Formulations variationnelles utilisant les bipotentiels

Marius Buliga[1] Géry de Saxcé[2] Claude Vallée[3]

[1] "Simion Stoilow" Institute of Mathematics of the Romanian Academy, Bucharest
[2] LML (UMR CNRS 8107), Villeneuve d'Ascq
[3] LMS (UMR CNRS 6610), Poitiers

10ème Colloque Franco-Roumain de Mathématiques Appliquées
Poitiers, 26-31 août 2010

Standard materials

- **Standard Material (Halphen & Nguyen Quoc Son, 1975)** uses Moreau work in Convex Analysis

Standard materials

- **Standard Material (Halphen & Nguyen Quoc Son, 1975)** uses Moreau work in Convex Analysis
- Φ – convex dissipation potential defining a constitutive law by:
  $y \in \partial\Phi(x)$ OR $x \in \partial\Phi^*(y)$ OR $\Phi(x) + \Phi^*(y) = \langle x, y \rangle$

Standard materials

- **Standard Material (Halphen & Nguyen Quoc Son, 1975)** uses Moreau work in Convex Analysis
- Φ – convex dissipation potential defining a constitutive law by:
  $$y \in \partial\Phi(x) \text{ OR } x \in \partial\Phi^*(y) \text{ OR } \Phi(x) + \Phi^*(y) = \langle x, y \rangle$$
- y is a stress variable, x a strain rate variable, and $\langle x, y \rangle$ denotes the duality product between them.

Standard materials

- X and Y are two spaces with duality $\langle \cdot, \cdot \rangle \rightarrow \mathbb{R}$

Standard materials

- X and Y are two spaces with duality $\langle \cdot, \cdot \rangle \to \mathbb{R}$
- $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ convex, proper, lsc function

Standard materials

- X and Y are two spaces with duality $\langle \cdot, \cdot \rangle \to \mathbb{R}$
- $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ convex, proper, lsc function
- $\Phi^* : X \to \mathbb{R} \cup \{+\infty\}$ is the Fenchel-Moreau conjugate:

$$\Phi^*(y) = \sup\{\langle x, y \rangle - \Phi(x) \mid x \in X\}$$

Standard materials

- X and Y are two spaces with duality $\langle \cdot, \cdot \rangle \to \mathbb{R}$
- $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ convex, proper, lsc function
- $\Phi^* : X \to \mathbb{R} \cup \{+\infty\}$ is the Fenchel-Moreau conjugate:

$$\Phi^*(y) = \sup\{\langle x, y \rangle - \Phi(x) \mid x \in X\}$$

- $\partial\Phi$ is the subdifferential of Φ:

$$\partial\Phi(x) = \{y \in X^* \mid \Phi(\xi) - \Phi(x) \geq \langle \xi - x, y \rangle\}$$

Standard materials

- Fenchel inequality: $\Phi(x) + \Phi^*(y) \ \geq \ \langle x, y \rangle$

Standard materials

- Fenchel inequality: $\Phi(x) + \Phi^*(y) \ \geq \ \langle x, y \rangle$

$$y \in \partial\Phi(x) \Longleftrightarrow x \in \partial\Phi^*(y) \Longleftrightarrow \Phi(x) + \Phi^*(y) = \langle x, y \rangle$$

Standard materials

- Fenchel inequality: $\Phi(x) + \Phi^*(y) \ \geq \ \langle x, y \rangle$

$$y \in \partial\Phi(x) \iff x \in \partial\Phi^*(y) \iff \Phi(x) + \Phi^*(y) = \langle x, y \rangle$$

- Remark that if we denote $\mathbf{b}(x, y) = \Phi(x) + \Phi^*(y)$ then the Fenchel inequality becomes: $\mathbf{b}(x, y) \geq \langle x, y \rangle$

Implicit standard materials. Bipotentials

- **b** = b(x,y) is a dissipation bipotential defining a implicit constitutive law

$$y \in \partial \mathbf{b}(\cdot, y)(x)$$

Implicit standard materials. Bipotentials

- **b** = b(x,y) is a dissipation bipotential defining a implicit constitutive law

$$y \in \partial \mathbf{b}(\cdot, y)(x)$$

- $\mathbf{b} : X \times Y \to \mathbb{R} \cup \{+\infty\}$ is bi-convex, bi-lsc $\mathbf{b}(x, y) \geq \langle x, y \rangle$

Implicit standard materials. Bipotentials

- $\mathbf{b} = \mathrm{b(x,y)}$ is a dissipation bipotential defining a implicit constitutive law

$$y \in \partial \mathbf{b}(\cdot, y)(x)$$

- $\mathbf{b} : X \times Y \to \mathbb{R} \cup \{+\infty\}$ is bi-convex, bi-lsc $\mathbf{b}(x,y) \geq \langle x, y \rangle$
- $y \in \partial \mathbf{b}(\cdot, y)(x) \iff x \in \partial \mathbf{b}(x, \cdot)(y) \iff \mathbf{b}(x,y) = \langle x, y \rangle$

Implicit standard materials. Bipotentials

- $\mathbf{b} : X \times Y \to \mathbb{R} \cup \{+\infty\}$ is bi-convex, bi-lsc $\mathbf{b}(x, y) \geq \langle x, y \rangle$

