Derived cones to reachable sets of second-order semilinear differential inclusions

Aurelian Cernea
University of Bucharest

DERIVED CONES

The concept of derived cone to an arbitrary subset of a normed space has been introduced by M.Hestenes (1966) and successfully used to obtain necessary optimality conditions in Control Theory.

However in the last 25-30 years this concept has been largely ignored in favor of other concepts of tangents cones, that may intrinsically be associated to a point of a given set: the cone of interior directions, the contingent, the quasitangent and, above all, Clarke's tangent cone.

Properties of derived cones may be used to obtain controllability and other results in the qualitative theory of control systems.

Let $(X, ||.||)$ be a normed space.

Definition. A subset $M \subset X$ is said to be a *derived set to* $E \subset X$ at $x \in E$ if for any finite subset $\{v_1, ..., v_k\} \subset M$, there exist $s_0 > 0$ and a continuous mapping $a(.) : [0, s_0]^k \to E$ such that $a(0) = x$ and $a(.)$ is (conically) differentiable at $s = 0$ with the derivative col$[v_1, ..., v_k]$ in the sense that

$$\lim_{\mathbf{R}_+^k \ni \theta \to 0} \frac{||a(\theta) - a(0) - \sum_{i=1}^k \theta_i v_i||}{||\theta||} = 0.$$

A subset $C \subset X$ is said to be a *derived cone* of E at x if it is a derived set and also a convex cone.

If M is a derived set then $M \bigcup \{0\}$ as well as the convex cone generated by M, defined by

$$cco(M) = \{\sum_{i=1}^{k} \lambda_j v_j; \lambda_j \geq 0,\, k \in N,\, v_j \in M,\, j = \overline{1,k}\}$$

is also a derived set, hence a derived cone.

The fact that the derived cone is a proper generalization of the classical concepts in Differential Geometry and Convex Analysis is illustrated by the following results:

If $E \subset \mathbf{R}^n$ is a differentiable manifold and T_xE is the tangent space in the sense of Differential Geometry to E at x

$$T_xE = \{v \in \mathbf{R}^n; \quad \exists\, c(.) : (-s,s) \to X, \quad C^1, \\ c(0) = x, \quad c'(0) = v\},$$

then T_xE is a derived cone; also, if $E \subset \mathbf{R}^n$ is a convex subset then the tangent cone in the sense of Convex Analysis defined by

$$TC_xE = cl\{t(y-x); \quad t \geq 0,\ y \in E\}$$

is also a derived cone.

Since any convex subcone of a derived cone is also a derived cone, such an object may not be uniquely associated to a point $x \in E$; moreover, simple examples show that even a maximal with respect to set-inclusion derived cone may not be uniquely defined: if the set $E \subset \mathbf{R}^2$ is defined by

$$E = C_1 \bigcup C_2,$$
$$C_1 = \{(x,x); x \geq 0\}, \quad C_2 = \{(x,-x), x \leq 0\},$$

then C_1 and C_2 are both maximal derived cones of E at the point $(0,0) \in E$.

The contingent, the quasitangent (intermediate) and Clarke's tangent cones, defined, respectively, by

$$K_xE = \{v \in X; \quad \exists\, s_m \to 0+, \quad \exists\, x_m \to x,\ x_m \in E : \frac{x_m - x}{s_m} \to v\},$$
$$Q_xE = \{v \in X; \quad \forall s_m \to 0+, \quad \exists\, x_m \to x,\ x_m \in E : \frac{x_m - x}{s_m} \to v\},$$
$$C_xE = \{v \in X; \quad \forall\, (x_m, s_m) \to (x, 0+), \quad x_m \in E, \exists\, y_m \in E : \frac{y_m - x_m}{s_m} \to v\}$$

The cone of interior directions defined by

$$I_xE := \{v \in X; \exists\, s_0, r > 0 : x + sB(v,r) \subset E \ \forall\, s \in [0, s_0)\}.$$

If $C \subset X$ is a derived cone to E at x then $C \subset Q_x E$.

If $C \subset I_x E$ is a convex cone then C is a derived cone.

If C is a derived cone with nonempty interior then $Int(C) \subset I_x E$.

Theorem 1. *Let $X = \mathbf{R}^n$. Then $x \in int(E)$ if and only if $C = \mathbf{R}^n$ is a derived cone at $x \in E$ to E.*

A CLASS OF SECOND-ORDER SEMI-LINEAR DIFFERENTIAL INCLUSIONS

$I = [0, T]$ and let X be a real separable Banach space with the norm $||.||$ and with the corresponding metric $d(.,.)$.

