A COSSERAT MODEL FOR THIN RODS MADE OF THERMOELASTIC MATERIALS WITH VOIDS

by
Mircea Bîrsan
Department of Mathematics, University of Iasi, Romania
and
Holm Altenbach
Department of Engineering Sciences, Martin–Luther University, Halle (Saale), Germany

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Outline

Motivation

- Theory of rods is a very old field of mechanics:
  Galilei and Bernoulli (XVII century),
  Euler and D'Alembert (XVIII century),
  Clebsch and Kirchhoff (XIX century).

- The modern studies on the mechanics of beams and rods are motivated by the new technologies and advanced materials in rod manufacturing.

- The necessity of elaborating adequate models and to extend the existing theories.

Classical approach

- Derivation from the three–dimensional theory by application of various kinematical and / or stress hypotheses.

- Examples : beam theories of Euler and Timoshenko.

- Requires mathematical techniques like:
  - ⋆ formal asymptotical expansions (Trabucho & Viaño, 1996);
  - ⋆ Γ–convergence analysis (Freddi, Morassi & Paroni, 2007);
  - ⋆ other variational methods (Sprekels & Tiba, 2009).

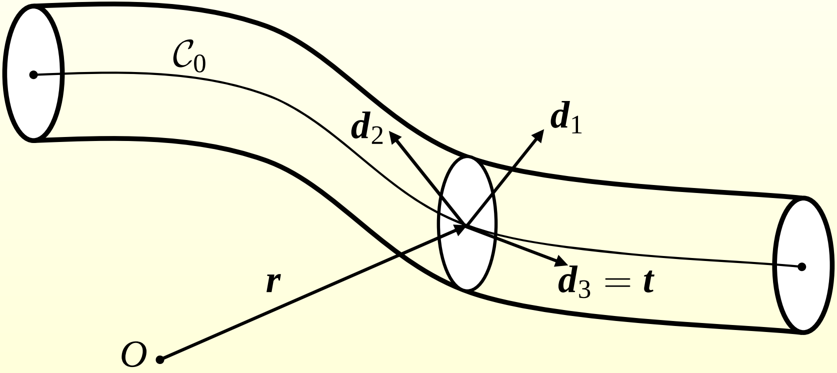
Direct approach

- Based on the deformable curve model.
- First proposed by Cosserat (1909).
- Green and Naghdi developed the theory of *Cosserat curves* (in 1970's) :
  - ⋆ the rod model consists in a curve with 2 deformable directors in each point ;
  - ⋆ presented in the book of Rubin (2000).
- Another direct approach is the *theory of directed curves* .

Kinematical model of directed curves

- Proposed by ZHILIN (2006, 2007):

  ⋆ the rod model consists in a deformable curve with a triad of rigidly rotating orthonormal vectors connected to each point.



Features of any direct approach :

- It does not require hypotheses about the through-the-thickness distributions of displacement and stress fields.
- No need for mathematical manipulations with three-dimensional equations.
- The basic laws of mechanics are applied directly to a one-dimensional continuum.
- To formulate the constitutive equations, we have to determine the structure of the elasticity tensors and to identify the effective properties.
- Use of the effective stiffness concept.

Basic field equations

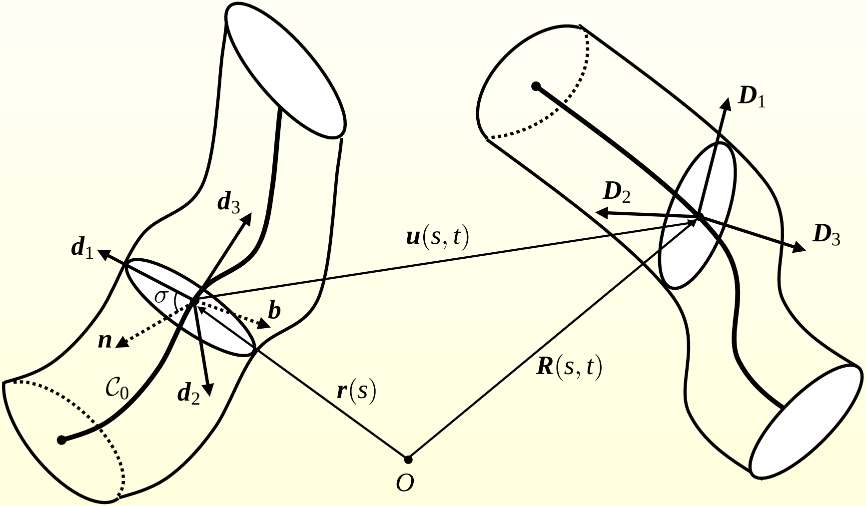
The reference configuration $\mathcal{C}_0$ of the rod is given by the vector fields:

$$\boldsymbol{r}(s), \quad \boldsymbol{d}_i(s), \quad i = 1, 2, 3,$$

where s is the arclength and the directors are:

$$\boldsymbol{d}_3 \equiv \boldsymbol{t} = \boldsymbol{r}'(s),$$
$$\boldsymbol{d}_1 = \boldsymbol{n}\cos\sigma + \boldsymbol{b}\sin\sigma, \quad \boldsymbol{d}_2 = -\boldsymbol{n}\sin\sigma + \boldsymbol{b}\cos\sigma,$$

and $\sigma = \sphericalangle(\boldsymbol{d}_1, \boldsymbol{n})$ is the *angle of natural twisting*.


$\boldsymbol{d}_1$
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O
$\boldsymbol{u}(s,t)$
$\boldsymbol{R}(s,t)$
$\boldsymbol{D}_1$
$\boldsymbol{D}_2$
$\boldsymbol{D}_3$

The motion of the rod is defined by the functions

$$\boldsymbol{R} = \boldsymbol{R}(s,t), \quad \boldsymbol{D}_i = \boldsymbol{D}_i(s,t), \quad i = 1,2,3, \;\; s \in [0,l].$$

The *displacement vector* : $\boldsymbol{u}(s,t) = \boldsymbol{R}(s,t) - \boldsymbol{r}(s)$,

and the *rotation tensor* : $\boldsymbol{P}(s,t) = \boldsymbol{D}_k(s,t) \otimes \boldsymbol{d}_k(s)$.

We denote by :

$\boldsymbol{V}$ the *velocity vector* : $\boldsymbol{V}(s,t) = \dot{\boldsymbol{R}}(s,t)$;

$\boldsymbol{\omega}$ the *angular velocity* : $\dot{\boldsymbol{P}}(s,t) = \boldsymbol{\omega}(s,t) \times \boldsymbol{P}(s,t)$.

We have $\boldsymbol{\omega} = \text{axial}\big(\dot{\boldsymbol{P}} \cdot \boldsymbol{P}^T\big) = -\frac{1}{2}\left[\dot{\boldsymbol{P}} \cdot \boldsymbol{P}^T\right]_\times$.

Porosity

We use the Nunziato–Cowin theory for elastic materials with voids (1979, 1983).

The mass density of the porous rod $\rho = \rho(s,t)$ is represented as the product :

$$\rho(s,t) = \nu(s,t)\ \gamma(s,t)\ ,$$

where $\gamma(s,t)$ is the mass density of the matrix elastic material.

The porosity variable is:

the volume fraction field : $\nu = \nu(s,t)$.

The field $\nu(s,t)$ describes the continuous distribution of voids along the rod. $(0 < \nu \leq 1)$

Temperature

The absolute temperature in the rod is :

$$\theta = \theta(s,t) > 0.$$

The basic laws of thermodynamics are applied directly to the deformable curve.

For instance, the Clausius-Duhem inequality for the entropy of the rod is

$$\int_{s_1}^{s_2} \rho_0\,\dot{\eta}\,\mathrm{d}s \;\geq\; \int_{s_1}^{s_2} \rho_0\,\frac{S}{\theta}\,\mathrm{d}s + \left(\frac{q}{\theta}\right)\Big|_{s_1}^{s_2}, \quad \forall\, s_1, s_2 \in [0, l].$$

Equations of motion

Equation of linear momentum :

$$\boldsymbol{N}'(s,t) + \rho_0 \boldsymbol{\mathcal{F}} = \rho_0 \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{V} + \boldsymbol{\Theta}_1 \cdot \boldsymbol{\omega}).$$

Equation of moment of momentum :

$$\boldsymbol{M}'(s,t) + \boldsymbol{R}' \times \boldsymbol{N}(s,t) + \rho_0 \boldsymbol{\mathcal{L}} = \\ = \rho_0 \big[\boldsymbol{V} \times \boldsymbol{\Theta}_1 \cdot \boldsymbol{\omega} + \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{V} \cdot \boldsymbol{\Theta}_1 + \boldsymbol{\Theta}_2 \cdot \boldsymbol{\omega}) \big].$$