Implicit standard materials. Bipotentials

- $\mathbf{b} : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ is bi-convex, bi-lsc $\mathbf{b}(x, y) \geq \langle x, y \rangle$
- $y \in \partial \mathbf{b}(\cdot, y)(x) \Longleftrightarrow x \in \partial \mathbf{b}(x, \cdot)(y) \Longleftrightarrow \mathbf{b}(x, y) = \langle x, y \rangle$

Implicit standard materials. Bipotentials

- $\mathbf{b} : X \times Y \to \mathbb{R} \cup \{+\infty\}$ is bi-convex, bi-lsc $\mathbf{b}(x, y) \geq \langle x, y \rangle$
- $y \in \partial \mathbf{b}(\cdot, y)(x) \iff x \in \partial \mathbf{b}(x, \cdot)(y) \iff \mathbf{b}(x, y) = \langle x, y \rangle$
- particular case: separable bipotential $\mathbf{b}(x, y) = \Phi(x) + \Phi^*(y)$

Implicit standard materials. Bipotentials

- $\mathbf{b} : X \times Y \to \mathbb{R} \cup \{+\infty\}$ is bi-convex, bi-lsc $\mathbf{b}(x, y) \geq \langle x, y \rangle$
- $y \in \partial \mathbf{b}(\cdot, y)(x) \iff x \in \partial \mathbf{b}(x, \cdot)(y) \iff \mathbf{b}(x, y) = \langle x, y \rangle$
- particular case: separable bipotential $\mathbf{b}(x, y) = \Phi(x) + \Phi^*(y)$
- funny example: Cauchy bipotential $\mathbf{b}(x, y) = \|x\| \|y\|$

Implicit standard materials. Bipotentials

- $\mathbf{b} : X \times Y \to \mathbb{R} \cup \{+\infty\}$ is bi-convex, bi-lsc $\mathbf{b}(x, y) \geq \langle x, y \rangle$
- $y \in \partial \mathbf{b}(\cdot, y)(x) \iff x \in \partial \mathbf{b}(x, \cdot)(y) \iff \mathbf{b}(x, y) = \langle x, y \rangle$
- particular case: separable bipotential $\mathbf{b}(x, y) = \Phi(x) + \Phi^*(y)$
- funny example: Cauchy bipotential $\mathbf{b}(x, y) = \|x\|\|y\|$

  $y = \lambda x, \ \lambda \geq 0 \iff x = \mu y, \ \mu \geq 0 \iff \|x\|\|y\| = \langle x, y \rangle$
- (Cauchy-Bunyakovsky-Schwarz inequality)

Applications in mechanics

- contact with friction - de Saxcé & Feng (1991)

$$b(v, f) = \begin{cases} \mu f_n \|v_t\| & \text{if } f \in K_\mu, v_n \leq 0 \\ +\infty & \text{otherwise} \end{cases}$$

Applications in mechanics

- contact with friction - de Saxcé & Feng (1991)

$$b(v,f) = \begin{cases} \mu f_n \|v_t\| & \text{if } f \in K_\mu, v_n \leq 0 \\ +\infty & \text{otherwise} \end{cases}$$

- non-associated Drücker-Prager model - Berga & de Saxcé (1994)

$$b(\dot{\varepsilon}^p, \sigma) = \begin{cases} C_1 \dot{\varepsilon}^p_m + C_2(\sigma_m - \frac{c}{\tan\phi})\|\dot{\varepsilon}^p\| & \text{if } \sigma \in K, \dot{\varepsilon}^p \in K' \\ +\infty & \text{otherwise} \end{cases}$$

$$\dot{\varepsilon}^p_m = tr\, \dot{\varepsilon}^p \,,\ \sigma_m = tr\, \sigma \,,\ C_1 = \frac{c}{\tan\phi} \,,\ C_2 = k_d(\tan\theta - \tan\phi)$$

Applications in mechanics

- contact with friction - de Saxcé & Feng (1991)

$$b(v, f) = \begin{cases} \mu f_n \|v_t\| & \text{if } f \in K_\mu, v_n \leq 0 \\ +\infty & \text{otherwise} \end{cases}$$

- non-associated Drücker-Prager model - Berga & de Saxcé (1994)

$$b(\dot{\varepsilon}^p, \sigma) = \begin{cases} C_1 \dot{\varepsilon}^p_m + C_2(\sigma_m - \frac{c}{\tan\phi})\|\dot{\varepsilon}^p\| & \text{if } \sigma \in K, \dot{\varepsilon}^p \in K' \\ +\infty & \text{otherwise} \end{cases}$$

$$\dot{\varepsilon}^p_m = tr\, \dot{\varepsilon}^p\ ,\ \sigma_m = tr\, \sigma\ ,\ C_1 = \frac{c}{\tan\phi}\ ,\ C_2 = k_d(\tan\theta - \tan\phi)$$

- cam-clay - de Saxcé (1995), coaxial laws - Vallée et al. (1997) Lemaitre plastic ductile damage law - Bodovillé (1999)

The construction problem

- $M_\Phi = \{(x, y) \mid y \in \partial\Phi(x)\}$ maximal cyclically monotone

The construction problem

- $M_\Phi = \{(x,y) \mid y \in \partial\Phi(x)\}$ maximal cyclically monotone
- Rockafellar theorem: The following are equivalent: - $M \subset X \times Y$ maximal cyclically monotone - $\exists \Phi$ convex, lsc $M = M_\Phi$

