Synchronization of weakly coupled oscillators and applications to physiology modeling

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Le système de van der Pol et Les oscillations de relaxation

Une part importante de la physiologie est fondée sur un bilan de charges électriques en présence. Ceci explique en particulier que parmi les principales contributions à la physiologie moderne figurent celles de spécialistes des sciences de l'ingénieur. L'article précurseur [Van der Pol, 1926], est un exemple historique particulièrement intéressant d'un point de vue pédagogique (voir aussi les articles liés [Van der Pol, 1931], [Van der Pol, Van der Mark, 1927,1928]).

Van der Pol considère un circuit électrique formé d'une impédance, d'une capacité et d'une triode montées en série. On obtient ainsi l'équation

$$L\frac{d^2v}{dt^2} + R(v)\frac{dv}{dt} + \frac{v}{C} = 0.$$

Van der Pol suppose (ce qui est naturel) que la résistance de la triode est une fonction paire du potentiel. Dans une première approximation, on peut supposer que c'est une fonction quadratique du potentiel. La forme normale de cette équation est donc

$$x'' - A(1 - x^2)x' + x = 0.$$

Dans le cas où A est petit, Van der Pol observe que la solution périodique, atteinte au bout d'un grand nombre d'oscillations, est très voisine d'une sinusoide. Dans le cas où A est grand, au contraire, la solution périodique est atteinte très rapidement mais elle est très éloignée d'une sinusoide. Il donne aux oscillations qui correspondent au cycle limite (dans le cas où A est grand) le nom d'oscillations de relaxation.

A la limite où le paramètre A dans l'équation

$$x'' - A(1 - x^2)x' + x = 0,$$

est grand, on peut considérer le système plan associé

$$x' = y$$

$$y' = A(1 - x^2)y + x,$$

et changer de variable (transformation de Liénard) en posant

$$y = Y - A(x - \frac{1}{3}x^3),$$

pour obtenir

$$\frac{1}{A}x' = \frac{1}{A}Y - (x - \frac{1}{3}x^3),$$

$$Y' = x.$$

On change alors Y en $A.Y$, le temps t en $A.t$, puis on pose : $\varepsilon = \frac{1}{A^2}$. On présente alors le système comme une dynamique lente-rapide

$$\varepsilon x' = Y - F(x),$$

$$Y' = x.$$

La dynamique portant sur la variable x est qualifiée de rapide. La cubique $Y = F(x) = x - \frac{1}{3}x^3$ est une variété invariante de la dynamique rapide formée de la réunion des points singuliers. Les points critiques de la fonction F sont donnés par $x = -1$ (qui correspond à un minimum local de la cubique et $x = +1$ qui correspond à un maximum local. Pour analyser le flot en première approximation on change à nouveau le temps t en t/ε et on obtient

$$x' = Y - F(x)$$

$$Y' = \varepsilon x.$$

En première approximation, en dehors de la cubique $Y - F(x) = 0$ le flot est donné par

$$x' = Y - F(x)$$

$$Y' = 0.$$

Donc les orbites sont des droites parallèles à l'axe des x et qui sont telles que x croît si $Y > F(x)$ et décroît si $Y < F(x)$. On obtient ainsi que le flot s'écarte de la branche de $Y = F(x)$ pour $-1 < x < 1$ qui correspond à des points singuliers instables de la dynamique rapide. Cette partie de la variété invariante est qualifiée d'instable. On obtient de même que le flot se rapproche des deux autres branches de la cubique (qui sont qualifiées de stables). Si on revient à la dynamique complète, on doit considérer la variété invariante lente $x = 0$. On peut comprendre les oscillations du système comme le résultat d'une hystérèse. A droite de la variété invariante lente, y croît (l'orbite étant très vite attirée par la branche stable de la cubique). On finit par arriver au maximum local de F. A ce point, la cubique devient instable et le système saute sur la branche stable de la cubique. Se faisant, il traverse $x = 0$ et donc y décroît le long de la branche stable jusqu'à arriver au minimum local. La perte de stabilité de la branche oblige l'orbite à sauter sur la branche stable de droite et le cycle recommence.

