On spatial behavior in a poroelastic material

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The mathematical model

- Theory of thermoelastic materials with voids was developed by

S. C. Cowin and J. W. Nunziato - Linear elastic materials with voids, J. Elasticity, vol. 13, pp. 125-147, 1983

J. W. Nunziato and S. C. Cowin - A nonlinear theory of elastic materials with voids, Arch. Rat. Mech. Anal., vol. 72, pp. 175-201, 1979

D. Ieşan - A theory of thermoelastic materials with voids, Acta Mechanica, vol. 60, pp. 67-89, 1986

in order to describe deformation of elastic bodies with small voids or vacuous pores which are distributed throughout the material. Such theory is one of the simple extensions of the classical theory of elasticity for the treatment of porous solids in which the matrix material is elastic and the interstices are void of material.

The theory of thermoporoelastic materials was intensively studied in the last years. An overview state of art can be found in the following books:

- R. De Boer - Theory of porous media: Highlights in historical development and current state. Springer-Verlag, Berlin, 1999;
- D. Ieşan - Thermoelastic models of continua, Kluwer Academic Publishers, Dordrecht, 2004;
- B. Straughan - Stability and wave motion in porous media. Applied Mathematical Sciences, vol. 165, Springer-Verlag, 2008.

Basic equations

The fundamental equations of the poroelastic model in concern consists of: the evolution equations

$$S_{rs,r} + \rho b_s = \varrho \ddot{u}_s \tag{2.1}$$

$$h_{i,i} + g + \varrho \ell = \varrho \kappa \ddot{\varphi} \tag{2.2}$$

in $\Omega \times (0, \infty)$, the constitutive equations

$$S_{rs} = C_{rsmn} e_{mn} + B_{rs} \varphi + D_{rsk} \varphi_{,k}$$

$$h_i = A_{ij} \varphi_{,j} + D_{rsi} e_{rs} + d_i \varphi$$

$$g = -B_{ij} e_{ij} - \xi \varphi - d_i \varphi_{,i} \tag{2.3}$$

in $\bar{\Omega} \times [0, \infty)$, the geometrical equations

$$e_{rs} = \frac{1}{2} \left(u_{r,s} + u_{s,r} \right), \tag{2.4}$$

in $\bar{\Omega} \times [0, \infty)$.

In terms of the displacement and volume fraction fields, the basic equations, for a homogeneous and isotropic body, become

$$\mu\Delta u_i + (\lambda + \mu)\, u_{j,ij} + b\varphi_{,i} + \varrho b_i = \varrho\ddot{u}_i, \tag{2.5}$$

$$\alpha\Delta\varphi - bu_{j,j} - \xi\varphi + \varrho\ell = \varrho\kappa\ddot{\varphi}. \tag{2.6}$$

while the specific internal energy ϵ is given by

$$\rho\epsilon = \frac{1}{2}\lambda e_{rr}e_{ss} + \mu e_{ij}e_{ij} + 2b\varphi e_{rr} + \xi\varphi^2 + \alpha\varphi_{,r}\varphi_{,r}. \tag{2.7}$$

The specific internal energy is positive definite in terms of e_{ij} and φ if and only if

$$\mu > 0, \quad \alpha > 0, \quad \xi > 0, \quad 2\mu + 3\lambda > 0, \quad (2\mu + 3\lambda)\,\xi > 3b^2. \tag{2.8}$$

State of plane strain

We consider the plane strain, parallel to the x_1, x_2-plane, characterized by

$$u_\alpha = u_\alpha(x_1, x_2), \quad u_3 = 0, \quad \varphi = \varphi(x_1, x_2), \quad (x_1, x_2) \in \Sigma. \tag{3.1}$$

It follows that the non-zero strain measures are given by

$$e_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right). \tag{3.2}$$

The non-zero dependent constitutive variables are $t_{\alpha\beta}$, h_α and g and, moreover, we have

$$t_{\alpha\beta} = \lambda e_{\rho\rho}\delta_{\alpha\beta} + 2\mu e_{\alpha\beta} + \beta\varphi\delta_{\alpha\beta}, \tag{3.3}$$

$$h_\rho = \alpha\varphi_{,\rho} \tag{3.4}$$

$$g = -\beta e_{\rho\rho} - \xi\varphi. \tag{3.5}$$

The equations of equilibrium reduce to

$$t_{\beta\alpha,\beta} + F_\alpha = 0, \quad h_{\rho,\rho} + g + G = 0 \quad \text{on } \Sigma. \tag{3.6}$$

The non-zero surface tractions acting at a point x on the curve Γ are given by

$$t_\alpha = t_{\beta\alpha} n_\beta, \quad h = h_\alpha n_\alpha, \tag{3.7}$$

where $n_\alpha = \cos(\mathbf{n}_x, x_\alpha)$ and $\mathbf{n}_x$ is the unit vector of the outward normal to Γ at x.