The construction problem

- $M_\Phi = \{(x,y) \mid y \in \partial\Phi(x)\}$ maximal cyclically monotone
- Rockafellar theorem: The following are equivalent: - $M \subset X \times Y$ maximal cyclically monotone - $\exists \Phi$ convex, lsc $M = M_\Phi$
- for $\mathbf{b}$ bipotential let $M_{\mathbf{b}} = \{(x,y) \mid \mathbf{b}(x,y) = \langle x, y \rangle\}$

The construction problem

- $M_\Phi = \{(x, y) \mid y \in \partial\Phi(x)\}$ maximal cyclically monotone
- Rockafellar theorem: The following are equivalent: - $M \subset X \times Y$ maximal cyclically monotone - $\exists\Phi$ convex, lsc $M = M_\Phi$
- for **b** bipotential let $M_{\mathbf{b}} = \{(x, y) \mid \mathbf{b}(x, y) = \langle x, y\rangle\}$
- if $\mathbf{b}(x, y) = \Phi(x) + \Phi^*(y)$ then $M_{\mathbf{b}} = M_\Phi$

The construction problem

- $M_\Phi = \{(x,y) \mid y \in \partial\Phi(x)\}$ maximal cyclically monotone
- Rockafellar theorem: The following are equivalent: - $M \subset X \times Y$ maximal cyclically monotone - $\exists \Phi$ convex, lsc $M = M_\Phi$
- for **b** bipotential let $M_{\mathbf{b}} = \{(x,y) \mid \mathbf{b}(x,y) = \langle x,y \rangle\}$
- if $\mathbf{b}(x,y) = \Phi(x) + \Phi^*(y)$ then $M_{\mathbf{b}} = M_\Phi$
- **PROBLEM:** Given $M \subset X \times Y$, find **b** bipotential s.t. $M = M_{\mathbf{b}}$

The construction problem

M. Buliga, G. de Saxcé, C. Vallée: Existence and construction of bipotentials for graphs of multivalued laws, J. Convex Analysis 15(1) (2008) 87-104.

- Cover M with cyclically monotone graphs M_λ

$$M \subset \bigcup_{\lambda \in \Lambda} M_\lambda$$

each M_λ gives Φ_λ convex

The construction problem

M. Buliga, G. de Saxcé, C. Vallée: Existence and construction of bipotentials for graphs of multivalued laws, J. Convex Analysis 15(1) (2008) 87-104.

- Cover M with cyclically monotone graphs M_λ

$$M \subset \bigcup_{\lambda \in \Lambda} M_\lambda$$

  each M_λ gives Φ_λ convex
- define $\mathbf{b}(x, y) = \inf_{\lambda \in \Lambda} \left(\Phi_\lambda(x) + \Phi_\lambda^*(y) \right)$

The construction problem

M. Buliga, G. de Saxcé, C. Vallée: Existence and construction of bipotentials for graphs of multivalued laws, J. Convex Analysis 15(1) (2008) 87-104.

- Cover M with cyclically monotone graphs M_λ

$$M \subset \bigcup_{\lambda \in \Lambda} M_\lambda$$

  each M_λ gives Φ_λ convex
- define $\mathbf{b}(x, y) = \inf_{\lambda \in \Lambda} \left(\Phi_\lambda(x) + \Phi^*_\lambda(y) \right)$
- **Theorem:** $M = M_{\mathbf{b}}$ and $\mathbf{b}$ bipotential **if** the family $\{\Phi_\lambda \mid \lambda \in \Lambda\}$ satisfies an implicit convexity inequality.

The construction problem

Example: the Cauchy bipotential $\mathbf{b}(x, y) = \|x\| \|y\|$
$M(b) = \{(x, y) \, : \, x = \lambda y \, , \, \lambda \geq 0\} \cup \{(0, y) \, : \, y \in \mathbb{R}^n\}$

$$\|x\| \|y\| = \inf_{\lambda \in [0, +\infty]} \left(\Phi_\lambda(x) + \Phi^*_\lambda(y) \right)$$

- for $\lambda \in (0, +\infty)$ take $\Phi_\lambda(x) = \frac{\lambda}{2} \|x\|^2$

The construction problem

Example: the Cauchy bipotential $\mathbf{b}(x, y) = \|x\|\|y\|$
$M(b) = \{(x, y) : x = \lambda y \, , \, \lambda \geq 0\} \cup \{(0, y) : y \in \mathbb{R}^n\}$

$$\|x\|\|y\| = \inf_{\lambda \in [0, +\infty]} \left(\Phi_\lambda(x) + \Phi^*_\lambda(y) \right)$$

- for $\lambda \in (0, +\infty)$ take $\Phi_\lambda(x) = \frac{\lambda}{2}\|x\|^2$
- for $\lambda = 0$ take $\Phi_\lambda(x) = 0$,

The construction problem

Example: the Cauchy bipotential $\mathbf{b}(x, y) = \|x\|\|y\|$
$M(b) = \{(x, y) : x = \lambda y \, , \, \lambda \geq 0\} \cup \{(0, y) : y \in \mathbb{R}^n\}$

$$\|x\|\|y\| = \inf_{\lambda \in [0,+\infty]} (\Phi_\lambda(x) + \Phi^*_\lambda(y))$$

- for $\lambda \in (0, +\infty)$ take $\Phi_\lambda(x) = \frac{\lambda}{2}\|x\|^2$
- for $\lambda = 0$ take $\Phi_\lambda(x) = 0$,
- for $\lambda = 0$ take $\Phi^*_\lambda(y) = 0$.