L'excitabilité et le système de FitzHugh-Nagumo

On considère ici le système de FitzHugh-Nagumo :

$$\varepsilon\dot{x} = -y + 4x - x^3 + I,$$
$$\dot{y} = b_0 x + b_1 y - c.$$

On choisit d'abord $I = 0$, $b_0 = 1, b_1 = 0$ et $c = c_0 = -2/\sqrt{3}$. La cubique $y = f(x) = -x^3 - 4x$ a un minimum local pour $x = c_0$ et un maximum local pour $x = -c_0$. Le système différentiel a un unique point singulier $(x_0, y_0) = (c_0, f(c_0))$. Si on modifie lègérement la deuxième équation et on considère

$$\varepsilon \dot{x} = -y + f(x),$$
$$\dot{y} = x - \Delta - c_0,$$

avec Δ petit, le système a toujours un unique point singulier (x_1, y_1). La linéarisation du champ de vecteurs au voisinage de ce point montre que le point singulier est un foyer attractif si $\Delta < 0$ et répulsif si $\Delta > 0$. Il y a donc un changement de stabilité si $\Delta = 0$.

On poursuit l'analyse avec ε petit et $\Delta > 0$. On prend d'abord une donnée initiale (x_0, y_0) proche de (x_1, y_1) avec une valeur de y_1 au dessus du "seuil" y_0 : $y_1 > y_0$. Dans ce cas l'orbite va très rapidement à la position d'équilibre. On prend ensuite une donnée initiale (x_0, y_0) proche de (x_1, y_1) avec une valeur de y_1 en dessous du "seuil" y_0 : $y_1 < y_0$. Dans ce cas, la première composante du champ de vecteurs est alors très grande et la deuxième composante peut être considérée comme nulle en première approximation. Ceci conduit au fait que l'orbite est très proche d'un segment de droite parallèle à $y = 0$ parcouru à très grande vitesse. Cette dynamique persiste jusqu'à ce que l'orbite percute la cubique $f(x, y) = 0$. Sur cette cubique, la dynamique lente devient prépondérante et fait lentement remonter le long de la branche stable $y = V_+(x)$. Arrivée au point singulier, la dynamique lente s'annule et la dynamique rapide reprend le dessus et ramène brutalement en sens contraire suivant un segment à nouveau parallèle à l'axe $y = 0$ jusqu'à percuter à nouveau la cubique. Une fois sur la cubique, la dynamique lente ramène doucement vers le point singulier. L'orbite a donc réalisée une grande incursion dans le portrait de phase avant de revenir au point singulier stable. Ce type de comportement est fondamental en physiologie et s'appelle l'excitabilité du point singulier stable.

Le système de FitzHugh-Nagumo est utilisé par exemple pour modéliser l'électrophysiologie des cellules cardiaques du noeud sinusal. Dans ce cadre, la question se pose d'analyser un système dynamique formé par N équations de FitzHugh-Nagumo couplées linéairement :

$$\varepsilon \dot{x}_i = -y_i + f(x_i)$$

$$\dot{y}_i = x_i - c_i - \delta y_i + c(x_i - x_{i+1})$$

On considère le couplage c faible en sorte qu'il est légitime de traiter le système par théorie des perturbations du système découplé ($c = 0$).

Il est facile de vérifier tout d'abord l'existence d'un borné absorbant pour le système découplé qui persiste pour le système complet.

Le système de FitzHugh-Nagumo forcé

On considère le système de FitzHugh-Nagumo forcé :

$$\begin{aligned} \varepsilon\dot{u} &= f(u) - v \\ \dot{v} &= u - c(t) - \delta v, \end{aligned} \tag{1}$$

avec

$$c(t) = -1 + \sin((0.01)t)$$

On observe le fait suivant :

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}(\varepsilon u^2 + v^2) = \varepsilon u\dot{u} + v\dot{v} \\ &= [f(u) - v]u + v[u - c(t) - \delta v] = f(u)u - \delta v^2 - c(t)v \\ &= [-\tfrac{u^4}{3} - \delta v^2] + u^2 - c(t)v. \end{aligned} \tag{2}$$