For convenience we have to note that the state of plane strain has to satisfy the following compatibility condition

$$e_{11,22}+e_{22,11}-2e_{12,12}=0. \tag{3.8}$$

The constitutive equations can be written as

$$\begin{aligned} e_{11} &= \frac{\lambda+2\mu}{4\mu\left(\lambda+\mu\right)}t_{11}-\frac{\lambda}{4\mu\left(\lambda+\mu\right)}t_{22}-\frac{\beta}{2\left(\lambda+\mu\right)}\varphi, \\ e_{22} &= -\frac{\lambda}{4\mu\left(\lambda+\mu\right)}t_{11}+\frac{\lambda+2\mu}{4\mu\left(\lambda+\mu\right)}t_{22}-\frac{\beta}{2\left(\lambda+\mu\right)}\varphi, \\ e_{12} &= \frac{1}{2\mu}t_{12}. \end{aligned} \tag{3.9}$$

Formulation of problem in terms of the Airy stress function and volume fraction

Throughout this section a rectangular region $R : 0 < x_1 < L, 0 < x_2 < \ell$ is considered. It is supposed to be occupied by an isotropic and homogeneous poroelastic material in an equilibrium state of plane strain, the edges $x_1 = L$, $x_2 = 0, \ell$, being traction-free, the remaining edge being (necessarily) subject to a self-equilibrated load.

The Airy stress function $\mathcal{A}(x_1, x_2)$ is introduced to simplify the analysis, and is such that the (relevant) stress components t_{11}, t_{12} and t_{22} are given by

$$t_{11} = \mathcal{A}_{,22}, \quad t_{22} = \mathcal{A}_{,11}, \quad t_{12} = -\mathcal{A}_{,12}. \tag{4.1}$$

The plane equilibrium equations and the compatibility equation give for the Airy stress function $\mathcal{A}(x_1, x_2)$ and the volumetric ratio $\varphi(x_1, x_2)$

$$\mathcal{A}_{,1111} + 2\mathcal{A}_{,1122} + \mathcal{A}_{,2222} - \frac{2\mu\beta}{\lambda + 2\mu}(\varphi_{,11} + \varphi_{,22}) = 0, \tag{4.2}$$

$$\alpha(\varphi_{,11} + \varphi_{,22}) - \left(\xi - \frac{\beta^2}{\lambda + \mu}\right)\varphi - \frac{\beta}{2(\lambda + \mu)}(\mathcal{A}_{,11} + \mathcal{A}_{,22}) = 0. \tag{4.3}$$

The foregoing equations hold in the rectangular region R, while on its boundary the arbitrariness inherent in $\mathcal{A}(x_1, x_2)$ may be used to give the simplified boundary conditions

$$\mathcal{A} = \mathcal{A}_{,2} = 0 \quad \text{on the edges} \quad x_2 = 0, \ell, \tag{4.4}$$

and

$$\mathcal{A} = \mathcal{A}_{,1} = 0 \quad \text{on the edge} \quad x_1 = L, \tag{4.5}$$

in the case of a finite strip. As regards the volumetric fraction we will consider either

$$\varphi = 0 \quad \text{on the edges} \quad x_2 = 0 \quad \text{and} \quad x_2 = \ell, \tag{4.6}$$

or

$$\varphi_{,2} = 0 \quad \text{on the edges} \quad x_2 = 0 \quad \text{and} \quad x_2 = \ell \tag{4.7}$$

and

$$\varphi = 0 \quad \text{on the edge} \quad x_1 = L, \tag{4.8}$$

when a finite strip is considered. In the limiting case when $L \to \infty$ conditions (4.5) and (4.8) are unnecessary.

We introduce the function

$$I\left(x_1\right)=\int_0^{\ell}\left[\left(-\mathcal{A}\mathcal{A}_{,11}+\mathcal{A}_{,1}^2+\mathcal{A}_{,2}^2\right)+\frac{2\alpha\mu\left(\lambda+\mu\right)}{\lambda+2\mu}\varphi^2\right]dx_2-$$

$$-\int_{R_{x_1}}\frac{2\mu\beta}{\lambda+2\mu}\left(\mathcal{A}\varphi_{,1}-\mathcal{A}_{,1}\varphi\right)da,\qquad x_1\in[0,L],\tag{4.9}$$

and note that

$$\frac{d^2I}{dx_1^2}\left(x_1\right)=\int_0^{\ell}\left\{\mathcal{A}_{,11}^2+2\mathcal{A}_{,12}^2+\mathcal{A}_{,22}^2+\frac{4\alpha\mu\left(\lambda+\mu\right)}{\lambda+2\mu}\left(\varphi_{,1}^2+\varphi_{,2}^2\right)+\right.$$

$$\left.+\frac{4\mu}{\lambda+2\mu}\left[\xi\left(\lambda+\mu\right)-\beta^2\right]\varphi^2\right\}dx_2\geq 0\quad\text{for all }\ x_1\in[0,L].\tag{4.10}$$