The construction problem

M. Buliga, G. de Saxcé, C. Vallée: Non maximal cyclically monotone graphs and construction of a bipotential for the Coulomb's dry friction law, J. Convex Analysis 17(1) (2010)

- if M is made by several pieces which are not maximal cyclically monotone, like in the case of Coulomb friction law, then apply (a slight generalization of) the previous result combined with the following one.

The construction problem

M. Buliga, G. de Saxcé, C. Vallée: Non maximal cyclically monotone graphs and construction of a bipotential for the Coulomb's dry friction law, J. Convex Analysis 17(1) (2010)

- if M is made by several pieces which are not maximal cyclically monotone, like in the case of Coulomb friction law, then apply (a slight generalization of) the previous result combined with the following one.
- define $\mathbf{b}(x, y) = \max\left(\Phi_1(x) + \Phi_1^*(y), \Phi_2(x) + \Phi_2^*(y)\right)$

The construction problem

M. Buliga, G. de Saxcé, C. Vallée: Non maximal cyclically monotone graphs and construction of a bipotential for the Coulomb's dry friction law, J. Convex Analysis 17(1) (2010)

- if M is made by several pieces which are not maximal cyclically monotone, like in the case of Coulomb friction law, then apply (a slight generalization of) the previous result combined with the following one.
- define $\mathbf{b}(x, y) = \max\left(\Phi_1(x) + \Phi_1^*(y), \Phi_2(x) + \Phi_2^*(y)\right)$
- **Theorem:** $\mathbf{b}$ bipotential **if and only if** Φ_1, Φ_2 satisfy a condition expressed in terms of inf-convolutions.

The construction problem

M. Buliga, G. de Saxcé, C. Vallée: Bipotentials for non monotone multivalued operators: fundamental results and applications, Acta Applicandae Mathematicae (2009), DOI 10.1007/s10440-009-9488-3.

- if M is only maximal monotone, not cyclically monotone then it admits a globally convex bipotential,

The construction problem

M. Buliga, G. de Saxcé, C. Vallée: Bipotentials for non monotone multivalued operators: fundamental results and applications, Acta Applicandae Mathematicae (2009), DOI 10.1007/s10440-009-9488-3.

- if M is only maximal monotone, not cyclically monotone then it admits a globally convex bipotential,
- related with the theory of selfdual lagrangians

The construction problem

M. Buliga, G. de Saxcé, C. Vallée: Bipotentials for non monotone multivalued operators: fundamental results and applications, Acta Applicandae Mathematicae (2009), DOI 10.1007/s10440-009-9488-3.

- if M is only maximal monotone, not cyclically monotone then it admits a globally convex bipotential,
- related with the theory of selfdual lagrangians
- related with minimax problems.

Blurred constitutive laws

G. de Saxcé, M. Buliga, C. Vallée, Blurred constitutive laws and bipotential convex covers, to appear in Mathematics and Mechanics of Solids

- (merci Michel Jean)

Blurred constitutive laws

G. de Saxcé, M. Buliga, C. Vallée, Blurred constitutive laws and bipotential convex covers, to appear in Mathematics and Mechanics of Solids

- (merci Michel Jean)
- **Theorem:** Let **b** be the bipotential which models Coulomb friction and $\varepsilon > 0$. Then the blurred Coulomb friction law

  $$distance(y, \partial \mathbf{b}(\cdot, y)(x)) \leq \varepsilon$$

  can be expressed as an implicit constitutive law with the help of a bipotential (and we construct it).

Blurred constitutive laws

M. Buliga, G. de Saxcé, C. Vallée, Blurred maximal cyclically monotone graphs and bipotentials, in revision at Journal of Mathematical Analysis and Applications

Theorem: Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a convex, lsc, proper function and $\varepsilon > 0$. If for any $y \in Y$ the set $\bigcup_{\|\bar{y} - y\| \leq \varepsilon} \partial \phi^*(\bar{y})$ is convex then the problem

$$distance(y, \partial \phi(x)) \leq \varepsilon$$

can be expressed as an implicit constitutive law with the help of the bipotential

$$b(x, y) = \phi(x) + \inf \left[\phi^*(y - a) + \langle x, a \rangle \right]$$

The model of Berga & de Saxcé

A. Berga, G. de Saxcé, Elastoplastic finite element analysis of soil problems with implicit standard material constitutive laws, *Rev. Eur. des Eléments Finis* **3**(3) (1994), 411-456

"*One of the advantages of the new formulation is to extend the classical Calculus of Variations to non associated constitutive laws. In the theoretical frame of the Implicit Standard Materials, a new functional, called bifunctional, is introduced, depending on both the displacement and stress field.*"

The model of Berga & de Saxcé

A. Berga, G. de Saxcé, Elastoplastic finite element analysis of soil problems with implicit standard material constitutive laws, *Rev. Eur. des Eléments Finis* **3**(3) (1994), 411-456