On note qu'alors :

$$-\delta v^2 - c(t)v \leq -\frac{\delta}{2}v^2 + 2 \mid v \mid -\frac{\delta}{2}v^2$$

$$= -\frac{\delta}{2}[v^2 - \frac{4}{\delta} \mid v \mid] - \frac{\delta}{2}v^2$$

$$-\frac{\delta}{2}[(\mid v \mid - \frac{2}{\delta}]^2 - \frac{4}{\delta^2}] - \frac{\delta}{2}v^2$$

$$\leq \frac{2}{\delta} - \frac{\delta}{2}v^2.$$

On remarque de plus que :

$$-\frac{u^4}{3} + u^2 + \frac{\delta\varepsilon}{2}u^2 =$$

$$-\frac{1}{3}[(u^2 - \frac{3}{2}(1 + \frac{\delta\varepsilon}{2})]^2 - \frac{9}{4}(1 + \frac{\delta\varepsilon}{2})^2$$

$$\leq \frac{3}{4}(1 + \frac{\delta\varepsilon}{2})^2.$$

Ceci donne :

$$[-\frac{u^4}{3} - \delta v^2] + u^2 - c(t)v \leq [\frac{3}{4}(1 + \frac{\delta\varepsilon}{2})^2 + \frac{2}{\delta}] - \frac{\delta}{2}(\varepsilon u^2 + v^2). \qquad (3)$$

On a ainsi obtenu que si B est suffisamment grand, plus précisèment si :

$$B > \frac{[\frac{3}{4}(1 + \frac{\delta\varepsilon}{2})^2]}{\frac{\delta}{2}(1 + \varepsilon)},$$

alors si $\varepsilon u^2 + v^2 > B$, on a

$$\frac{1}{2}\frac{d}{dt}(\varepsilon u^2 + v^2) < 0.$$

On ainsi démontré que la boule $\varepsilon u^2 + v^2 \leq B$ attire toutes les trajectoires. Ceci démontre que l'équation différentielle définit un système dynamique global. Notez l'importance du term δ dans les majorations. Une autre façon de procéder consiste à écrire :

$$\begin{array}{l} \frac{1}{2}\frac{d}{dt}(\varepsilon u^2 + v^2) + \frac{\delta}{2}(\varepsilon u^2 + v^2) \leq \\ M = [\frac{3}{4}(1 + \frac{\delta\varepsilon}{2})^2 + \frac{2}{\delta}], \end{array} \tag{4}$$

et ceci implique :

$$(\varepsilon u^2 + v^2) \leq \frac{2M}{\delta} + (\varepsilon u^2(0) + v^2(0))\mathrm{e}^{-\frac{\delta}{2}t}.$$

Les orbites sont donc bornées et elles définissent un système dynamique global.

On se retrouve donc avec une situation où le système initial présente une variété invariante d'orbites périodiques (dans notre cas un tore de dimension N d'orbites périodiques). La question de la synchronisation du système complet se pose déjà en terme d'accrochage des fréquences, c'est à dire de persistance pour le système perturbé d'au moins une orbite périodique attractive.

Les travaux utilisés dans ce contexte sont ceux de Malkin (1956). On doit aussi remarquer la contribution essentielle de Maurice Roseau (1966).

Dans ce contexte, nous avons écrit une nouvelle version du théorème de Roseau qui est un peu plus générale et fait le lien avec d'autres contextes comme la méthode de moyennisation de Moser.

We deal with nonlinear periodic differential systems depending on a small parameter. The unperturbed system has an invariant manifold of periodic solutions. We provide sufficient conditions in order that some of the periodic orbits of this invariant manifold persist after the perturbation. These conditions are not difficult to check, as we show in some applications. The key tool for proving the main result is the Lyapunov–Schmidt reduction method applied to the Poincaré–Andronov mapping.

We consider the problem of bifurcation of T–periodic solutions for a differential system of the form,

$$x'(t) = F_0(t, x) + \varepsilon F_1(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{5}$$

where ε is a small parameter, $F_0, F_1 : R\times\Omega \to R^n$ and $R : R\times\Omega\times(-_{f},_{f}) \to R^n$ are C^2 functions, T–periodic in the first variable, and Ω is an open subset of R^n. One of the main hypotheses is that the unperturbed system

$$x'(t) = F_0(t, x), \tag{6}$$

has a manifold of periodic solutions. This problem was solved before by Malkin (1956) and Roseau (1966) (see [4]). We will give here a new and shorter proof. In addition, we will give a series of corollaries in some particular cases. In order to describe these cases we introduce some notation. We denote the projection onto the first k coordinates by $\pi : R^k \times R^{n-k} \to R^k$ and the one onto the last $(n-k)$ coordinates by $\pi^\perp : R^k \times R^{n-k} \to R^{n-k}$. For the n–dimensional functions F_0 and x we denote $F_0^1 = \pi F_0$, $F_0^2 = \pi^\perp F_0$ and $u = \pi x$, $v = \pi^\perp x$, respectively.