Equation of equilibrated force :

$$h'(s,t) - g(s,t) + \rho_0 p = \rho_0 \frac{\mathrm{d}}{\mathrm{d}t} (\varkappa \dot{\nu}) .$$

Energy balance

Equation of energy balance :

$$\rho_0 \dot{\mathcal{U}} = \mathcal{P} + \rho_0 S + q' ,$$

with $\quad \mathcal{P} = \boldsymbol{N} \cdot (\boldsymbol{V}' + \boldsymbol{R}' \times \boldsymbol{\omega}) + \boldsymbol{M} \cdot \boldsymbol{\omega}' + g\dot{\nu} + h\dot{\nu}' .$

Entropy inequality :

$$\rho_0 \, \theta \, \dot{\eta} \;\geq\; \rho_0 S + q' - \frac{\theta'}{\theta} \, q \, .$$

Introduce the Helmholtz free energy function :

$$\Psi = \mathcal{U} - \theta \, \eta \, .$$

Vectors of deformation

Vector of extension–shear $\boldsymbol{\mathcal{E}}$:

$$\boldsymbol{\mathcal{E}} = \boldsymbol{R}' - \boldsymbol{P} \cdot \boldsymbol{t} .$$

Vector of bending–twisting $\boldsymbol{\Phi}$ given by :

$$\boldsymbol{P}' = \boldsymbol{\Phi} \times \boldsymbol{P} \qquad \text{or} \qquad \boldsymbol{\Phi} = \text{axial}\big(\boldsymbol{P}' \cdot \boldsymbol{P}^T\big) .$$

The energetic vectors of deformation $\boldsymbol{\mathcal{E}}_*$, $\boldsymbol{\Phi}_*$:

$$\boldsymbol{\mathcal{E}}_* = \boldsymbol{P}^T \cdot \boldsymbol{\mathcal{E}} , \qquad \boldsymbol{\Phi}_* = \boldsymbol{P}^T \cdot \boldsymbol{\Phi} .$$

Then the function $\mathcal{P}$ reduces to :

$$\mathcal{P} = (\boldsymbol{N} \cdot \boldsymbol{P}) \cdot \dot{\boldsymbol{\mathcal{E}}}_* + (\boldsymbol{M} \cdot \boldsymbol{P}) \cdot \dot{\boldsymbol{\Phi}}_* + g\,\dot{\nu} + h\,\dot{\nu}' .$$

Constitutive equations

The energy function Ψ depends on :

$$\Psi = \Psi\left(\boldsymbol{\mathcal{E}}_*, \boldsymbol{\Phi}_*, \nu, \nu', \theta\right).$$

We have: $\boldsymbol{N} = \dfrac{\partial(\rho_0\Psi)}{\partial\boldsymbol{\mathcal{E}}_*} \cdot \boldsymbol{P}^T, \quad \boldsymbol{M} = \dfrac{\partial(\rho_0\Psi)}{\partial\boldsymbol{\Phi}_*} \cdot \boldsymbol{P}^T,$

$$\eta = -\frac{\partial\Psi}{\partial\theta}, \qquad g = \frac{\partial(\rho_0\Psi)}{\partial\nu}, \qquad h = \frac{\partial(\rho_0\Psi)}{\partial(\nu')}.$$

The heat flux :

$$q = q\left(\boldsymbol{\mathcal{E}}_*, \boldsymbol{\Phi}_*, \nu, \nu', \theta, \theta'\right).$$

The expression of the energy function Ψ :