In view of the end boundary conditions, we deduce that $I(L) = 0$ and $\frac{dI}{dx_1}(L) = 0$. Then the relation (4.10) implies

$$\frac{dI}{dx_1}(x_1) = -\int_{R_{x_1}} \left[\mathcal{A}_{,11}^2 + 2\mathcal{A}_{,12}^2 + \mathcal{A}_{,22}^2 + \frac{4\alpha\mu(\lambda+\mu)}{\lambda+2\mu}\left(\varphi_{,1}^2 + \varphi_{,2}^2\right) + \right.$$

$$\left. + \frac{4\mu}{\lambda+2\mu}\left[\xi(\lambda+\mu) - \beta^2\right]\varphi^2\right] da \leq 0 \quad \text{for all } \; x_1 \in [0, L], \tag{4.11}$$

and

$$I(x_1) = \int_{x_1}^{L} d\eta \int_{R_\eta} \left[\mathcal{A}_{,11}^2 + 2\mathcal{A}_{,12}^2 + \mathcal{A}_{,22}^2 + \frac{4\alpha\mu(\lambda+\mu)}{\lambda+2\mu}\left(\varphi_{,1}^2 + \varphi_{,2}^2\right) + \right.$$

$$\left. + \frac{4\mu}{\lambda+2\mu}\left[\xi(\lambda+\mu) - \beta^2\right]\varphi^2\right] da \geq 0 \quad \text{for all } \; x_1 \in [0, L]. \tag{4.12}$$

This proves that $I(x_1)$ is a measure of the solution $(\mathcal{A}, \varphi)$.

On the other hand, we obtain the following second-order differential inequality

$$\frac{d^2 I}{dx_1^2}(x_1) - \frac{\varkappa_1}{\varkappa_2}\frac{dI}{dx_1}(x_1) - \frac{1}{\varkappa_2} I(x_1) \geq 0 \quad \text{for all } \; x_1 \in [0, L], \tag{4.13}$$

where

$$\varkappa_1 = \frac{\ell\,|\beta|}{4\pi}\sqrt{\frac{2\mu}{\lambda + 2\mu}} \max\left(\frac{\ell}{\pi\sqrt{2\alpha(\lambda+\mu)}}, \frac{1}{\xi(\lambda+\mu) - \beta^2}\right), \tag{4.14}$$

$$\varkappa_2 = \max\left(\frac{\ell^2}{2\pi^2}, \frac{\alpha(\lambda+\mu)}{2\left[\xi(\lambda+\mu) - \beta^2\right]}\right). \tag{4.15}$$

By a well-known Comparison Principle, it follows that $I(x_1)$ is bounded above by the solution of the differential equation corresponding to the differential inequality (4.13) with the same boundary conditions, that is the function $G(x_1)$ satisfying

$$\frac{d^2G}{dx_1^2}(x_1) - \frac{\varkappa_1}{\varkappa_2}\frac{dG}{dx_1}(x_1) - \frac{1}{\varkappa_2}G(x_1) = 0 \quad \text{for all } x_1 \in [0, L], \tag{4.16}$$

with

$$G(0) = I(0), \quad G(L) = I(L). \tag{4.17}$$

On this basis we obtain

$$0 \leq I(x_1) \leq \frac{1 - e^{-(\nu_1+\nu_2)(L-x_1)}}{1 - e^{-(\nu_1+\nu_2)L}} I(0)\, e^{-\nu_2 x_1} + \frac{1 - e^{-(\nu_1+\nu_2)x_1}}{1 - e^{-(\nu_1+\nu_2)L}} I(L)\, e^{-\nu_1(L-x_1)} \leq$$

$$\leq I(0)\, e^{-\nu_2 x_1} + I(L)\, e^{-\nu_1(L-x_1)} \quad \text{for all } x_1 \in [0, L], \tag{4.18}$$

where

$$\nu_1 = \frac{1}{2\varkappa_2}\left(\varkappa_1 + \sqrt{\varkappa_1^2 + 4\varkappa_2}\right), \quad \nu_2 = \frac{1}{2\varkappa_2}\left(-\varkappa_1 + \sqrt{\varkappa_1^2 + 4\varkappa_2}\right). \tag{4.19}$$

Thus, we have

$$0 \leq I(x_1) \leq I(0)\, e^{-\nu_2 x_1} \quad \text{for all } x_1 \in [0, L]. \tag{4.20}$$

Throughout this section we suppose the strip to be occupied by a smoothly varying inhomogeneous isotropic poroelastic material in an equilibrium state of plane strain under self–equilibrated traction and equilibrated force applied on the edge $x_1 = 0$, while the other three edges $x_1 = L$, $x_2 = 0$, $x_2 = \ell$ are traction free and subjected to zero volumetric fraction or zero equilibrated force. We introduce the notations

$$\epsilon = \frac{\lambda + 2\mu}{4\mu(\lambda + \mu)}, \quad \varepsilon = \frac{\lambda}{4\mu(\lambda + \mu)}, \quad \tau = \frac{\beta}{2(\lambda + \mu)} \tag{5.1}$$

and we will consider the following types of inhomogeneities:

(i) ϵ, ε, α, ξ and τ are smooth functions of x_1, when the differential system is substituted by the following one

$$(\epsilon \mathcal{A}_{,11})_{,11} + 2(\epsilon \mathcal{A}_{,12})_{,12} + (\epsilon \mathcal{A}_{,22})_{,22} - \varepsilon'' \mathcal{A}_{,22} - (\tau\varphi)_{,11} - (\tau\varphi)_{,22} = 0,$$

$$(\alpha\varphi_{,1})_{,1} + (\alpha\varphi_{,2})_{,2} - \left(\xi - \frac{2\tau^2}{\epsilon - \varepsilon}\right)\varphi - \tau(\mathcal{A}_{,11} + \mathcal{A}_{,22}) = 0, \quad \text{in } R_0, \tag{5.2}$$

where a prime is used to denote the derivative with respect to x_1;

(ii) ϵ, ε, α, ξ and τ are smooth functions of x_2, when the differential system is substituted by the following one

$$\left(\epsilon\mathcal{A}_{,11}\right)_{,11} + 2\left(\epsilon\mathcal{A}_{,12}\right)_{,12} + \left(\epsilon\mathcal{A}_{,22}\right)_{,22} - \ddot{\varepsilon}\mathcal{A}_{,11} - \left(\tau\varphi\right)_{,11} - \left(\tau\varphi\right)_{,22} = 0,$$

$$\left(\alpha\varphi_{,1}\right)_{,1} + \left(\alpha\varphi_{,2}\right)_{,2} - \left(\xi - \frac{2\tau^2}{\epsilon - \varepsilon}\right)\varphi - \tau\left(\mathcal{A}_{,11} + \mathcal{A}_{,22}\right) = 0, \quad \text{in } R_0, \quad (5.3)$$

where a superposed dot denotes the derivative with respect to x_2.

Inhomogeneous poroelastic material of type (i)

We introduce the function

$$J\left(x_1\right)=\int_0^{\ell}\left[\epsilon\left(-\mathcal{A}\mathcal{A}_{,11}+\mathcal{A}_{,1}^2+\mathcal{A}_{,2}^2\right)+\tau\varphi\mathcal{A}+\frac{\alpha}{2}\varphi^2\right]dx_2+$$

$$+\int_{R_{x_1}}\left[\epsilon'\left(\mathcal{A}_{,1}^2+\mathcal{A}_{,2}^2\right)+2\tau\varphi\mathcal{A}_{,1}+\frac{\alpha'}{2}\varphi^2\right]da,\quad x_1\in[0,L], \tag{5.4}$$

and note that we have

$$\frac{d^2J}{dx_1^2}\left(x_1\right)\geq\int_0^{\ell}\left[\epsilon\left(\mathcal{A}_{,11}^2+2\mathcal{A}_{,12}^2\right)+\alpha\left(\varphi_{,1}^2+\varphi_{,2}^2\right)+\left(\xi-\frac{2\tau^2}{\epsilon-\varepsilon}\right)\varphi^2+\right.$$

$$\left.+\left(\varepsilon''+\frac{4\pi^2}{\ell^2}\epsilon\right)\mathcal{A}_{,2}^2\right]dx_2\geq 0, \tag{5.5}$$

provided

$$\varepsilon''\geq-\frac{4\pi^2}{\ell^2}\epsilon. \tag{5.6}$$

Further, we have

$$J(x_1) \geq \int_{x_1}^{L} d\eta \int_{R_\eta} \left[\epsilon \left(\mathcal{A}_{,11}^2 + 2\mathcal{A}_{,12}^2 \right) + \alpha \left(\varphi_{,1}^2 + \varphi_{,2}^2 \right) + \left(\xi - \frac{2\tau^2}{\epsilon - \varepsilon} \right) \varphi^2 + \right.$$

$$\left. + \left(\varepsilon'' + \frac{4\pi^2}{\ell^2} \epsilon \right) \mathcal{A}_{,2}^2 \right] da \geq 0. \qquad (5.7)$$

The constitutive restriction (5.6) is assumed - as a minimum - henceforward, and, in these circumstances, the positive definiteness referred to, qualifies (5.7) as a suitable global measure of solution in R_{x_1}.
We can determine the positive parameters δ and γ so that

$$\frac{d^2 J}{dx_1^2}(x_1) - \gamma \frac{dJ}{dx_1}(x_1) - \delta J(x_1) \geq 0 \quad \text{for all } x_1 \in [0, L], \qquad (5.8)$$

provided appropriate assumptions are imposed on the elastic coefficients.