The first step in the proof of the main result is to reduce the problem of bifurcation of T–periodic solutions of system (5) to the bifurcation of fixed points of the Poincaré–Andronov mapping, or equivalently, of the zeros of some convenient map $g : D(g) \times (-\varepsilon_0, \varepsilon_0) \to R^n$ (where $D(g)$ is some open subset of Ω). Since, in general, it is not possible to apply directly the Implicit Function Theorem for the function g, we will use the Lyapunov–Schmidt reduction theory, but not in its general form (like in [3]). This theory here is made simpler by assuming that the Jacobian matrix of $g(\cdot, 0)$ has a particular form. The corresponding hypothesis for the differential system is that some fundamental matrix solution of the linearized system of (6) around each of its periodic solutions has a particular form.

For $z \in \Omega$ we denote by $x(\cdot, z,) : [0, t_{(z,\varepsilon)}) \to R^n$ the solution of (5) with $x(0, z,) = z$. From Theorem 8.3 of [1] we deduce that, whenever $t_{(z_0,0)} > T$ for some $z_0 \in \Omega$ there exists a neighborhood of $(z_0, 0)$ in $\Omega \times (-_f,_f)$ such that, for all (z, ε) in this neighborhood, $t_{(z,\varepsilon)} > T$. In this work, one of the main assumptions is the existence of T–periodic solutions of system (5) for $\varepsilon = 0$. Under this assumption there exists an open subset

D of Ω and a sufficiently small $\varepsilon_0 > 0$ such that, for all $(z, \varepsilon) \in D \times (-_0,_0)$, the solution $x(\cdot, z,)$ is defined on the interval $[0, T]$. Hence, we can consider the function $f : D \times (-_0,_0) \to R^n$, given by

$$f(z,) = x(T, z,) - z. \tag{7}$$

Then, every $(z,)$ such that

$$f(z,) = 0 \tag{8}$$

provides the periodic solution $x(\cdot, z,)$ of (5).

The converse is also true, i.e. for every T–periodic solution of (5), if we denote by z its value at $t = 0$ then (8) holds. Then, the problem of finding a T–periodic solution of (5), can be replaced by the problem of finding zeros of the finite–dimensional function $f(\cdot,)$ given by (7).

We denote the linearization of (6) by

$$y' = P(t,z)y, \tag{9}$$

where

$$P(t,z) = D_x F_0(t, x(t,z,0)), \tag{10}$$

and let $Y(\cdot, z)$ be some fundamental matrix solution of (9).

The next theorem is our main result. Various consequences of it will be given in the next sections. In the proof we apply Theorem 2.1 to the function (7) after a suitable change of coordinates.

Theorem 1. *Let $\beta_0 : \overline{V} \to R^{n-k}$ be a C^2 function, where $V \subset R^k$ is open and bounded. We assume that*

(i) $\mathcal{Z} = \left\{ z_\alpha = (\alpha, \beta_0(\alpha)), \quad \alpha \in \overline{V} \right\} \subset D$ and that for each $z_\alpha \in \mathcal{Z}$, the unique solution x_α of (6) with $x(0) = z_\alpha$, is T–periodic ;

(ii) for each $z_\alpha \in \mathcal{Z}$, there exists a fundamental matrix solution of (9), $Y_\alpha(t) = Y(t, z_\alpha)$ such that the matrix $Y_\alpha^{-1}(0) - Y_\alpha^{-1}(T)$ has in the upper right corner the null $k \times (n-k)$ matrix, while in the lower right corner has the $(n-k) \times (n-k)$ matrix Δ_α, with $\det(\Delta_\alpha) \neq 0$.

We consider the function $f_1 : \overline{V} \to R^k$ given by

$$f_1(\alpha) = \pi \int_0^T Y_\alpha^{-1}(t) F_1(t, x_\alpha(t)) dt. \tag{11}$$

If there exists $a \in V$ with $f_1(a) = 0$ and $\det\left(\left(df_1/d\alpha\right)(a)\right) \neq 0$, then there exists a T–periodic solution $\varphi(\cdot,)$ of system (5) such that $\varphi(0,) \to z_a$ as $\to 0$.