$$\begin{aligned}\rho_0\Psi \;=\;& \Psi_0 + \boldsymbol{N}_0\cdot\boldsymbol{\mathcal{E}}_* + \boldsymbol{M}_0\cdot\boldsymbol{\Phi}_* + \frac{1}{2}\boldsymbol{\mathcal{E}}_*\cdot\boldsymbol{A}\cdot\boldsymbol{\mathcal{E}}_* \\ &+\boldsymbol{\mathcal{E}}_*\cdot\boldsymbol{B}\cdot\boldsymbol{\Phi}_* + \frac{1}{2}\boldsymbol{\Phi}_*\cdot\boldsymbol{C}\cdot\boldsymbol{\Phi}_* + \boldsymbol{\Phi}_*\cdot(\boldsymbol{\mathcal{E}}_*\cdot\boldsymbol{D})\cdot\boldsymbol{\Phi}_* \\ &+\frac{1}{2}K_1\,\nu^2 + \frac{1}{2}K_2(\nu')^2 + K_3\,\nu\,\nu' + (\boldsymbol{K}_4\cdot\boldsymbol{\mathcal{E}}_*)\,\nu \\ &+(\boldsymbol{K}_5\cdot\boldsymbol{\Phi}_*)\,\nu + (\boldsymbol{K}_6\cdot\boldsymbol{\mathcal{E}}_*)\,\nu' + (\boldsymbol{K}_7\cdot\boldsymbol{\Phi}_*)\,\nu' \\ &-(\boldsymbol{G}_1\cdot\boldsymbol{\mathcal{E}}_*)\,\theta - (\boldsymbol{G}_2\cdot\boldsymbol{\Phi}_*)\,\theta - G_3\,\nu\,\theta - G_4\,\nu'\,\theta - \frac{1}{2}G\,\theta^2,\end{aligned}$$

The elasticity tensors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ have been analysed by Zhilin (2006).
We have to determine the structure of the tensors K_1 , ... , $\boldsymbol{K}_7$ and $\boldsymbol{G}_1$,..., G_4 , which describe the poro-thermoelastic properties.

Structure of constitutive tensors

We choose the directors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ such that

$$\int_\Sigma \rho^* x \,\mathrm{d}x\mathrm{d}y = \int_\Sigma \rho^* y \,\mathrm{d}x\mathrm{d}y = 0, \qquad \int_\Sigma \rho^* xy \,\mathrm{d}x\mathrm{d}y = 0.$$

We assume the symmetry of the material cross–section with respect to $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$.
We require that the following tensors belong to the symmetry group of each constitutive tensor:

$$\boldsymbol{Q} = \mathbf{1} - 2\,\boldsymbol{d}_1 \otimes \boldsymbol{d}_1 \quad \text{and} \quad \boldsymbol{Q} = \mathbf{1} - 2\,\boldsymbol{d}_2 \otimes \boldsymbol{d}_2\,.$$

We express any constitutive tensor $\boldsymbol{f}$ as the decomposition : $\boldsymbol{f} = \boldsymbol{f}^0(\sigma) + \boldsymbol{f}^1(\sigma) \cdot \boldsymbol{\tau}$.
We find :

$$\boldsymbol{G}_1 = G_1\,\boldsymbol{t} + \frac{1}{R_c}\left(G_1^1 \cos\sigma \boldsymbol{d}_1 + G_1^2 \sin\sigma \boldsymbol{d}_2\right),$$

$$\boldsymbol{G}_2 = \frac{G_2}{R_t}\,\boldsymbol{t} + \frac{1}{R_c}\left(G_2^1 \sin\sigma \boldsymbol{d}_1 + G_2^2 \cos\sigma \boldsymbol{d}_2\right),$$

$$\boldsymbol{K}_4 = K_4\,\boldsymbol{t} + \frac{1}{R_c}\left(K_4^1\,\boldsymbol{d}_1 \cos\sigma + K_4^2\,\boldsymbol{d}_2 \sin\sigma\right),$$

$$\boldsymbol{K}_5 = \frac{K_5}{R_t}\,\boldsymbol{t} + \frac{1}{R_c}\left(K_5^1\,\boldsymbol{d}_1 \sin\sigma + K_5^2\,\boldsymbol{d}_2 \cos\sigma\right),$$

$$\boldsymbol{K}_6 = K_6\,\boldsymbol{t} + \frac{1}{R_c}\left(K_6^1\,\boldsymbol{d}_1 \cos\sigma + K_6^2\,\boldsymbol{d}_2 \sin\sigma\right),$$

$$\boldsymbol{K}_7 = \frac{K_7}{R_t}\,\boldsymbol{t} + \frac{1}{R_c}\left(K_7^1\,\boldsymbol{d}_1 \sin\sigma + K_7^2\,\boldsymbol{d}_2 \cos\sigma\right).$$