In fact, we set

$$0 < \delta < \min\left(\frac{2\pi^2}{\ell^2}, \delta_1, \delta_2\right), \tag{5.9}$$

where

$$\delta_1 = \min\left(\frac{m_1}{2}, m_2\right) \tag{5.10}$$

$$m_1 \equiv \min_{x_1\in[0,L]} \frac{\varepsilon''}{\epsilon} + \frac{4\pi^2}{\ell^2} > 0, \quad m_2 \equiv \min_{x_1\in[0,L]} \frac{2}{\alpha}\left(\xi - \frac{2\tau^2}{\epsilon-\varepsilon}\right) > 0, \tag{5.11}$$

$$\delta_2 = \min_{x_1\in[0,L]} \left\{ \left(\varepsilon'' + \frac{4\pi^2}{\ell^2}\epsilon\right)\left(\xi - \frac{2\tau^2}{\epsilon-\varepsilon}\right) \Big/ \left[\epsilon\left(\xi - \frac{2\tau^2}{\epsilon-\varepsilon}\right) + \frac{\alpha}{4}\left(\varepsilon'' + \frac{4\pi^2}{\ell^2}\epsilon\right) + \right.\right.$$
$$\left.\left. + \sqrt{\left[\epsilon\left(\xi - \frac{2\tau^2}{\epsilon-\varepsilon}\right) - \frac{\alpha}{4}\left(\varepsilon'' + \frac{4\pi^2}{\ell^2}\epsilon\right)\right]^2 + \frac{\ell^2\tau^2}{4\pi^2}\left(\xi - \frac{2\tau^2}{\epsilon-\varepsilon}\right)\left(\varepsilon'' + \frac{4\pi^2}{\ell^2}\epsilon\right)}\,\right]\right\} \tag{5.12}$$

and γ is so that

$$\gamma > \delta \max\left(\gamma_1, \gamma_2, \gamma_3, \gamma_4\right), \tag{5.13}$$

where

$$\gamma_1 = \max_{x_1 \in [0,L]} \frac{|\epsilon'|}{\varepsilon'' + \frac{4\pi^2}{\ell^2}\epsilon}, \quad \gamma_2 = \frac{\ell^2}{2\pi^2} \max_{x_1 \in [0,L]} \frac{|\epsilon'|}{\epsilon}, \quad \gamma_3 = \frac{1}{2} \max_{x_1 \in [0,L]} \frac{|\alpha'|}{\xi - \frac{2\tau^2}{\epsilon - \varepsilon}},$$

$$\gamma_4 = \frac{\ell^2}{4\pi^2} \max_{x_1 \in [0,L]} \left\{ \frac{1}{\epsilon\left(\xi - \frac{2\tau^2}{\epsilon-\varepsilon}\right)} \left[\epsilon' \left(\xi - \frac{2\tau^2}{\epsilon - \varepsilon}\right) + \frac{\pi^2}{\ell^2}\epsilon\alpha' + \right.\right.$$

$$\left.\left. + \sqrt{\left[\epsilon'\left(\xi - \frac{2\tau^2}{\epsilon - \varepsilon}\right) - \frac{\pi^2}{\ell^2}\epsilon\alpha'\right]^2 + \frac{8\pi^2}{\ell^2}\tau^2\epsilon\left(\xi - \frac{2\tau^2}{\epsilon - \varepsilon}\right)} \right] \right\}. \tag{5.14}$$

Thus, we have

$$0 \leq J\left(x_1\right) \leq J\left(0\right) e^{-\nu_2^* x_1} \quad \text{for all } \; x_1 \in [0, L], \tag{5.15}$$

where

$$\nu_2^* = \frac{1}{2}\left(-\gamma + \sqrt{\gamma^2 + 4\delta}\right). \tag{5.16}$$

Such estimate is possible only if the relation (5.11) holds true!

Application to functionally graded materials

As an example, we consider a poroelastic material essentially characterized by

$$\epsilon\left(x_1\right)=E_0 e^{-p x_1}, \quad \varepsilon\left(x_1\right)=e_0 e^{-p x_1}, \quad \alpha\left(x_1\right)=A_0 e^{-p x_1},$$
$$\tau\left(x_1\right)=T_0 e^{-p x_1}, \quad \xi=X_0 e^{-p x_1}, \tag{5.17}$$

where E_0, A_0, X_0, e_0 and T_0 are prescribed constants. Then we have

$$E_0>0, \quad A_0>0, \quad X_0>0, \quad E_0-e_0>0, \quad T_0^2<\frac{1}{2} X_0\left(E_0-e_0\right) \tag{5.18}$$

and

$$e_0>-\frac{4 \pi^2}{\ell^2 p^2} E_0. \tag{5.19}$$

and we can take

$$0<\delta<\min \left(\frac{2 \pi^2}{\ell^2}, \frac{2 \pi^2}{\ell^2}+\frac{p^2 e_0}{2 E_0}, \frac{2}{A_0}\left(X_0-\frac{2 T_0^2}{E_0-e_0}\right),\left[\left(p^2 e_0+\frac{4 \pi^2}{\ell^2 p^2} E_0\right) \cdot\right.\right.$$
$$\left.\cdot\left(X_0-\frac{2 T_0^2}{E_0-e_0}\right)\right] / \left\{E_0\left(X_0-\frac{2 T_0^2}{E_0-e_0}\right)+\frac{A_0}{4}\left(p^2 e_0+\frac{4 \pi^2}{\ell^2 p^2} E_0\right)+\right.$$
$$+\left[\left(E_0\left(X_0-\frac{2 T_0^2}{E_0-e_0}\right)-\frac{A_0}{4}\left(p^2 e_0+\frac{4 \pi^2}{\ell^2 p^2} E_0\right)\right)^2+\right.$$