A family $\{C(t);\ t \in \mathbf{R}\}$ of bounded linear operators from X into X is a strongly continuous cosine family if

(i) $C(0) = Id$,

(ii) $C(t+s) + C(t-s) = 2C(t)C(s)\ \forall t, s \in \mathbf{R}$,

(iii) the map $t \to C(t)x$ is strongly continuous $\forall x \in X$.

The strongly continuous sine family $\{S(t);\ t \in \mathbf{R}\}$ associated to a strongly continuous cosine family $\{C(t);\ t \in \mathbf{R}\}$ is defined by

$$S(t)y := \int_0^t C(s)yds, \quad y \in X, t \in \mathbf{R}.$$

The infinitesimal generator $A : X \to X$ of a cosine family $\{C(t);\ t \in \mathbf{R}\}$ is defined by

$$Ay = (\frac{d^2}{dt^2})C(t)y|_{t=0}.$$

$F(.,.) : I \times X \to \mathcal{P}(X)$ is a set-valued map

$$x'' \in Ax + F(t,x), \quad x(0) = x_0, \quad x'(0) = x_1. \tag{1}$$

A continuous mapping $x(.) \in C(I,X)$ is called a *mild solution* of problem (1) if there exists a (Bochner) integrable function $f(.) \in L^1(I,X)$ such that:

$$f(t) \in F(t,x(t)) \quad a.e.\,(I),$$

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-u)f(u)du \quad \forall t \in I,$$

i.e., $f(.)$ is a (Bochner) integrable selection of the set-valued map $F(.,x(.))$ and $x(.)$ is the mild solution of the Cauchy problem

$$x'' = Ax + f(t) \quad x(0) = x_0, \quad x'(0) = x_1.$$

We shall call $(x(.), f(.))$ a *trajectory-selection pair* of (1).

Corresponding to each type of tangent cone, say $\tau_x E$ one may introduce a *set-valued directional derivative* of a multifunction $G(.) : E \subset X \to \mathcal{P}(X)$ (in particular of a single-valued mapping) at a point $(x, y) \in Graph(G)$ in the direction $v \in \tau_x E$

$$\tau_y G(x; v) = \{w \in X;\ (v, w) \in \tau_{(x,y)} Graph(G)\}.$$

A set-valued map, $A(.) : X \to \mathcal{P}(X)$ is said to be a *convex* (respectively, closed convex) *process* if $Graph(A(.)) \subset X \times X$ is a convex (respectively, closed convex) cone.

AN EXAMPLE

The Cauchy problem associated to a nonlinear wave equation

$$\begin{aligned}
&\frac{\partial^2 z}{\partial t^2}(t,x) - \frac{\partial^2 z}{\partial x^2}(t,x) \in G(z(t,x)), \quad (0,T)\times(0,\pi),\\
&z(t,0) = z(t,\pi) = 0, \quad \text{in} \quad (0,T),\\
&z(0,x) = z_0(x), \quad \frac{\partial z}{\partial t}(0,x) = z_1(x) \text{ a.e. } (0,\pi),
\end{aligned}$$

where $G(.) : \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map, $z_0(.) \in H_0^1(0,\pi)$, $z_1(.) \in L^2(0,\pi)$.

This problem may be rewritten as (1) with $X = L^2(0,\pi)$, $A : D(A) \subset X \to X$, $Az = \frac{d^2}{dt^2}z$, $D(A) = H_0^1(0,\pi) \cap H^2(0,\pi)$
$F(.) : X \to \mathcal{P}(X)$, $F(z) := Sel\ G(z(.))$.

$Sel\ G(z(.))$ is the set of all $f \in L^2(0,\pi)$ satisfying $f(x) \in G(z(x))$ a.e. for $x \in (0,\pi)$.

$Au = \sum_{n=1}^{\infty} n^2 < u, u_n > u_n, \quad u \in D(A),$

$u_n(t) = (\frac{2}{\pi})^{\frac{1}{2}} \sin(nt),\ n = 1, 2, ...$

A is the infinitesimal generator of the cosine family $\{C(t); t \in \mathbf{R}\}$ defined by
$C(t)u = \sum_{n=1}^{\infty} \cos nt < u, u_n > u_n,\ u \in X.$

The sine family associated is given by
$S(t)u = \sum_{n=1}^{\infty} \frac{1}{n} \sin nt < u, u_n > u_n,\ u \in X.$