" *The exact solution of the Boundary Value Problem corresponds to the simultaneous minimization of the bifunctional, firstly with respect to kinematically admissible displacement fields, when the stress field is equal with the exact one, and secondly with respect to statically admissible stress fields, when the displacement field is the exact one. The two minimization problems are the direct extension of the dual variational principles of displacements and stresses.*"

The model of Berga & de Saxcé

non-associated Drücker-Prager model

- $\varepsilon = D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$, $\varepsilon = \varepsilon^e + \varepsilon^p$

The model of Berga & de Saxcé

non-associated Drücker-Prager model

- $\varepsilon = D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$, $\varepsilon = \varepsilon^e + \varepsilon^p$
- $\sigma = S\varepsilon^e \quad , \quad b_e(\varepsilon^e, \sigma) = \frac{1}{2}\langle \varepsilon^e, S\varepsilon^e\rangle + \frac{1}{2}\langle S^{-1}\sigma, \sigma\rangle$

The model of Berga & de Saxcé

non-associated Drücker-Prager model

- $\varepsilon = D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$, $\varepsilon = \varepsilon^e + \varepsilon^p$
- $\sigma = S\varepsilon^e$, $b_e(\varepsilon^e, \sigma) = \frac{1}{2}\langle \varepsilon^e, S\varepsilon^e \rangle + \frac{1}{2}\langle S^{-1}\sigma, \sigma \rangle$
- $\dot{\varepsilon}^p \in \partial b_p(\dot{\varepsilon}^p, \cdot)(\sigma)$
  $b_p(\dot{\varepsilon}^p, \sigma) = \begin{cases} C_1 \dot{\varepsilon}^p_m + C_2(\sigma_m - \frac{c}{\tan\phi}) \|\dot{\varepsilon}^p\| & \text{if } \sigma \in K, \dot{\varepsilon}^p \in K' \\ +\infty & \text{otherwise} \end{cases}$

The model of Berga & de Saxcé

non-associated Drücker-Prager model

- $\varepsilon = D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$, $\varepsilon = \varepsilon^e + \varepsilon^p$
- $\sigma = S\varepsilon^e \quad , \quad b_e(\varepsilon^e, \sigma) = \frac{1}{2}\langle \varepsilon^e, S\varepsilon^e\rangle + \frac{1}{2}\langle S^{-1}\sigma, \sigma\rangle$
- $\dot{\varepsilon}^p \in \partial b_p(\dot{\varepsilon}^p, \cdot)(\sigma)$
  $b_p(\dot{\varepsilon}^p, \sigma) =$
  $\begin{cases} C_1\dot{\varepsilon}^p_m + C_2(\sigma_m - \frac{c}{\tan\phi})\|\dot{\varepsilon}^p\| & \text{if } \sigma \in K, \dot{\varepsilon}^p \in K' \\ +\infty & \text{otherwise} \end{cases}$
- u is $CA(u_0)$: $u = u_0$ on $\partial_0\Omega$
  σ is $SA(f_v, f_s)$: $div\,\sigma + f_v = 0$ in Ω, $\sigma \cdot n = f_s$ on $\partial_1\Omega$
  $\partial\Omega = \partial_1\Omega \cup \partial_2\Omega$ (disjoint union, ...)

The model of Berga & de Saxcé

- discretized in time problem: knowing the fields of displacement, deformation (elastic and plastic), stress, the increments of the imposed boundary conditions and volume force, find the increments Δu, $\Delta\sigma$, $\Delta\varepsilon^e$, $\Delta\varepsilon^p$

The model of Berga & de Saxcé

- discretized in time problem: knowing the fields of displacement, deformation (elastic and plastic), stress, the increments of the imposed boundary conditions and volume force, find the increments Δu, $\Delta\sigma$, $\Delta\varepsilon^e$, $\Delta\varepsilon^p$
- $\Delta\varepsilon = D(\Delta u)\, , \, \Delta\varepsilon = \Delta\varepsilon^e + \Delta\varepsilon^p$

The model of Berga & de Saxcé

- discretized in time problem: knowing the fields of displacement, deformation (elastic and plastic), stress, the increments of the imposed boundary conditions and volume force, find the increments Δu, $\Delta\sigma$, $\Delta\varepsilon^e$, $\Delta\varepsilon^p$
- $\Delta\varepsilon = D(\Delta u)\,,\ \Delta\varepsilon = \Delta\varepsilon^e + \Delta\varepsilon^p$
- $\Delta\sigma = S\Delta\varepsilon^e$
  $\Delta\varepsilon^p \in \partial\Delta b_p(\Delta\varepsilon^p,\cdot)(\Delta\sigma)$
  $\Delta b_p(\Delta\varepsilon^p,\Delta\sigma) = (\Delta t) b_p(\frac{1}{\Delta t}\Delta\varepsilon^p, \sigma^0 + \Delta\sigma) - \langle \Delta\varepsilon^p, \sigma^0\rangle$

