Nonlinear elliptic differential-functional equations

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A problem in the Calculus of Variations

$$\text{To minimize} \qquad J : u \mapsto \int_\Omega L(\nabla u(x))\, dx + \left(\int_\Omega G(x, u(x))\, dx \right)^\beta$$

$$u_{|\partial\Omega} = \phi$$

Framework

- $\Omega \subset \mathbb{R}^n$, bounded open set,
- $\phi : \partial\Omega \to \mathbb{R}$ Lipschitz continuous,
- $L : \mathbb{R}^n \to \mathbb{R}$ and $G : \Omega \times \mathbb{R} \to \mathbb{R}$ are C^1.

The Euler equation

$$\operatorname{div}\left[a(\nabla u)\right] + F[u](x) = 0$$

$$a(\xi) = \nabla L(\xi)\,,\ F[u](x) = \beta \left(\int_\Omega G(x,u)\, dx \right)^{\beta - 1} G_u(x,u)$$

A nonlinear elliptic equation

$$\int_\Omega a(\nabla u).\nabla\eta - F[u]\eta = 0 \quad \forall\, \eta \in C_c^\infty(\Omega),$$

$$u \in W^{1,1}(\Omega)\,, tr\, u|_\Omega = \phi\,, a(\nabla u) \text{ and } F[u] \in L^1_{loc}(\Omega)$$

Assumptions on a

- $a \in C^0(\mathbb{R}^n, \mathbb{R}^n)$ and $\exists\ \mu > 0$ s.t.

$$(a(\xi) - a(\xi')).(\xi - \xi') \geq \mu|\xi - \xi'|^2$$

Assumptions on F

- $\forall M > 0,\ |u|_\infty \leq M \ \Rightarrow |F[u]| \leq C(M),$
- when $u_n \to u$ uniformly on $\overline{\Omega}$, then $F[u_n] \to F[u]$ a.e.
- growth assumptions:

$$F[u](x)\, sgn\, u(x) \leq C|u|^\beta_{2^*}|u(x)|^{\gamma-1} \text{ with } \beta + \gamma < 2.$$

$$\int_\Omega a(\nabla u)(x).\nabla\eta(x) - F[u](x)\eta(x) = 0 \quad \eta \in C_c^\infty(\Omega)$$

Existence and regularity do not follow from the classical Schauder's theory:

- a is not even C^1 but merely continuous,
- even if a were smooth, it would not necessarily satisfy

$$\mu(1+|\xi|)^\tau|\zeta|^2 \leq \sum_{i,j} \frac{\partial a^i}{\partial x i_j}(\xi)\zeta_i\zeta_j \leq \nu(1+|\xi|)^\tau|\zeta|^2 \quad (\tau > -1)$$

Existence does not follow either from Visik's theory

- requires the additional growth assumption

$$|a(\xi)| \leq \nu|\xi|^{p-1} + \nu'$$

- so that for $\eta \in W^{1,p}(\Omega)$, $a(\nabla u) \in L^{p'}$

$$\Longrightarrow a(\nabla u).\nabla\eta \in L^1(\Omega).$$

Hartman-Stampacchia's strategy

Quasi solution $u \in Lip_\phi(\Omega,K)$ is a K quasi solution $(K > 0)$ if

$$\int_\Omega a(\nabla u).(\nabla v - \nabla u) - F[u](v-u) \geq 0 \quad \forall v \in Lip_\phi(\Omega,K).$$

Theorem

For each K, there exists a K quasi solution.

A priori bounds

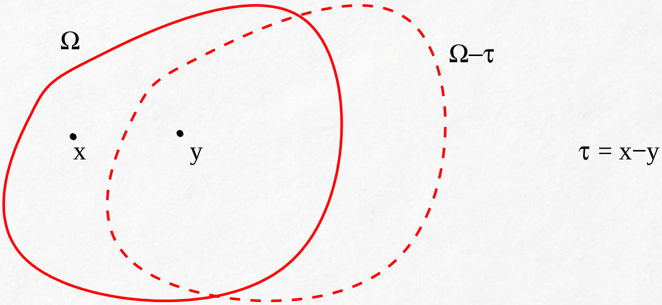
- $L^\infty(\Omega)$ a priori bound on the K quasi solutions,
- $L^\infty(\Omega)$ a priori bound on the gradients of the K quasi solutions.