For rods without natural twisting ($\sigma = \text{const}$), we consider that the symmetry groups include:

$$\boldsymbol{Q} = \mathbf{1} - 2\, \boldsymbol{d}_1 \otimes \boldsymbol{d}_1, \quad \boldsymbol{Q} = \mathbf{1} - 2\, \boldsymbol{d}_2 \otimes \boldsymbol{d}_2, \quad \boldsymbol{Q} = \mathbf{1} - 2\, \boldsymbol{t} \otimes \boldsymbol{t}.$$

We obtain :

$$\boldsymbol{G}_1 = G_1\, \boldsymbol{t}\,, \qquad \boldsymbol{G}_2 = \frac{G_2}{R_t}\, \boldsymbol{t} + \frac{1}{R_c}\left(G_2^1 \sin\sigma\, \boldsymbol{d}_1 + G_2^2 \cos\sigma\, \boldsymbol{d}_2\right),$$

$$G_4 = 0\,, \qquad K_3 = 0, \qquad \boldsymbol{K}_4 = K_4\, \boldsymbol{t}, \qquad \boldsymbol{K}_7 = \boldsymbol{0},$$

$$\boldsymbol{K}_5 = \frac{K_5}{R_t}\, \boldsymbol{t} + \frac{1}{R_c}\left(K_5^1\, \boldsymbol{d}_1 \sin\sigma + K_5^2\, \boldsymbol{d}_2 \cos\sigma\right),$$

$$\boldsymbol{K}_6 = \frac{1}{R_c}\left(K_6^1\, \boldsymbol{d}_1 \cos\sigma + K_6^2\, \boldsymbol{d}_2 \sin\sigma\right).$$

The expressions of the elasticity tensors are

$$\boldsymbol{A} = A_1 \boldsymbol{d}_1 \otimes \boldsymbol{d}_1 + A_2 \boldsymbol{d}_2 \otimes \boldsymbol{d}_2 + A_3 \boldsymbol{t} \otimes \boldsymbol{t}\,,$$

$$\begin{aligned}\boldsymbol{B} = \frac{1}{R_t}\big(B_1 \boldsymbol{d}_1 \otimes \boldsymbol{d}_1 + B_2 \boldsymbol{d}_2 \otimes \boldsymbol{d}_2 + B_3 \boldsymbol{t} \otimes \boldsymbol{t}\big) \\ + \frac{1}{R_c}\Big[\big(B_{23} \boldsymbol{d}_2 \otimes \boldsymbol{d}_3 + B_{32} \boldsymbol{d}_3 \otimes \boldsymbol{d}_2\big)\cos\sigma \\ + \big(B_{13} \boldsymbol{d}_1 \otimes \boldsymbol{d}_3 + B_{31} \boldsymbol{d}_3 \otimes \boldsymbol{d}_1\big)\sin\sigma\Big],\end{aligned}$$

$$\boldsymbol{C} = C_1 \boldsymbol{d}_1 \otimes \boldsymbol{d}_1 + C_2 \boldsymbol{d}_2 \otimes \boldsymbol{d}_2 + C_3 \boldsymbol{t} \otimes \boldsymbol{t}\,.$$

The values A_i , B_i , C_i for the elastic stiffness can be determined by solving problems in the linear theory.

Linear theory

In the linear theory, there exists the *vector of small rotations* $\psi(s,t)$ such that :

$$\boldsymbol{P}(s,t) = \mathbf{1} + \psi(s,t) \times \mathbf{1}\,,$$

We have $\boldsymbol{\omega}(s,t) = \dot{\psi}(s,t)\,,\quad \boldsymbol{\Phi}(s,t) = \psi'(s,t).$

Denote by T and φ the *variations of temperature and porosity fields* :

$$T(s,t) = \theta(s,t) - \theta_0\,, \qquad \varphi(s,t) = \nu(s,t) - \nu_0(s).$$

We assume that $\boldsymbol{u}\,,\ \psi\,,\ T\,,\ \varphi$ are infinitesimal.

The vectors of deformation become :

$$\boldsymbol{e} \equiv \boldsymbol{u}' + \boldsymbol{t} \times \boldsymbol{\psi} = \boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}_* , \qquad \boldsymbol{\kappa} \equiv \boldsymbol{\psi}' = \boldsymbol{\Phi} = \boldsymbol{\Phi}_* .$$

The constitutive equations :

$$\boldsymbol{N} = \frac{\partial(\rho_0 \Psi)}{\partial \boldsymbol{e}} , \qquad \boldsymbol{M} = \frac{\partial(\rho_0 \Psi)}{\partial \boldsymbol{\kappa}} ,$$

$$\eta = -\frac{\partial \Psi}{\partial T} , \quad \boldsymbol{g} = \frac{\partial(\rho_0 \Psi)}{\partial \varphi} , \quad \boldsymbol{h} = \frac{\partial(\rho_0 \Psi)}{\partial(\varphi')} .$$

The heat flux is expressed by :

$$q = K\, T' ,$$

with K the thermal conductivity of the rod.