MAIN RESULTS

Hypothesis 1. i) $F(.,.) : I \times X \to \mathcal{P}(X)$ *has nonempty closed values and is measurable.*

ii) *There exists* $L(.) \in L^1(I, \mathbf{R}_+)$ *such that, for any* $t \in I, F(t,.)$ *is* $L(t)$*-Lipschitz.*

Hypothesis 2. S *be a separable metric space,* $X_0, X_1 \subset X$ *are closed sets,* $a_0(.) : S \to X_0$, $a_1(.) : S \to X_1$ *and* $c(.) : S \to (0, \infty)$ *are given continuous mappings.*

The continuous mappings $g(.) : S \to L^1(I, X)$, $y(.) : S \to C(I, X)$ *are given such that*

$$(y(s))''(t) = Ay(s)(t) + g(s)(t),$$

$$y(s)(0) \in X_0, \quad (y(s))'(0) \in X_1.$$

There exists a continuous function $p(.) : S \to L^1(I, \mathbf{R}_+)$ *such that*

$$d(g(s)(t), F(t, y(s)(t))) \le p(s)(t) \quad a.e.\,(I),\ \forall\, s \in S.$$

Theorem 2. *Assume Hypotheses 1 and 2.*

Then there exist $M > 0$ *and the continuous functions* $x(.) : S \to L^1(I, X)$, $h(.) : S \to C(I, X)$ *such that for any* $s \in S$ $(x(s)(.), h(s)(.))$ *is a trajectory-selection of (1) satisfying for any* $(t, s) \in I \times S$

$$x(s)(0) = a_0(s), \quad (x(s))'(0) = a_1(s),$$

$$||x(s)(t) - y(s)(t)|| \leq M[c(s) + ||a_0(s) - y(s)(0)||$$

$$+||a_1(s) - (y(s))'(0)|| + \int_0^t p(s)(u)du].$$

Our object of study is the reachable set of (1)

$$x'' \in Ax + F(t,x), \quad x(0) \in X_0, \quad x'(0) \in X_1. \tag{1}$$

$$R_F(T, X_0, X_1) := \{x(T); x(.) \text{ mild sol. of (1)}\}.$$

Hypothesis 3. i) *Hypothesis 2 is satisfied and* $X_0 \subset X$ *is a closed set.*

ii) $(z(.), f(.)) \in C(I,X) \times L^1(I,X)$ *is a trajectory-selection pair of (1) and a family* $P(t,.) : X \to \mathcal{P}(X)$*,* $t \in I$ *of convex processes satisfying for almost all* $t \in I$ *the condition*

$$P(t,u) \subset Q_{f(t)}F(t,.)(z(t);u) \quad \forall\, u \in dom(P(t,.)),$$

is assumed to be given and defines the variational inclusion

$$v'' \in Av + P(t,v). \tag{2}$$

Remark. For any set-valued map $F(.,.)$, one may find an infinite number of families of convex process $P(t,.)$, $t \in I$, satisfying this condition; in fact any family of closed convex subcones of the quasitangent cones,

$$\overline{P}(t) \subset Q_{(z(t),f(t))} graph(F(t,.)),$$

defines the family of closed convex process

$$P(t,u) = \{v \in X;\ (u,v) \in \overline{P}(t)\}, \quad u, v \in X,\ t \in I$$

that satisfy this condition. One is tempted, of course, to take as an "intrinsic" family of such closed convex process, for example Clarke's convex-valued directional derivatives $C_{f(t)}F(t,.)(z(t);.)$.

Theorem 3. *Assume that Hypothesis 3 is satisfied and let $C_0 \subset X$ be a derived cone to X_0 at $z(0)$ and $C_1 \subset X$ be a derived cone to X_1 at $z'(0)$. Then, the reachable set of (2), $R_P(T, C_0, C_1)$ is a derived cone to $R_F(T, X_0, X_1)$ at $z(T)$.*

APPLICATION

An application of Theorem 3 concerns local controllability of the semilinear differential inclusion in (1) along a reference trajectory, $z(.)$ at time T, in the sense that

$$z(T) \in Int(R_F(T, X_0, X_1)).$$

Theorem 4. *Let $X = \mathbf{R}^n$, $z(.), F(.,.)$ and $P(.,.)$ satisfy Hypothesis 3 and let $C_0 \subset X$ be a derived cone to X_0 at $z(0)$ and $C_1 \subset X$ be a derived cone to X_1 at $z'(0)$. If, the variational semilinear differential inclusion in (2) is controllable at T in the sense that $R_P(T, C_0, C_1) = \mathbf{R}^n$, then the differential inclusion (1) is locally controllable along $z(.)$ at time T.*

Proof. Theorem 1 and Theorem 3.

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