Convergence of the quasi solutions to a Lipschitz solution

L^∞ bound on ∇u: Rado-Haar Lemma

For $\tau \in \mathbb{R}^n$ s.t. $\Omega \cap (\Omega - \tau) \neq \emptyset$, use as a test function of the quasi solution u

$$u_\tau(x) := u(x + \tau).$$



A maximum principle on the gradient

$$|u(x) - u(y)| \leq \sup_{\substack{x' \in \Omega, y' \in \partial\Omega \\ |x'-y'| \leq |x-y|}} |u(x') - \phi(y')| + C|x - y|.$$

Lower and upper barrier

Definition

$v : \Omega \to \mathbb{R}$ *is a lower barrier at* $\gamma \in \partial\Omega$ *if there exists* $Q > 0$ *s.t.*

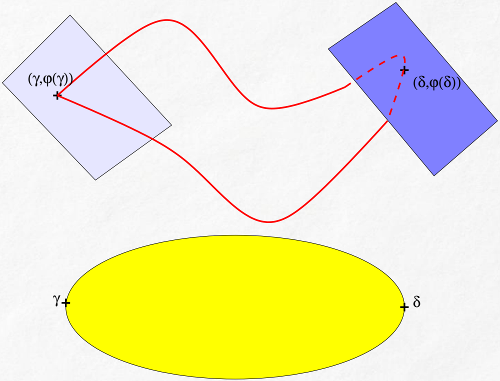
- $v \in Lip\,(\Omega,Q)$,
- $v(\gamma) = \phi(\gamma)$,
- v *is not larger than any* K *quasi solution on* Ω, *for any* $K \geq Q$.

Rado Haar Lemma $+$ **barriers** $\Longrightarrow$ **Lipschitz a priori bound**

Definition (Bounded Slope Condition)

$\phi : \partial\Omega \to \mathbb{R}$ *satisfies the bounded slope condition if it is the restriction of a convex function defined on* $\mathbb{R}^n$ *and also the restriction of a concave function defined on* $\mathbb{R}^n$.

The bounded slope condition



An existence theorem

$$\operatorname{div}[a(\nabla u)] + F[u](x) = 0$$

$$u_{|\partial\Omega} = \phi$$

Theorem (Hartman-Stampacchia)

Assume that

- ϕ *satisfies the bounded slope condition,*
- $(a(\xi) - a(\xi')).(\xi - \xi') \geq \mu|\xi - \xi'|^2$, $\mu > 0$,
- $F[u]$ *locally bounded, continuous + growth assumptions.*

Then there exists a solution which is $W^{1,\infty}(\Omega)$.

The bounded slope condition: A restrictive condition

- Ω necessarily convex,
- ϕ is affine on the faces of $\partial\Omega$,
- ϕ is $C^{1,1}$ if Ω is $C^{1,1}$

The lower bounded slope condition

Definition

ϕ satisfies the lower bounded slope condition if ϕ is the restriction to $\partial\Omega$ of a convex function defined on $\mathbb{R}^n$.

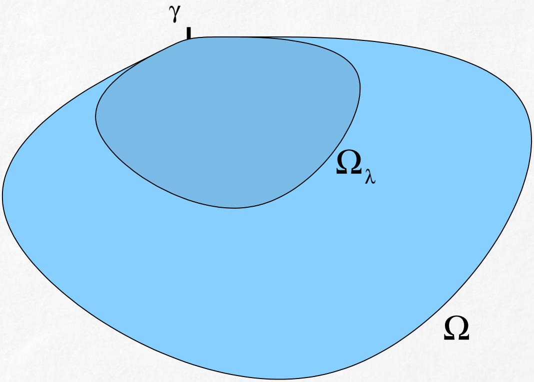
Less restrictive than the full bounded slope condition

- does not imply the convexity of Ω
- ϕ is semiconvex when Ω is convex and $C^{1,1}$

A general principle due to Clarke

A lower barrier is enough to obtain *local* Lipschitz estimates when Ω is convex.

Dilations instead of translations



$$\Omega_\lambda := \lambda(\Omega - \gamma) + \gamma \ , \ u_\lambda(x) = \lambda u(\frac{x-\gamma}{\lambda} + \gamma).$$

Existence of locally Lipschitz solutions

Theorem

Assume that

- Ω *is convex,*
- ϕ *satisfies the lower bounded slope condition,*
- $(a(\xi) - a(\xi')).(\xi - \xi') \geq \mu|\xi - \xi'|^2$, $\mu > 0$,
- $F[u]$ *locally bounded, continuous + growth assumptions.*

Then there exists a solution which is $L^\infty \cap W^{1,2} \cap W^{1,\infty}_{loc}(\Omega)$. *Moreover, if* Ω *is a polyhedron, then* u *is Hölder continuous on* $\overline{\Omega}$.

Implicit barriers and continuity I

Definition

An implicit barrier is a barrier obtained as the solution of an auxiliary problem stated on a larger domain $\Omega_0 \supset \Omega$.

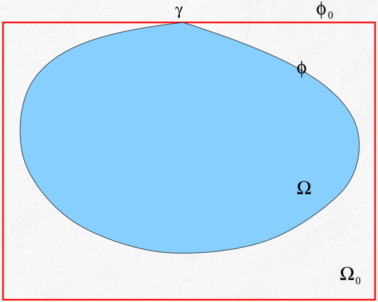
Let u be the solution obtained previously and $\gamma \in \partial\Omega$. We consider

$$(E_0) \quad \operatorname{div}[a(\nabla v)](x) + F[u](x) = 0 \ , \ v_{|\partial\Omega_0} = \phi_0$$

where Ω_0 a cube enclosing Ω with $\gamma \in \partial\Omega_0$,

$$\phi_0 = \phi(\gamma) + K_\phi|x-\gamma| + L|x-\gamma|^2 \ \ \text{(for a suitable large } L \text{)}$$

Implicit barriers and continuity II



$$\phi_0 = \phi(\gamma) + K_\phi|x-\gamma| + L|x-\gamma|^2$$

- L large enough $\implies$ ϕ_0 lower barrier for (E_0)
- 'the' solution u_0 of $(E_0) \geq \phi_0 \geq \phi$ and is continuous
- u_0 is an implicit upper barrier at γ: $u_0 \geq u$ on Ω.