The equations of motion become :

$$\boldsymbol{N}' + \rho_0 \boldsymbol{\mathcal{F}} = \rho_0 \left(\ddot{\boldsymbol{u}} + \boldsymbol{\Theta}_1^0 \cdot \ddot{\boldsymbol{\psi}}\right),$$
$$\boldsymbol{M}' + \boldsymbol{t} \times \boldsymbol{N} + \rho_0 \boldsymbol{\mathcal{L}} = \rho_0 \left(\ddot{\boldsymbol{u}} \cdot \boldsymbol{\Theta}_1^0 + \boldsymbol{\Theta}_2^0 \cdot \ddot{\boldsymbol{\psi}}\right),$$
$$h' - g + \rho_0 \, p = \rho_0 \, \varkappa \, \ddot{\varphi} \, .$$

The reduced energy balance equation :

$$q' + \rho_0 \, S \; = \; \rho_0 \, \theta_0 \, \dot{\eta} \, .$$

The entropy inequality reduces to :

$$K \; \geq \; 0 \, .$$

To formulate the boundary–initial–value problem we adjoin boundary conditions :

$$\begin{aligned}
&\boldsymbol{u}(\bar{s},t)=\bar{\boldsymbol{u}}(t) \quad \text{or} \quad \boldsymbol{N}(\bar{s},t)=\bar{\boldsymbol{N}}(t),\\
&\boldsymbol{\psi}(\bar{s},t)=\bar{\boldsymbol{\psi}}(t) \quad \text{or} \quad \boldsymbol{M}(\bar{s},t)=\bar{\boldsymbol{M}}(t),\\
&\varphi(\bar{s},t)=\bar{\varphi}(t) \quad \text{or} \quad h(\bar{s},t)=\bar{h}(t),\\
&T(\bar{s},t)=\bar{T}(t) \quad \text{or} \quad q(\bar{s},t)=\bar{q}(t), \quad \text{for} \quad s\in\{0,l\}.
\end{aligned}$$

and initial conditions :

$$\begin{aligned}
&\boldsymbol{u}(s,0)=\boldsymbol{u}_0(s), \quad \dot{\boldsymbol{u}}(s,0)=\boldsymbol{V}_0(s),\\
&\boldsymbol{\psi}(s,0)=\boldsymbol{\psi}_0(s), \quad \dot{\boldsymbol{\psi}}(s,0)=\boldsymbol{\omega}_0(s),\\
&\varphi(s,0)=\varphi_0(s), \quad \dot{\varphi}(s,0)=\lambda_0(s),\\
&T(s,0)=T_0(s), \quad \text{for} \quad s\in[0,l].
\end{aligned}$$

Uniqueness of Solution

Introduce the function :

$$U(t) = \int_{\mathcal{C}_0} \rho_0 \big(\Psi + \eta T\big)\, \mathrm{d}s$$

and the kinetic energy :

$$K(t) = \frac{1}{2}\int_{\mathcal{C}_0} \rho_0 \big(\dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{u}} + 2\dot{\boldsymbol{u}}\cdot\boldsymbol{\Theta}_1^0\cdot\dot{\boldsymbol{\psi}} + \dot{\boldsymbol{\psi}}\cdot\boldsymbol{\Theta}_2^0\cdot\dot{\boldsymbol{\psi}} + \varkappa\,\dot{\varphi}^2\big)\mathrm{d}s.$$

We prove :

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[K(t)+U(t)\big] = \int_{\mathcal{C}_0}\Big[\rho_0\Big(\boldsymbol{\mathcal{F}}\cdot\dot{\boldsymbol{u}} + \boldsymbol{\mathcal{L}}\cdot\dot{\boldsymbol{\psi}} + p\,\dot{\varphi} + \frac{1}{\theta_0}\,S\,T\Big) - \frac{K}{\theta_0}\big(T'\big)^2\Big]\mathrm{d}s$$
$$+\Big(\boldsymbol{N}\cdot\dot{\boldsymbol{u}} + \boldsymbol{M}\cdot\dot{\boldsymbol{\psi}} + h\,\dot{\varphi} + \frac{1}{\theta_0}\,q\,T\Big)\Big|_0^l . \qquad (1)$$