The model of Berga & de Saxcé

- discretized in time problem: knowing the fields of displacement, deformation (elastic and plastic), stress, the increments of the imposed boundary conditions and volume force, find the increments Δu, $\Delta \sigma$, $\Delta \varepsilon^e$, $\Delta \varepsilon^p$
- $\Delta \varepsilon = D(\Delta u)$, $\Delta \varepsilon = \Delta \varepsilon^e + \Delta \varepsilon^p$
- $\Delta \sigma = S \Delta \varepsilon^e$
  $\Delta \varepsilon^p \in \partial \Delta b_p(\Delta \varepsilon^p, \cdot)(\Delta \sigma)$
  $$\Delta b_p(\Delta \varepsilon^p, \Delta \sigma) = (\Delta t) b_p(\frac{1}{\Delta t} \Delta \varepsilon^p, \sigma^0 + \Delta \sigma) - \langle \Delta \varepsilon^p, \sigma^0 \rangle$$
- Δu is $CA(\Delta u_0)$, $\Delta \sigma$ is $SA(\Delta f_v, \delta f_s)$

The model of Berga & de Saxcé

discretized in time problem (Pdisc): knowing the fields of displacement, deformation (elastic and plastic), stress, the increments of the imposed boundary conditions and volume force, find the increments u, σ, ε^e, ε^p

$$\varepsilon = D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right) \ , \ \varepsilon = \varepsilon^e + \varepsilon^p$$

$$\sigma = S\varepsilon^e \quad , \quad b_e(\varepsilon^e, \sigma) = \tfrac{1}{2}\langle \varepsilon^e, S\varepsilon^e \rangle + \tfrac{1}{2}\langle S^{-1}\sigma, \sigma \rangle$$

$\dot{\varepsilon}^p \in \partial b_p(\dot{\varepsilon}^p, \cdot)(\sigma)$ (with a b_p computed from the old plastic bipotential and the input fields)

u is $CA(u_0)$, σ is $SA(f_v, f_s)$

The model of Berga & de Saxcé

- define the inf-convolution:
  $b(\varepsilon, \sigma) = \inf \{ b_e(\varepsilon^e, \sigma) + b_p(\varepsilon^p, \sigma) \, : \, \varepsilon^e + \varepsilon^p = \varepsilon \}$
  (due to the nice form of b_e this is like a Moreau-Yosida regularization w.r.t. ε, so b is smooth in ε)

The model of Berga & de Saxcé

- define the inf-convolution:
  $b(\varepsilon, \sigma) = \inf \{ b_e(\varepsilon^e, \sigma) + b_p(\varepsilon^p, \sigma) \, : \, \varepsilon^e + \varepsilon^p = \varepsilon \}$
  (due to the nice form of b_e this is like a Moreau-Yosida regularization w.r.t. ε, so b is smooth in ε)
- **Proposition:** If b_p is a bipotential then (Pdisc) is equivalent with the following problem (P):
  $\varepsilon = D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$, u is $CA(u_0)$, σ is $SA(f_v, f_s)$
  $\sigma \in \partial b(\cdot, \sigma)(\varepsilon)$

The model of Berga & de Saxcé (revised)

$b(\varepsilon, \sigma) = \inf \{ b_e(\varepsilon^e, \sigma) + b_p(\varepsilon^p, \sigma) \, : \, \varepsilon^e + \varepsilon^p = \varepsilon \}$

Properties of b:

- $b(\varepsilon, \sigma) \geq \langle \varepsilon, \sigma \rangle$

The model of Berga & de Saxcé (revised)

$b(\varepsilon,\sigma) = \inf\{b_e(\varepsilon^e,\sigma) + b_p(\varepsilon^p,\sigma) : \varepsilon^e + \varepsilon^p = \varepsilon\}$

Properties of b:

- $b(\varepsilon,\sigma) \geq \langle \varepsilon, \sigma \rangle$
- $b(\cdot,\sigma)$ is convex and smooth

The model of Berga & de Saxcé (revised)

$b(\varepsilon, \sigma) = \inf \{ b_e(\varepsilon^e, \sigma) + b_p(\varepsilon^p, \sigma) \, : \, \varepsilon^e + \varepsilon^p = \varepsilon \}$

Properties of b:

- $b(\varepsilon, \sigma) \geq \langle \varepsilon, \sigma \rangle$
- $b(\cdot, \sigma)$ is convex and smooth
- b is not a bipotential!

The model of Berga & de Saxcé (revised)

$b(\varepsilon, \sigma) = \inf \left\{ b_e(\varepsilon^e, \sigma) + b_p(\varepsilon^p, \sigma) \,:\, \varepsilon^e + \varepsilon^p = \varepsilon \right\}$

Properties of b:

- $b(\varepsilon, \sigma) \geq \langle \varepsilon, \sigma \rangle$
- $b(\cdot, \sigma)$ is convex and smooth
- b is not a bipotential!
- $b(\varepsilon, \cdot)$ is lsc but not convex!
  Nevertheless (Pdisc) is equivalent with (P) because b_p is a bipotential.