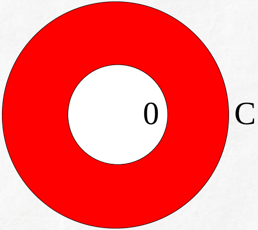
Theorem

The solution u is Hölder continuous on $\overline{\Omega}$.

Nonconvex domains: a simple case

The radial case on an annulus

- $\Omega = B(0{,}2) \setminus \overline{B}(0{,}1)$ in $\mathbb{R}^n$
- $a(\xi) = l(|\xi|)\xi/|\xi|$
- $l : \mathbb{R}^+ \to \mathbb{R}^+$ bijective
- $F[u] = 0$
- $\phi = 0$ on $\partial B(0{,}1)$ and $\phi = C > 0$ on $\partial B(0{,}2)$



The solution is

$$u(x) = \int_1^{|x|} l^{-1}\left(\frac{\lambda}{r^{n-1}}\right) dr$$

for a suitable $\lambda \in \mathbb{R}$.

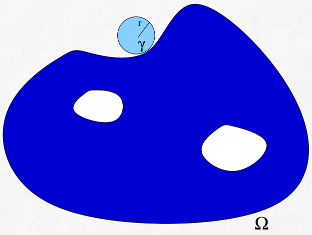
An existence result on nonconvex domains

Theorem

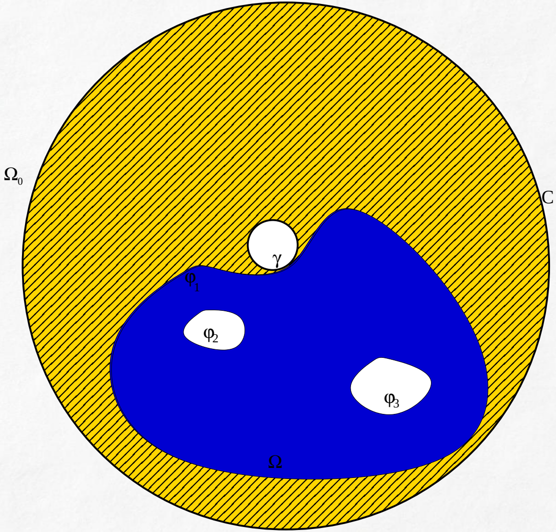
Assume that

- Ω *satisfies a uniform exterior sphere condition,*
- ϕ *is constant on each connected component of* $\partial\Omega$,
- $a(p) = l(|p|)p/|p|$ *continuous with* $l(t) - l(s) \geq \mu(t - s)$, $\forall s < t$,
- $F[u]$ *is locally bounded, continuous +growth assumptions.*

Then there exists a Lipschitz solution.



Sketch of the proof



A variational problem

$$\text{To minimize } J : u \mapsto \int_\Omega f(|\nabla u(x)|)\,dx$$
$$u_{|\partial\Omega} = \phi$$

Theorem

Assume that

- f *strictly convex,* $f(|\xi|)/|\xi| \to +\infty$ *when* $|\xi| \to +\infty$*,*
- ϕ *is Lipschitz continuous,*
- Ω *satisfies the uniform exterior sphere condition.*

Then the solution u *is continuous on* $\overline{\Omega}$*.*

Sketch of the proof

The key lemma

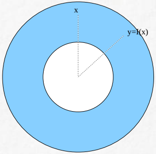
Lemma

Let $u \in W^{1,1}(B(0,R) \setminus \overline{B}(0,r))$. If there exists $Q > 0$ such that

$$\forall r < |x| = |y| < R, \qquad |u(x) - u(y)| \leq Q|x - y|,$$

then u is continuous on $\overline{B(0,R)} \setminus B(0,r)$.

Estimates on the spheres If u is a minimum, compare u and $u \circ I$ where I is the rotation which maps x to y.



$$|u(x) - u(y)| \leq \max_{\gamma \in \partial\Omega} |\phi(\gamma) - \phi(I(\gamma))|$$

A counterexample of Marcellini and Giaquinta

$$u(x_1,\ldots,x_n) := c_n \frac{x_n^2}{\sqrt{\sum_{i=1}^{n-1} x_i^2}}, \Omega := \{\sqrt{\sum_{i=1}^{n-1} x_i^2} < 1, x_n > 1\}$$

solution of

$$\sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}(u_{x_i}) + \frac{\partial}{\partial x_n}(u_{x_n}^3) = 0.$$

Take

$$a(p) = (p_1,\ldots,p_{n-1},h(p_n))$$

with $h(p_n) = p_n^3$ when $p_n > 2c_n$.