Theorem 1. *For any two moments* $t, z \geq 0$*, let*

$$Q(t,z) = \int_{\mathcal{C}_0} \rho_0 \Big(\boldsymbol{\mathcal{F}}(t) \cdot \dot{\boldsymbol{u}}(z) + \boldsymbol{\mathcal{L}}(t) \cdot \dot{\boldsymbol{\psi}}(z) + p(t)\, \dot{\varphi}(z) - \frac{1}{\theta_0} S(t) T(z) \Big) \mathrm{d}s$$
$$+ \Big(\boldsymbol{N}(t) \cdot \dot{\boldsymbol{u}}(z) + \boldsymbol{M}(t) \cdot \dot{\boldsymbol{\psi}}(z) + h(t)\, \dot{\varphi}(z) - \frac{1}{\theta_0} q(t) T(z) \Big) \Big|_0^l ,$$

Then, for any $t \geq 0$*, we have :*

$$2\big[U(t) - K(t)\big] = \int_0^t \big[Q(t+\tau, t-\tau) - Q(t-\tau, t+\tau)\big] \mathrm{d}\tau$$
$$+ \int_{\mathcal{C}_0} \Big[\boldsymbol{N}(0) \cdot \boldsymbol{e}(2t) + \boldsymbol{M}(0) \cdot \boldsymbol{\kappa}(2t) + g(0) \varphi(2t) + h(0) \varphi'(2t)$$
$$+ \rho_0\, \eta(2t)\, T(0) \Big] \mathrm{d}s \; - \int_{\mathcal{C}_0} \rho_0 \Big[\dot{\boldsymbol{u}}(2t) \cdot \big(\dot{\boldsymbol{u}}(0) + \boldsymbol{\Theta}_1^0 \cdot \dot{\boldsymbol{\psi}}(0) \big)$$
$$+ \dot{\boldsymbol{\psi}}(2t) \cdot \big(\dot{\boldsymbol{u}}(0) \cdot \boldsymbol{\Theta}_1^0 + \boldsymbol{\Theta}_2^0 \cdot \dot{\boldsymbol{\psi}}(0) \big) + \varkappa\, \dot{\varphi}(2t)\, \dot{\varphi}(0) \Big] \mathrm{d}s.$$

We show :

Theorem 2. (*Uniqueness*)
Assume that the mass density ρ_0 *, the inertia coefficient* $\varkappa$ *and the constitutive coefficient* G *are positive.*
Then the boundary–initial–value problem for porous thermoelastic rods has at most one solution.

Proof : based on relation (1) and Theorem 1.

Korn Inequality and Existence results

Theorem 3.

Assume that $\mathbf{r}(s)$ *is of class* $C^3[0,l]$. *For every* $\boldsymbol{y} = \big(u_i(s), \psi_i(s)\big) \in \boldsymbol{H}^1[0,l]$ *we define the components of the deformation vectors* $e_i(\boldsymbol{y})$ *and* $\kappa_i(\boldsymbol{y})$ *in the Frenet vector basis* $\{\,\boldsymbol{t}\,,\,\boldsymbol{n}\,,\,\boldsymbol{b}\,\}$. *Then, there exists a constant* $c_1 > 0$ *such that*

$$\int_{\mathcal{C}} \big[u_i u_i + \psi_i \psi_i + e_i(\boldsymbol{y})\, e_i(\boldsymbol{y}) + \kappa_i(\boldsymbol{y})\kappa_i(\boldsymbol{y})\big]\mathrm{d}s \geq \geq c_1 \int_{\mathcal{C}} \big(u_i u_i + \psi_i \psi_i + u_i' u_i' + \psi_i' \psi_i'\big)\mathrm{d}s, \tag{2}$$

for any $\boldsymbol{y} = \big(u_i, \psi_i\big) \in \boldsymbol{H}^1[0,l]$.

Relation (2) is a *Korn inequality "without boundary conditions"*.
The proof relies on a corollary of the closed graph theorem.

