displaystyle\sum_{i,k=1}^{n} a_{ik}(x,y) \frac{\partial \chi_0(x,y)}{\partial y_k} n_i = 0 \quad on \ Y^*. \end{cases}$$

The homogenized matrix $A^0 = (a^0_{ij})_{1\leq i,j\leq n}$ is elliptic and given by

$$a^0_{ij}(x) = \frac{1}{|Y|}\int_{Y^*}(a_{ij} - \sum_{k=1}^{n} a_{ik}\frac{\partial \chi_j}{\partial y_k})(x,y)\,dy,$$

where the "corrector" functions $\chi_j$ for $(j = 1,\dots,n)$ belong to $L^\infty(\Omega; H^1_{per}(Y^*))$ and are (a.e. $x$ in $\Omega$), the solutions of

$$\begin{cases} -\displaystyle\sum_{i,k=1}^{n}\frac{\partial}{\partial y_i}\Big(a_{ik}(x,y)\big(\frac{\partial \chi_j(x,y)}{\partial y_k} - \delta_{jk}\big)\Big) = 0 \quad \text{in } Y^*, \\ \displaystyle\sum_{i,k=1}^{n} a_{ik}(x,y)\big(\frac{\partial \chi_j(x,y)}{\partial y_k} - \delta_{jk}\big)n_i = 0 \quad \text{on } Y^*, \\ \mathcal{M}_{Y^*}(\chi_j)(x,\cdot) = 0, \qquad \chi_j(x,\cdot) \quad Y\text{-periodic}. \end{cases}$$

# A general corrector result

Using the unfolding method, a general corrector result can be proved. The standard corrector result

$$\Big\|\nabla u_\varepsilon - \nabla u_0 - \sum_{i=1}^{n} \frac{\partial u_0}{\partial x_i} \nabla_y \chi_i\Big(\Big\{\frac{\cdot}{\varepsilon}\Big\}_Y\Big)\Big\|_{L^1(\Omega_\varepsilon^*)} \to 0.$$

is then a simple corollary. To do so, we need to recall the definition of the averaging operator $\mathcal{U}_\varepsilon^*$, the adjoint operator to $\mathcal{T}_\varepsilon^*$ (and its left inverse).

For $p$ in $[1, +\infty]$, the averaging operator $\mathcal{U}_\varepsilon^* : L^p(\Omega \times Y^*) \mapsto L^p(\Omega_\varepsilon^*)$ is defined as

$$\mathcal{U}_\varepsilon^*(\Phi)(x) = \begin{cases} \dfrac{1}{|Y|} \displaystyle\int_Y \Phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon z, \Big\{\frac{x}{\varepsilon}\Big\}_Y\Big)\, dz & \text{a.e. for } x \in \widehat{\Omega}_\varepsilon^*, \\ 0 & \text{a.e. for } x \in \Lambda_\varepsilon^*. \end{cases}$$

## Theorem

*The following strong convergence holds:*

$$\|\nabla u_\varepsilon - \nabla u_0 - \sum_{i=1}^{n} \mathcal{U}_\varepsilon\Big(\frac{\partial u_0}{\partial x_i}\Big)\,\mathcal{U}_\varepsilon^*(\nabla_y \chi_i)\|_{L^2(\Omega_\varepsilon^*)} \to 0.$$

*where $\mathcal{U}_\varepsilon$ and $\mathcal{U}_\varepsilon^*$ are the average operators.*
*In the case where the matrix field $A$ does not depend on $x$, the following corrector result holds:*

$$\|u_\varepsilon - u_0 - \varepsilon \sum_{i=1}^{n} \mathcal{Q}_\varepsilon\Big(\frac{\partial u_0}{\partial x_i}\Big)\chi_i\Big(\Big\{\frac{\cdot}{\varepsilon}\Big\}_Y\Big)\|_{H^1(\Omega_\varepsilon^*)} \to 0.$$

# A problem with nonlinear boundary conditions

Consider the homogenization of the following problem (modeling chemical reactive flows):

$$\begin{cases} -\mathrm{div}(A^\varepsilon \nabla u^\varepsilon) = f \quad \text{in } \Omega^\varepsilon, \\ A^\varepsilon \nabla u^\varepsilon \cdot n + a\varepsilon^\gamma h(u^\varepsilon) = g^\varepsilon \quad \text{on } \partial T^\varepsilon_{int}, \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega^\varepsilon \setminus \partial T^\varepsilon_{int}, \end{cases}$$

where $a \in \mathbb{R}^+$, $\gamma$ is a real parameter, $A^\varepsilon = (a^\varepsilon_{ij}(x))_{1 \le i,j \le N}$ is a matrix field in $\mathcal{M}(\alpha, \beta, \Omega)$, $f \in L^2(\Omega)$, $g^\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right)$ where $g$ is a $Y-$periodic function in $L^2(\partial T)$.