The model of Berga & de Saxcé (revised)

Problem (P):

- $\varepsilon = D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$, u is $CA(u_0)$, σ is $SA(f_v, f_s)$
  $\sigma \in \partial b(\cdot, \sigma)(\varepsilon)$

The model of Berga & de Saxcé (revised)

Problem (P):

- $\varepsilon = D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$, u is $CA(u_0)$, σ is $SA(f_v, f_s)$
  $\sigma \in \partial b(\cdot, \sigma)(\varepsilon)$
- $b(\varepsilon, \sigma) \geq \langle \varepsilon, \sigma \rangle$

The model of Berga & de Saxcé (revised)

Problem (P):

- $\varepsilon = D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$, u is $CA(u_0)$, σ is $SA(f_v, f_s)$
  $\sigma \in \partial b(\cdot, \sigma)(\varepsilon)$

- $b(\varepsilon, \sigma) \geq \langle \varepsilon, \sigma \rangle$
- $b(\cdot, \sigma)$ is convex and smooth ($\partial b(\cdot, \sigma)(\varepsilon)$ is univalued)

The model of Berga & de Saxcé (revised)

Problem (P):

- $\varepsilon = D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$, u is $CA(u_0)$, σ is $SA(f_v, f_s)$
  $\sigma \in \partial b(\cdot, \sigma)(\varepsilon)$

- $b(\varepsilon, \sigma) \geq \langle \varepsilon, \sigma \rangle$
- $b(\cdot, \sigma)$ is convex and smooth ($\partial b(\cdot, \sigma)(\varepsilon)$ is univalued)
- $b(\varepsilon, \cdot)$ is lsc but not convex

The model of Berga & de Saxcé (revised)

Problem (P):

$$\varepsilon = D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right) \text{ , } u \text{ is } CA(0) \text{ , } \sigma \text{ is } SA(0,0)$$

$$\sigma \in \partial b(\cdot, \sigma)(\varepsilon)$$

- $b(\varepsilon, \sigma) \geq \langle \varepsilon, \sigma \rangle$

The model of Berga & de Saxcé (revised)

Problem (P):

$$\varepsilon = D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right) \text{ , } u \text{ is } CA(0) \text{ , } \sigma \text{ is } SA(0,0)$$

$$\sigma \in \partial b(\cdot, \sigma)(\varepsilon)$$

- $b(\varepsilon, \sigma) \geq \langle \varepsilon, \sigma \rangle$
- $b(\cdot, \sigma)$ is convex and $\partial b(\cdot, \sigma)(\varepsilon)$ is univalued

The model of Berga & de Saxcé (revised)

Problem (P):

$$\varepsilon = D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right) \text{ , } u \text{ is } CA(0) \text{ , } \sigma \text{ is } SA(0,0)$$

$$\sigma \in \partial b(\cdot, \sigma)(\varepsilon)$$

- $b(\varepsilon, \sigma) \geq \langle \varepsilon, \sigma \rangle$
- $b(\cdot, \sigma)$ is convex and $\partial b(\cdot, \sigma)(\varepsilon)$ is univalued
- $b(\varepsilon, \cdot)$ is lsc but not convex

The model of Berga & de Saxcé (revised)

Bifunctional: $B(\varepsilon, \sigma) = \int_\Omega b(\varepsilon, \sigma) \, \mathrm{d}x$

Algorithm: the solution (u, σ) of (P) is the limit of the sequence

0. initialize $u^0 \in CA(0)$, $\sigma^0 \in SA(0,0)$

The model of Berga & de Saxcé (revised)

Bifunctional: $B(\varepsilon, \sigma) = \int_{\Omega} b(\varepsilon, \sigma) \, \mathrm{d}x$

Algorithm: the solution (u, σ) of (P) is the limit of the sequence

0. initialize $u^0 \in CA(0)$, $\sigma^0 \in SA(0,0)$
1. given u^n, σ^n

The model of Berga & de Saxcé (revised)

Bifunctional: $B(\varepsilon, \sigma) = \int_\Omega b(\varepsilon, \sigma) \, \mathrm{d}x$

Algorithm: the solution (u, σ) of (P) is the limit of the sequence

0. initialize $u^0 \in CA(0)$, $\sigma^0 \in SA(0,0)$
1. given u^n, σ^n

a. (global step) find u^{n+1} such that

$$B(D(u^{n+1}), \sigma^n) \leq B(D(v), \sigma^n) \quad \forall v \in CA(0)$$

The model of Berga & de Saxcé (revised)

Bifunctional: $B(\varepsilon, \sigma) = \int_\Omega b(\varepsilon, \sigma) \, \mathrm{d}x$

Algorithm: the solution (u, σ) of (P) is the limit of the sequence

0. initialize $u^0 \in CA(0)$, $\sigma^0 \in SA(0,0)$
1. given u^n, σ^n

a. (global step) find u^{n+1} such that

$$B(D(u^{n+1}), \sigma^n) \leq B(D(v), \sigma^n) \quad \forall v \in CA(0)$$

b (local step) take $\sigma^{n+1} \in \partial b(\cdot, \sigma^n)(D(u^{n+1}))$ (integration by parts shows that $\sigma^n \in SA(0,0)$ for any n, in a weak sense)

The model of Berga & de Saxcé (revised)

- Variational formulation: à la Nayroles
  X is the space of ϵ, for example $X = L^2(\Omega, M_{sym}^{n\times n}(\mathbb{R}))$
  Y is the space of σ, for example $Y = L^2(\Omega, M_{sym}^{n\times n}(\mathbb{R}))$
  they are in duality by $\langle \varepsilon, \sigma \rangle_1 = \int_\Omega \varepsilon : \sigma \, dx$

The model of Berga & de Saxcé (revised)

- Variational formulation: à la Nayroles
  X is the space of ϵ, for example $X = L^2(\Omega, M^{n\times n}_{sym}(\mathbb{R}))$
  Y is the space of σ, for example $Y = L^2(\Omega, M^{n\times n}_{sym}(\mathbb{R}))$
  they are in duality by $\langle \varepsilon, \sigma \rangle_1 = \int_\Omega \varepsilon : \sigma \, dx$
- U is the space of u, let's take $U = W^{1,2}(\Omega, \mathbb{R}^n)$