and

$$\gamma > \delta \max\left(\frac{pE_0}{p^2 e_0 + \frac{4\pi^2}{\ell^2}E_0}, \frac{p\ell^2}{2\pi^2}, \frac{pA_0}{2\left(X_0 - \frac{2T_0^2}{E_0 - e_0}\right)}, \frac{\ell^2}{4\pi^2 E_0\left(X_0 - \frac{2T_0^2}{E_0 - e_0}\right)} \cdot \right.$$

$$\cdot \left[-\frac{\pi^2 pA_0E_0}{\ell^2} - pE_0\left(X_0 - \frac{2T_0^2}{E_0 - e_0}\right) + \right.$$

$$\left.\left. + \sqrt{\left[\frac{\pi^2 pA_0E_0}{\ell^2} - pE_0\left(X_0 - \frac{2T_0^2}{E_0 - e_0}\right)\right]^2 + \frac{8\pi^2 T_0^2 E_0}{\ell^2}\left(X_0 - \frac{2T_0^2}{E_0 - e_0}\right)}\,\right]\right). \tag{5.21}$$

An arch-like region

We consider a curvilinear strip of the form of an arch-like region R, which in polar coordinates r and θ is described by

$$R : a < r < b, \qquad 0 < \theta < \omega. \tag{6.1}$$

Here a, b and ω ($< 2\pi$) are prescribed positive constants.
The curvilinear strip is made of a homogeneous and isotropic elastic material with voids and is subject to zero body force and zero equilibrated force. The right edge $\theta = 0$ is subject to a prescribed self-equilibrated traction and an equilibrated force, while the other three edges $\theta = \omega$, $r = a$ and $r = b$ are traction free and subjected to zero volume fraction or zero equilibrated force.

If u_r and u_θ are the radial and transversal components of the plane displacement vector in a plane polar reference frame, then the geometrical measures of deformation are

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{1}{r}u_\theta\right)$$

$$e_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) \tag{6.2}$$

By eliminating u_r and u_θ in the above relation we obtain the Saint-Venant compatibility condition in the form

$$r\frac{\partial^2}{\partial r^2}\left(re_{\theta\theta}\right) + \left(\frac{\partial^2}{\partial \theta^2} - r\frac{\partial}{\partial r}\right)e_{rr} - 2\frac{\partial^2}{\partial r \partial \theta}\left(re_{r\theta}\right) = 0 \tag{6.3}$$

The constitutive equations are

$$\begin{aligned}\tau_{rr} &= (\lambda + 2\mu)\, e_{rr} + \lambda e_{\theta\theta} + \beta\varphi \\ \tau_{\theta\theta} &= \lambda e_{rr} + (\lambda + 2\mu)\, e_{\theta\theta} + \beta\varphi \\ \tau_{r\theta} &= 2\mu e_{r\theta}\end{aligned} \tag{6.4}$$

$$\begin{gathered} h_r = \alpha\frac{\partial\varphi}{\partial r} \qquad h_\theta = \frac{\alpha}{r}\frac{\partial\varphi}{\partial\theta} \\ g = -\beta\,(e_{rr} + e_{\theta\theta}) - \xi\varphi. \end{gathered} \tag{6.5}$$

Relation (6.4) can be written as

$$\begin{aligned} e_{rr} &= \epsilon\tau_{rr} - \varepsilon\tau_{\theta\theta} - \eta\varphi \\ e_{\theta\theta} &= -\varepsilon\tau_{rr} + \epsilon\tau_{\theta\theta} - \eta\varphi \\ e_{r\theta} &= (\epsilon + \varepsilon)\,\tau_{r\theta}\end{aligned} \tag{6.6}$$

The equilibrium equations corresponding to a plane strain state (u_r, u_θ, φ) reduce to

$$\begin{aligned}\frac{\partial\tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\tau_{rr}-\tau_{\theta\theta}}{r} &= 0\\ \frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{2}{r}\tau_{r\theta} &= 0\end{aligned} \tag{6.7}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rh_r\right) + \frac{1}{r}\frac{\partial h_\theta}{\partial\theta} + g = 0 \tag{6.8}$$

and the state of plane stress is represented in terms of the Airy stress function $\mathcal{A}$ by

$$\begin{aligned}\tau_{rr} &= \frac{1}{r^2}\frac{\partial^2\mathcal{A}}{\partial\theta^2} + \frac{1}{r}\frac{\partial\mathcal{A}}{\partial r}\\ \tau_{\theta\theta} &= \frac{\partial^2\mathcal{A}}{\partial r^2}\\ \tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\mathcal{A}}{\partial\theta}\right)\end{aligned} \tag{6.9}$$