Assume that $h : \mathbb{R} \mapsto \mathbb{R}$ is such that

- $h$ is a continuously differentiable function, monotonously non-decreasing, $h(0) = 0$. Suppose moreover that there exist $C \geq 0$ and $q$ with $0 \leq q \leq +\infty$ if $n = 2$, and $0 \leq q \leq \dfrac{n}{n-2}$ if $n > 2$, such that

$$\left| \frac{\partial h}{\partial s} \right| \leq C \left( 1 + |s|^{q-1} \right), \quad \forall s \in \mathbb{R}.$$

The variational formulation is: find $u^\varepsilon \in V^\varepsilon$

$$\begin{cases} \displaystyle\int_{\Omega^\varepsilon} A^\varepsilon \nabla u^\varepsilon \nabla v \, dx + a\varepsilon^\gamma \int_{\partial T^\varepsilon} h(u^\varepsilon) v \, d\sigma(x) \\ \displaystyle\quad = \int_{\Omega^\varepsilon} fv \, dx + \int_{\partial T^\varepsilon} g^\varepsilon v \, d\sigma(x) \qquad \text{for every } v \in V^\varepsilon, \end{cases}$$

where

$$V_\varepsilon = \{ \varphi \in H^1(\Omega^\varepsilon) \,|\, \varphi = 0 \text{ on } \partial\Omega^\varepsilon \setminus \partial T^\varepsilon_{int} \}.$$

Difficulty: the passage to the limit in the term
$a\varepsilon^\gamma \int_{\partial T^\varepsilon} h(u^\varepsilon) v \, d\sigma(x)$.

**Theorem**. *Given $w^\varepsilon$ in $V_\varepsilon$, with*

$$\|w^\varepsilon\|_{H^1(\Omega^\varepsilon)} \leq C,$$

*suppose there is $w \in H_0^1(\Omega)$ such that*

$$\widetilde{w^\varepsilon} \rightharpoonup \theta w \quad \text{weakly in } L^2(\Omega).$$

*Then,*

$$\mathcal{T}_\varepsilon^b(h(w^\varepsilon)) \rightharpoonup h(w) \quad \text{weakly in } L^{q_1}_{loc}(\Omega; W^{1-\frac{1}{q_1},q_1}(\partial T)),$$

*where*

$$q_1 = \frac{2n}{q(n-2)+n}.$$

# Limit problems

Case $\gamma = 1$

- If $\mathcal{M}_{\partial T}(g) = 0$,

$$\widetilde{u^\varepsilon} \rightharpoonup \theta u^0 \quad \text{weakly in } L^2(\Omega),$$

where $u^0 \in H_0^1(\Omega)$ is the unique solution of the problem

$$-\text{div}(A^0(x)\nabla u^0) + a\frac{|\partial T|}{|Y|}h(u^0) = \theta f \qquad \text{in } \Omega.$$

- If $\mathcal{M}_{\partial T}(g) \neq 0$ and $h$ is positively homogeneous of degree 1,

$$\varepsilon \widetilde{u^\varepsilon} \rightharpoonup \theta u^0 \quad \text{weakly in } L^2(\Omega),$$

where $u^0 \in H_0^1(\Omega)$ is the unique solution of the problem

$$-\text{div}(A^0(x)\nabla u^0) + a\frac{|\partial T|}{|Y|}h(u^0) = \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g) \quad \text{in } \Omega.$$

# Case $\gamma > 1$

- If $\mathcal{M}_{\partial T}(g) = 0$,

$$\widetilde{u^\varepsilon} \rightharpoonup \theta u^0 \quad \text{weakly in } L^2(\Omega),$$

where $u^0 \in H_0^1(\Omega)$ is the unique solution of the problem

$$-\mathrm{div}(A^0(x)\nabla u^0) = \theta f \qquad \text{in } \Omega,$$

- If $\mathcal{M}_{\partial T}(g) \neq 0$ and $h$ is positively homogeneous of degree 1,

$$\varepsilon\widetilde{u^\varepsilon} \rightharpoonup \theta u^0 \quad \text{weakly in } L^2(\Omega),$$

where $u^0 \in H_0^1(\Omega)$ is the unique solution of the problem

$$-\mathrm{div}(A^0(x)\nabla u^0) = \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g) \qquad \text{in } \Omega.$$

## Case $-1 \leq \gamma < 1$ and $a$ strictly positive

- If $\mathcal{M}_{\partial T}(g) = 0$, then the sequence $\{\varepsilon^{\gamma-1}\widetilde{u^\varepsilon}\}$ is bounded in $L^2(\Omega)$, and if $\{\varepsilon_k\}$ is a subsequence such that

$$\varepsilon_k^{\gamma-1}\widetilde{u^\varepsilon}_k \rightharpoonup \theta u^0 \quad \text{weakly in } L^2(\Omega), \;\; \text{then } h\left(u^0\right) = \frac{1}{a}\frac{|Y^\star|}{|\partial T|}f.$$

If the Nemytskii operator associated to $h$ is invertible from $L^{2q}(\Omega)$ on $L^2(\Omega)$, then $u^0 = h^{-1}\left(\frac{1}{a}\frac{|Y^\star|}{|\partial T|}f\right)$.

- If $\mathcal{M}_{\partial T}(g) \neq 0$ and $h$ is positively homogeneous of degree 1, then the sequence $\{\varepsilon^{\gamma}\widetilde{u^\varepsilon}\}$ is bounded in $L^2(\Omega)$, and if $\{\varepsilon_k\}$ is a subsequence such that

$$\varepsilon_k^{\gamma}\widetilde{u^\varepsilon}_k \rightharpoonup \theta u^0 \quad \text{weakly in } L^2(\Omega), \;\; \text{then } h\left(u^0\right) = \frac{1}{a}\mathcal{M}_{\partial T}(g).$$

## Case $\gamma < -1$

$$\widetilde{u^\varepsilon} \rightarrow 0 \quad \text{strongly in } H_0^1(\Omega).$$