The model of Berga & de Saxcé (revised)

- Variational formulation: à la Nayroles
  X is the space of ϵ, for example $X = L^2(\Omega, M^{n\times n}_{sym}(\mathbb{R}))$
  Y is the space of σ, for example $Y = L^2(\Omega, M^{n\times n}_{sym}(\mathbb{R}))$
  they are in duality by $\langle \varepsilon, \sigma \rangle_1 = \int_\Omega \varepsilon : \sigma \, \mathrm{d}x$
- U is the space of u, let's take $U = W^{1,2}(\Omega, \mathbb{R}^n)$
- P is the space of $f = (f_v, f_s)$ (pairs of volume force, surface force) seen in duality with U by
  $\langle u, f \rangle_2 = \int_\Omega u \cdot f_v \, \mathrm{d}x + \int_{\partial_0 \Omega} u \cdot f_s$

The model of Berga & de Saxcé (revised)

- Variational formulation: à la Nayroles
  $D : U \rightarrow X$ linear and continuous, $D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$

The model of Berga & de Saxcé (revised)

- Variational formulation: à la Nayroles
  $D : U \rightarrow X$ linear and continuous, $D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$
- $U_0 \subset U$ is the space CA(0), $X_0 = D(U_0)$

The model of Berga & de Saxcé (revised)

- Variational formulation: à la Nayroles
  $D : U \to X$ linear and continuous, $D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$
- $U_0 \subset U$ is the space CA(0), $X_0 = D(U_0)$
- $Y_0 \subset Y$ is the space SA(0,0), defined as the space of all $\sigma \in Y$ such that:

$$\forall u \in U_0 \quad \langle \sigma, D(u) \rangle = 0$$

The model of Berga & de Saxcé (revised)

- Variational formulation: à la Nayroles
  $D : U \to X$ linear and continuous, $D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$
- $U_0 \subset U$ is the space CA(0), $X_0 = D(U_0)$
- $Y_0 \subset Y$ is the space SA(0,0), defined as the space of all $\sigma \in Y$ such that:

$$\forall u \in U_0 \quad \langle \sigma, D(u) \rangle = 0$$

- Integrate the relation $\sigma \in \partial b(\cdot, \sigma)(D(u))$ from (P) to get:
  $u \in U_0 \, , \, \sigma \in Y_0 \, , \, \forall \epsilon \in X \quad B(D(u), \sigma) \leq B(\varepsilon, \sigma) - \langle \varepsilon, \sigma \rangle_1$

The model of Berga & de Saxcé (revised)

- Variational formulation: à la Nayroles

$$u \in U_0\,,\ \sigma \in Y_0\,,\ \forall \epsilon \in X \quad B(D(u), \sigma) \leq B(\varepsilon, \sigma) - \langle \varepsilon, \sigma \rangle_1$$

The model of Berga & de Saxcé (revised)

- Variational formulation: à la Nayroles

$$u \in U_0\,,\ \sigma \in Y_0\,,\ \forall \epsilon \in X \quad B(D(u), \sigma) \leq B(\varepsilon, \sigma) - \langle \varepsilon, \sigma \rangle_1$$

- Remark that:
$\forall v \in U_0 \quad B(D(u), \sigma) \leq B(D(v), \sigma)$ (that is 1a.)

The model of Berga & de Saxcé (revised)

- Variational formulation: à la Nayroles

$$u \in U_0\,,\; \sigma \in Y_0\,,\; \forall \epsilon \in X \quad B(D(u), \sigma) \leq B(\varepsilon, \sigma) - \langle \varepsilon, \sigma \rangle_1$$

- Remark that:
  $\forall v \in U_0 \quad B(D(u), \sigma) \leq B(D(v), \sigma)$ (that is 1a.)
- $\sigma \in \partial_1 B(\cdot, \sigma)(D(u))$ (that is 1b.). Indeed, that means

$$\forall \varepsilon \in X \quad B(\varepsilon, \sigma) \geq B(D(u), \sigma) + \langle \varepsilon - D(u), \sigma \rangle_1$$

Conclusion

- Variational formulation:

  We have a fixed point problem in $\sigma \in Y_0$:

$$\sigma \in \partial_1 B(\cdot, \sigma)(D(U_0))$$

Conclusion

- Variational formulation:

  We have a fixed point problem in $\sigma \in Y_0$:

$$\sigma \in \partial_1 B(\cdot, \sigma)(D(U_0))$$

- alternatively:

$$D(U_0) \cap \partial_1 \left(B(\cdot, \sigma)\right)^* (\sigma) \neq \emptyset$$

Conclusion

- Variational formulation:

  We have a fixed point problem in $\sigma \in Y_0$:

$$\sigma \in \partial_1 B(\cdot, \sigma)(D(U_0))$$

- alternatively:

$$D(U_0) \cap \partial_1 \left(B(\cdot, \sigma)\right)^* (\sigma) \neq \emptyset$$

- (remark that $\left(B(\cdot, \sigma)\right)^* (\sigma)$ has an integral expression as a sum of b_e and $\left(b_p(\cdot, \sigma)\right)^* (\sigma)$)