Proof. We need to study the zeros of the function (7), or, equivalently, of

$$g(z, \varepsilon) = Y^{-1}(T, z) f(z, \varepsilon).$$

We have that $g\left(z_{\alpha},0\right)=0$, because $x(\cdot,z_{\alpha},0)$ is T–periodic, and we shall prove that

$$G_{\alpha}=\frac{dg}{dz}\left(z_{\alpha},0\right)=Y_{\alpha}^{-1}(0)-Y_{\alpha}^{-1}(T). \tag{12}$$

For this we need to know $\left(\partial x/\partial z\right)\left(\cdot,z,0\right)$. Since it is the matrix solution of (9) with $\left(\partial x/\partial z\right)\left(0,z,0\right)=I_n$, we have that $\left(\partial x/\partial z\right)\left(t,z,0\right)=Y(t,z)Y^{-1}(0,z)$. Moreover,

$$\frac{df}{dz}(z,0)=\frac{\partial x}{\partial z}(T,z,0)-I_n=Y(T,z)Y^{-1}(0,z)-I_n$$

and

$$\frac{dg}{dz}(z,0)=Y^{-1}(0,z)-Y^{-1}(T,z)+\left(\frac{\partial Y^{-1}}{\partial z_1}(T,z)f(z,0),...,\frac{\partial Y^{-1}}{\partial z_n}(T,z)f(z,0)\right),$$

which, for $z_{\alpha}\in\mathcal{Z}$, reduces to (12).

We have

$$\frac{\partial g}{\partial \varepsilon}(z,0) = Y^{-1}(T,z)\frac{\partial x}{\partial \varepsilon}(T,z,0).$$

The function $(\partial x/\partial \varepsilon)(\cdot, z, 0)$ is the unique solution of the IVP

$$y' = D_x F_0(t, x(t,z,0))y + F_1(t, x(t,z,0)), \quad y(0) = 0.$$

Hence,

$$\frac{\partial x}{\partial \varepsilon}(t,z,0) = Y(t,z)\int_0^t Y^{-1}(s,z)F_1(s,x(s,z,0))ds. \tag{13}$$

Now, we have

$$\frac{\partial g}{\partial \varepsilon}(z,0) = \int_0^T Y^{-1}(s,z)F_1(s,x(s,z,0))ds,$$

Hence,

$$\frac{\partial\,(\pi g)}{\partial \varepsilon}(z_\alpha, 0) = f_1(\alpha),$$

where f_1 is given by (11). Applying Theorem 2.1, there exists $\alpha_\varepsilon \in V$ such that $g(z_{\alpha_\varepsilon}, \varepsilon) = 0$ and, further, $f(z_{\alpha_\varepsilon}, \varepsilon) = 0$, which assures that $\varphi(\cdot, \varepsilon) = x(\cdot, z_{\alpha_\varepsilon}, \varepsilon)$ is a T–periodic solution of (5).

Case (i) : Perturbations of an isochronous system and the first order averaging method

In this section we assume that there exists an open set V with $\overline{V} \subset D$ and such that for each $z \in \overline{V}$, $x(\cdot, z, 0)$ is T–periodic (we recall that $x(\cdot, z, 0)$ is the solution of the unperturbed system (6) with $x(0) = z$). An answer to the problem of bifurcation of T–periodic solutions from $x(\cdot, z, 0)$ is given in the following result. It is obtained as a consequence of Theorem 3.1 by considering $k = n$.

(Perturbations of an isochronous system) We assume that there exists an open set V with $\overline{V} \subset D$ and such that for each $z \in \overline{V}$, $x(\cdot, z, 0)$ is T–periodic and we consider the function $f_1 : \overline{V} \to R^n$ given by

$$f_1(z) = \int_0^T Y^{-1}(t, z) F_1(t, x(t, z, 0)) dt. \tag{14}$$

If there exists $a \in V$ with $f_1(a) = 0$ and $\det\left(\left(df_1/d\alpha\right)(a)\right) \neq 0$, then there exists a T–periodic solution $\varphi(\cdot,)$ of system (5) such that $\varphi(0,) \to a$ as $\to 0$.

A particular case is when F_0 is identically zero, i.e. the system (6) becomes $x' = 0$ and hence all its solutions are constant, $x(t, z, 0) = z$ for all $t \in R$. Of course, the linearized system is the same, and we take as its fundamental matrix solution $Y(t, z) = I_n$, the identity matrix, for all $t \in R$ and $z \in \overline{V}$. It is easy to see now that the well known averaging method (see, for example [9, 2]) is obtained as consequence of the above Corollary.