The basic equations for the Airy stress function $\mathcal{A}$ and the volume fraction φ are

$$\epsilon\left[r\frac{\partial^2}{\partial r^2}\left(r\frac{\partial^2\mathcal{A}}{\partial r^2}\right)+2r\frac{\partial^2}{\partial r\partial\theta}\left(\frac{1}{r}\frac{\partial^2\mathcal{A}}{\partial r\partial\theta}\right)+\frac{\partial^2}{\partial\theta^2}\left(\frac{1}{r^2}\frac{\partial^2\mathcal{A}}{\partial\theta^2}\right)-r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\mathcal{A}}{\partial r}\right)\right.$$
$$\left.+\frac{4}{r^2}\frac{\partial^2\mathcal{A}}{\partial\theta^2}\right]-\eta r^2\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}\right)=0 \qquad (6.10)$$

$$\alpha\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}\right)-\eta\left(\frac{\partial^2\mathcal{A}}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\mathcal{A}}{\partial\theta^2}+\frac{1}{r}\frac{\partial\mathcal{A}}{\partial r}\right)-(\xi-2\beta\eta)\,\varphi=0 \qquad (6.11)$$

Concerning the above differential system we will consider two different boundary value problems.

Problem A

Here we consider the edge $\theta = 0$ to be subjected to a given traction, while the other three edges are free of tractions, that is we have the following boundary conditions

$$\mathcal{A}(a,\theta) = \frac{\partial \mathcal{A}}{\partial r}(a,\theta) = 0, \ \ \mathcal{A}(b,\theta) = \frac{\partial \mathcal{A}}{\partial r}(b,\theta) = 0, \ \ \theta \in [0,\omega] \quad (6.12)$$

$$\mathcal{A}(r,\omega) = \frac{\partial \mathcal{A}}{\partial \theta}(r,\omega) = 0, \ \ r \in [a,b] \quad (6.13)$$

$$\mathcal{A}(r,0) = \int_a^r (r-\varrho)\,\tau_{\theta\theta}(\varrho,0)\,d\varrho, \ \ \frac{1}{r}\frac{\partial \mathcal{A}}{\partial \theta}(r,0) = -\int_a^r \tau_{r\theta}(\varrho,0)\,d\varrho \quad (6.14)$$

$$\begin{aligned} &\varphi(a,\theta) = \varphi(b,\theta) = 0, \ \ \theta \in [0,\omega] \\ &\varphi(r,\omega) = 0, \ \ r \in [a,b] \end{aligned} \quad (6.15)$$

$$\varphi(r,0) = f_0(r), \ \ r \in [a,b] \quad (6.16)$$

where the applied self-equilibrated tractions $\tau_{r\theta}(\rho,0)$ and $\tau_{\theta\theta}(\rho,0)$ satisfy the conditions of global equilibrium.

Problem B

Here we consider the edge $r = a$ to be subjected to a prescribed traction, while the other three edges are free of loads, that is we associate with the differential system the following boundary conditions

$$\mathcal{A}(r,0) = \frac{\partial \mathcal{A}}{\partial \theta}(r,0) = 0, \quad \mathcal{A}(r,\omega) = \frac{\partial \mathcal{A}}{\partial \theta}(r,\omega) = 0, \quad r \in [a,b] \tag{6.17}$$

$$\mathcal{A}(b,\theta) = \frac{\partial \mathcal{A}}{\partial r}(b,\theta) = 0, \quad \theta \in [0,\omega] \tag{6.18}$$

$$\mathcal{A}(a,\theta) = a^2 \int_0^\theta \left[\tau_{rr}(a,s) + \int_0^s \tau_{r\theta}(a,\sigma)\, d\sigma\right] \sin(\theta - s)\, ds$$

$$\frac{\partial \mathcal{A}}{\partial r}(a,\theta) = a \int_0^\theta \left[\tau_{rr}(a,s) + \int_0^s \tau_{r\theta}(a,\sigma)\, d\sigma\right] \sin(\theta - s)\, ds - a \int_0^\theta \tau_{r\theta}(a,\sigma)\, d\sigma \tag{6.19}$$

where the applied self-equilibrated tractions $\tau_{rr}(a,\sigma)$ and $\tau_{r\theta}(a,\sigma)$ satisfy the global equilibrium conditions.