(The first order averaging method) We assume that $F_0(t,x)$ is identically zero and we consider the function $f_1 : R^n \to R^n$ given by

$$f_1(z) = \int_0^T F_1(t,z)dt. \tag{15}$$

If there exists $a \in V$ with $f_1(a) = 0$ and $\det\left(\left(df_1/d\alpha\right)(a)\right) \neq 0$, then there exists a T–periodic solution $\varphi(\cdot,)$ of system (5) such that $\varphi(0,) \to a$ as $\to 0$.

Case (ii) : Perturbations of a linear system

In this section we consider the system (5) with $F_0(t,x) = P(t)x + q(t)$, i.e. the unperturbed system (6) is the linear system $x' = P(t)x + q(t)$. Before stating the main result as a consequence of Theorem 3.1, we need two lemmas from linear systems theory.

Let $P : R \to \mathcal{M}_n$ be a continuous and T–periodic function and consider the system

$$y' = P(t)y. \tag{16}$$

The following statements are equivalent :

(i) the system (16) has k T–periodic linearly independent solutions.

(ii) there exists a fundamental matrix of solutions, $Y(t)$, of (16) such that $Y^{-1}(t)$ has in its first k lines only T–periodic functions.

Proof. We consider the adjoint system

$$y' = -P^*(t)y, \tag{17}$$

where $P^*(t)$ is the transpose matrix of $P(t)$. A nonsingular $n \times n$ matrix $Y(t)$ is a fundamental matrix solution for (16) if and only if $Y_a(t) = \left(Y^{-1}(t)\right)^*$ is a fundamental matrix for (17) (Lemma 7.1 page 62, [5]).

The systems (16) and (17) have the same number of linearly independent T–periodic solutions (Lemma 1.3 page 410 [5]). Hence, (i) is equivalent to the fact that (17) has k T–periodic linearly independent solutions. Moreover, this is equivalent to the existence of some fundamental matrix of solutions for (17), denoted Y_a, that has in the first k columns only T–periodic functions. Further, using that $Y^{-1}(t) = Y_a^*(t)$, this is equivalent to (ii).

Let $P : R \to \mathcal{M}_n$ and $q : R \to R^n$ be continuous and T–periodic functions. We assume that the system (16) has k T–periodic linearly independent solutions and we denote by $Y(t)$ its fundamental matrix of solutions as given by Lemma (ii). In addition, we assume that

(i) $\pi \int_0^T Y^{-1}(s)q(s)ds = 0$,
(ii) $\det(\Delta) \neq 0$, where Δ is the $(n-k) \times (n-k)$ matrix from the lower right corner of the $n \times n$ matrix $Y^{-1}(0) - Y^{-1}(T)$.

Then there exists $\beta_0 : R^k \to R^{n-k}$ such that, for all $\alpha \in R^k$, $z_\alpha =$

$(\alpha, \beta_0(\alpha))$ satisfies

$$[Y^{-1}(T) - Y^{-1}(0)]z = \int_0^T Y^{-1}(s)q(s)ds. \tag{18}$$

Moreover, for all $\alpha \in R^k$, the unique solution of

$$x' = P(t)x + q(t), \tag{19}$$

with $x(0) = z_\alpha$, is T–periodic.

Proof. Since the matrix $Y^{-1}(T) - Y^{-1}(0)$ has the first k lines identically 0 and we have (i), the first k equations in the system (18) are the trivial ones, i.e $0 = 0$. Using (ii) we obtain the solution of this system as $z_\alpha = (\alpha, \beta_0(\alpha))$ for all $\alpha \in R^k$.

Denoting by $x(\cdot, z)$ the solution of (19) with $x(0) = z$ and $f_0(z) = x(T, z) - z$, we have that

$$Y^{-1}(T)f_0(z) = -[Y^{-1}(T) - Y^{-1}(0)]z + \int_0^T Y^{-1}(s)q(s)ds.$$

Then, every zero of f_0 is a solution of the linear algebraic system (18). The last part of the conclusion follows now from the correspondence between the zeros of f_0 and the T–periodic solutions of (19).

As a consequence of Theorem 3.1 it is easy to obtain the following Corollary. This result is known as the Theorem of Malkin (see [4]).

Consider the system (5) with $F_0(t, x) = P(t)x + q(t)$ and assume that all the hypotheses of Lemma are satisfied. Let the function $f_1 : R^k \to R^k$

be given by

$$f_1(\alpha) = \pi \int_0^T Y^{-1}(t) F_1(t, x_\alpha(t)) dt.$$

If there exists $a \in V$ with $f_1(a) = 0$ and $\det\left(\left(df_1/d\alpha\right)(a)\right) \neq 0$, then there exists a T–periodic solution $\varphi(\cdot,)$ of system (5) such that $\varphi(0,) \to z_a$ as $\to 0$.

Case (iii)

In this section we consider the system

$$\begin{array}{l} u'(t) = F_0^1(t, u) + F_1^1(t, u, v) +^2 R^1(t, u, v,), \\ v'(t) = F_0^2(t, u, v) + F_1^2(t, u, v) +^2 R^2(t, u, v,), \end{array} \tag{20}$$

where $F_0 = (F_0^1, F_0^2)$, $F_1 = (F_1^1, F_1^2)$ and $R = (R^1, R^2)$ satisfy the hypotheses stated in the Introduction, and the splitting is with respect to the projectors $(\pi, \pi^{\perp})$. We assume that there exists an open set V with $\overline{V} \subset \pi\Omega$ such that, for each $\alpha \in \overline{V}$, the unique solution u_α of $u'(t) = F_0^1(t, u)$ satisfying $u(0) = \alpha$ is T–periodic, and the system $v' = F_0^2(t, u_\alpha(t), v)$ has a unique T–periodic solution. Before stating the main results, we give the following lemma.

Let $P : R \to \mathcal{M}_n$ be a continuous and T–periodic function such that, for all $t \in R$, the matrix $P(t)$ has in the upper right corner the null $k \times (n-k)$ matrix and it has the block form

$$P(t) = \begin{pmatrix} A(t) & & 0 \\ B(t) & C(t) & \end{pmatrix}.$$

Then there exists $Y(t)$ a fundamental matrix of solutions of the system

$$y' = P(t)y, \tag{21}$$

such that $Y^{-1}(t)$ has in the upper right corner the null $k \times (n-k)$ matrix. Moreover,

$$Y^{-1}(t) = \begin{pmatrix} U^{-1}(t) & 0 \\ W(t) & V^{-1}(t) \end{pmatrix},$$

where $U(t)$ and $V(t)$, respectively, are fundamental matrices solutions of $u' = A(t)u$ and $v' = C(t)v$.

Proof. For $y \in R^n$ we define $u = \pi y \in R^k$ and $v = \pi^{\perp} y \in R^{n-k}$. Then, the adjoint system, $y' = -P^*(t)y$, can be written as

$$u' = -A^*(t)u - B^*(t)v, \quad v' = -C^*(t)v. \tag{22}$$

Denoting $U_a(t)$ and $V_a(t)$, respectively, some fundamental matrix solutions for $u' = -A^*(t)u$ and $v' = -C^*(t)v$, we see that

$$Y_a(t) = \begin{pmatrix} U_a(t) & W_a(t) \\ 0_{(n-k)\times k} & V_a(t) \end{pmatrix}$$

is a fundamental matrix solution for (22). Hence, the fundamental matrix of solutions of (21), $Y(t)$, satisfying $Y^{-1}(t) = Y_a^*(t)$, has the required property.

Let $U_\alpha(t)$ and $V_\alpha(t)$, respectively, be fundamental matrix solutions for the systems $u' = A_\alpha(t)u$ and $v' = C_\alpha(t)v$, where $A_\alpha(t) = D_u F_0^1(t, u_\alpha(t))$ and $C_\alpha(t) = D_v F_0^2(t, u_\alpha(t), v_\alpha(t))$. The following corollary of Theorem 3.1 is the main result of this section.

Assume that there exists an open set V with $\overline{V} \subset \pi\Omega$ such that, for each $\alpha \in \overline{V}$, $u_\alpha(\cdot)$ is T–periodic, and the system $v' = F_0^2(t, \alpha, v)$ has a unique T–periodic solution, denoted v_α. Moreover, assume that the matrix $\Delta_\alpha = V_\alpha^{-1}(0) - V_\alpha^{-1}(T)$ has $\det(\Delta_\alpha) \neq 0$ and consider the function

$f_1 : \overline{V} \to R^k$ given by

$$f_1(\alpha) = \int_0^T U_\alpha^{-1}(t) F_1(t, u_\alpha(t), v_\alpha(t)) dt. \tag{23}$$

If there exists $a \in V$ with $f_1(a) = 0$ and $\det((df_1/d\alpha)(a)) \neq 0$, then there exists a T–periodic solution $\varphi(\cdot,)$ of system (20) such that $\varphi(0,) \to (a, v_a(0))$ as $\to 0$.