As regards the volumetric fraction we will consider

$$\begin{aligned} &\varphi(r,0)=\varphi(r,\omega)=0, \quad r\in[a,b] \\ &\varphi(b,\theta)=0, \quad \theta\in[0,\omega] \end{aligned} \tag{6.20}$$

$$\varphi(a,\theta)=g_0(\theta), \quad \theta\in[0,\omega] \tag{6.21}$$

To study the spatial behavior, we use the change of variable

$$r=e^t \tag{6.22}$$

and the following change of functions

$$\mathcal{A}(r,\theta)=e^t\psi(t,\theta), \quad \varphi(r,\theta)=\phi(t,\theta) \tag{6.23}$$

For **Problem A** we introduce the following functional

$$E(\theta) = \int_{a_1}^{b_1} \left\{ \epsilon \left[\left(\frac{\partial^2 \psi}{\partial t^2} \right)^2 + 2 \left(\frac{\partial^2 \psi}{\partial t \partial \theta} \right)^2 + \left(\frac{\partial^2 \psi}{\partial \theta^2} + \psi \right)^2 + 2 \left(\frac{\partial \psi}{\partial t} \right)^2 \right] \right.$$

$$\left. + \alpha \left[\left(\frac{\partial \phi}{\partial t} \right)^2 + \left(\frac{\partial \phi}{\partial \theta} \right)^2 \right] + (\xi - 2\beta\eta) e^{2t} \phi^2 \right\} dt \qquad (6.24)$$

and note that we can determine the positive constants γ_1 and γ_2 in order to be satisfied the following second-order differential inequality

$$\mathcal{E}(\theta) \leq \gamma_1 \frac{d^2 \mathcal{E}}{d\theta^2}(\theta) - \gamma_2 \frac{d\mathcal{E}}{d\theta}(\theta) \quad \text{for all } \theta \in [0, \omega] \qquad (6.25)$$

Thus, we get the following estimate

$$\mathcal{E}(\theta) \leq \mathcal{E}(0) e^{-\kappa_2 \theta} \quad \text{for all } \theta \in [0, \omega] \qquad (6.26)$$

For **Problem B** we introduce the following functional

$$F(t)=\int_0^{\omega}\left\{\epsilon\left[\left(\frac{\partial^2\psi}{\partial t^2}\right)^2+2\left(\frac{\partial^2\psi}{\partial t\partial\theta}\right)^2+\left(\frac{\partial^2\psi}{\partial\theta^2}\right)^2+2\left(\frac{\partial\psi}{\partial t}\right)^2-2\left(\frac{\partial\psi}{\partial\theta}\right)^2+\right.\right.$$

$$\left.+\alpha\left[\left(\frac{\partial\phi}{\partial t}\right)^2+\left(\frac{\partial\phi}{\partial\theta}\right)^2\right]+(\xi-2\beta\eta)\,e^{2t}\phi^2\right\}d\theta,\quad t\in[a_1,b_1]\qquad(6.27)$$

and we can establish the following differential inequality

$$\mathcal{F}(t)\leq\chi_1\frac{d^2\mathcal{F}}{dt^2}(t)-\chi_2\frac{d\mathcal{F}}{dt}(t)\quad\text{for all }\ t\in[a_1,b_1]\qquad(6.28)$$

where

$$\chi_1=\max\left(\frac{1}{2\tau},\frac{3}{2}+\frac{|\eta|}{2\sqrt{\epsilon(\xi-2\beta\eta)}},\frac{|\eta|}{2\sqrt{\epsilon(\xi-2\beta\eta)}}+\frac{\alpha}{2e^{2a_1}(\xi-2\beta\eta)}\right)$$

$$\chi_2=\max\left(\frac{2|\eta|}{\sqrt{\epsilon(\xi-2\beta\eta)}},\frac{|\eta|}{\left(\frac{\pi^2}{\omega^2}+1\right)\sqrt{\epsilon(\xi-2\beta\eta)}}\right)\qquad(6.29)$$

By applying the Comparison Principle to the above differential inequality, we obtain the following estimate

$$0 \leq \mathcal{F}(t) \leq \mathcal{F}(a_1)\, e^{-\varkappa_2 (t-a_1)} \quad \text{for all } \; t \in [a_1, b_1] \tag{6.30}$$

where

$$\varkappa_2 = \frac{1}{2\chi_1}\left(-\chi_2 + \sqrt{\chi_2^2 + 4\chi_1}\right) \tag{6.31}$$

In terms of the initial variables (r, θ), the decay estimate (6.30) can be written as follows

$$0 \leq \mathcal{F}_1(r) \leq \mathcal{F}_1(a)\left(\frac{a}{r}\right)^{\varkappa_2} \quad \text{for all } \; r \in [a, b] \tag{6.32}$$

where

$$\mathcal{F}_1(r) = \int_r^b \frac{1}{s} F_1(\ln s)\, ds \tag{6.33}$$

$$F_1(s) = \int_s^{b_1} F(\sigma)\, d\sigma \tag{6.34}$$

provided the angle ω is so that

$$0 < \omega < \pi\sqrt{2}. \tag{6.35}$$

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- C. D'Apice and S. Chiriţă, On Saint-Venant principle for a linear poroelastic material in plane strain. *Journal of Mathematical Analysis and Applications*, vol. 363, pp. 454–467, 2010.
- S. Chiriţă and C. D'Apice, On Saint Venant's principle in a poroelastic arch-like region. *Mathematical Methods in the Applied Science* Article first published online: 11 MAR 2010 DOI: 10.1002/mma.1294.

THANK YOU!