Proof. We consider the function $\beta_0 : \overline{V} \to R^{n-k}$ given by $\beta_0(\alpha) = v_\alpha(0)$. Then the set $\mathcal{Z} = \left\{ z_\alpha = (\alpha, \beta_0(\alpha)), \quad \alpha \in \overline{V} \quad \right\}$ satisfies hypothesis (i) of Theorem 3.1.

The matrix $P(t, z)$ given by (10) has in the upper right corner the null $k \times (n-k)$ matrix because F_0^1 does not depend on v. Then, by Lemma 6.1, there exists $Y(t)$ a fundamental matrix of solutions of the system (9) such that $Y^{-1}(t, z)$ has in the upper right corner the null $k \times (n-k)$

matrix. In particular, this is true for the matrix $Y_\alpha^{-1}(0) - Y_\alpha^{-1}(T)$. Since, also by Lemma 6.1, this matrix has in the lower right corner the matrix $V_\alpha^{-1}(0) - V_\alpha^{-1}(T)$, we see that also hypothesis (ii) is fulfilled. The form of the function f_1 follows from the specific form of $Y_\alpha^{-1}(t)$.

For the particular case when F_0^1 is identically zero, the result is given in the following corollary.

Assume that $F_0^1(t,u)$ is identically zero and that the system $v' = F_0^2(t,\alpha,v)$ has a unique T–periodic solution, denoted v_α. Moreover, assume that the matrix $\Delta_\alpha = V_\alpha^{-1}(0) - V_\alpha^{-1}(T)$ has $\det(\Delta_\alpha) \neq 0$, and consider the function $f_1 : \overline{V} \to R^k$ given by

$$f_1(\alpha) = \int_0^T F_1(t,\alpha,v_\alpha(t))dt. \tag{24}$$

If there exists $a \in V$ with $f_1(a) = 0$ and $\det\left(\left(df_1/d\alpha\right)(a)\right) \neq 0$, then there exists a T–periodic solution $\varphi(\cdot,)$ of system (20) such that $\varphi(0,) \to (a, v_a(0))$ as $\to 0$.

Remarks

1- Weaker versions of the theorem presented here have been used in applications. Let us, for instance, mention the repeated use of Malkin's theorem to establish synchronization of weakly coupled oscillators. We mention references linked with mathematical physiology. Synchronization of the electrical activity of cardiac cells in the sinusal node explains the formation of the cardiac rythm (see for instance [7], p. 427). Also, it is now believed that synchronization of electrical neurons plays a key role in explaining brain activity in neurosciences (see for instance [6]). There are many other applications to mechanics and physics, which are, in some sense, more classical.

2- There are possible applications to frequency locking, as it appears, for instance in the periodically forced Van der Pol oscillator.

Consider the perturbed equation

$$\frac{dx}{dt} = F_0(x) + F_1(x, t,),$$

where the unperturbed part displays a periodic solution of period T. Assume that the perturbation is periodic of period $T' = pT(1 + \delta())/q$. Perform the change of variables $t = \tau(1 + \delta())$, which transforms the equation into

$$\frac{dx}{d\tau} = F_0(x) + G(x, \tau,),$$

with G periodic of period $T'' = \frac{p}{q}T$ relatively to the time τ. The preceding theorem shows, under some conditions, the existence of periodic solutions for the perturbed system of period pT and hence of period qT'. This "adaptation" of the oscillation on a multiple of the period of the forcing term was observed for the first time by van der Pol.

3- Finally, consider the special case of Hamiltonian dynamics in dimension $n = 2m$:

$$H(p, q,) = H_0(p, q) + H_1(p, q) + O(^2).$$

It is interesting to note that in the case where the unperturbed dynamics is isochronous (all orbits of the associated Hamiltonian system $H_0(p, q)$ are periodic of same period T), the bifurcation function takes the special form

$$f_1(p, q) = \left(\frac{\partial \overline{H}_1}{\partial q}(p, q), \frac{\partial \overline{H}_1}{\partial p}(p, q) \right),$$

where $\overline{H}_1(p, q)$ is the Hamiltonian $H_1(p, q)$ averaged along the periodic orbits of H_0. Our theorem extends in this case a well–known theorem of J.

Moser (see [8]).

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