To prove a *Korn inequality "with boundary conditions"*, we consider the closed subspace

$$\boldsymbol{V} = \left\{ \left(u_i, \psi_i\right) \in \boldsymbol{H}^1[0, l] \mid u_i = 0 \text{ on } \Gamma_u\,, \;\; \psi_i = 0 \text{ on } \Gamma_\psi \right\},$$

in the sense of traces.

Theorem 4.

Assume that the hypotheses of Theorem 3 are satisfied and that Γ_u and Γ_ψ are nonempty sets. Then, there exists a constant $c_2 > 0$ such that

$$\int_{\mathcal{C}} \big[e_i(\boldsymbol{y})\, e_i(\boldsymbol{y}) + \kappa_i(\boldsymbol{y})\kappa_i(\boldsymbol{y}) \big] \mathrm{d}s \ \geq$$

$$\geq c_2 \int_{\mathcal{C}} \big(u_i\, u_i + \psi_i\, \psi_i + u_i'\, u_i' + \psi_i'\, \psi_i' \big) \mathrm{d}s, \quad \forall \boldsymbol{y} \in \boldsymbol{V}.$$

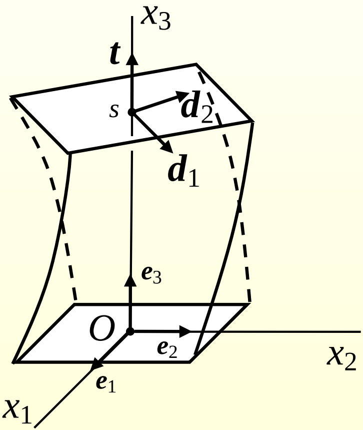
Proof : based on Theorem 3 and the Lemma on infinitesimal rigid displacements.

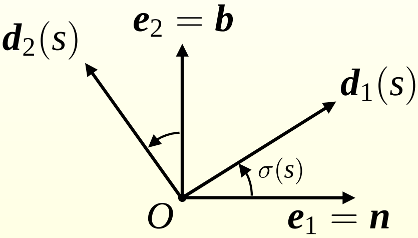
The inequality of Korn type from *Theorem 4* can be used to prove existence results for the equations of rods written in a weak variational form :

- Dynamical equations: we employ the semigroup of linear operators theory
- Equilibrium equations: we employ the Lax–Milgram lemma

Straight porous rods

We consider the case when the middle curve $\mathcal{C}_0$ is straight, but has natural twisting.





The tensors of inertia become :

$$\rho_0 \boldsymbol{\Theta}_1^0 = \mathbf{0}, \quad \rho_0 \boldsymbol{\Theta}_2^0 = I_1\, \boldsymbol{d}_1 \otimes \boldsymbol{d}_1 + I_2\, \boldsymbol{d}_2 \otimes \boldsymbol{d}_2 + (I_1 + I_2)\, \boldsymbol{t} \otimes \boldsymbol{t},$$

where $I_1 = \int_\Sigma \rho^* y^2 \mathrm{d}x\mathrm{d}y,\ I_2 = \int_\Sigma \rho^* x^2 \mathrm{d}x\mathrm{d}y.$

We decompose by $\boldsymbol{t}$ and the normal plane

$$\boldsymbol{u} = u\, \boldsymbol{t} + \boldsymbol{w} \qquad \text{and} \qquad \boldsymbol{\psi} = \psi\, \boldsymbol{t} + \boldsymbol{t} \times \boldsymbol{\vartheta},$$

where u is the longitudinal displacement,
$\boldsymbol{w}$ is the vector of transversal displacement,
ψ is the torsion,
$\boldsymbol{\vartheta}'$ is the vector of bending deformation.

The vector of *transverse shear* : $\quad \boldsymbol{\gamma} = \boldsymbol{w}' - \boldsymbol{\vartheta}$.

We decompose also the force vector $\boldsymbol{N}$ and the moment vector $\boldsymbol{M}$

$$\boldsymbol{N} = F\,\boldsymbol{t} + \boldsymbol{Q} \qquad \text{and} \qquad \boldsymbol{M} = H\,\boldsymbol{t} + \boldsymbol{t} \times \boldsymbol{L},$$

where F is the longitudinal force,
$\boldsymbol{Q}$ is the vector of transversal force,
H is the torsion moment
$\boldsymbol{L}$ is the vector of bending moment.

The boundary–initial–value problem decouples into 2 problems :

Extension - torsion problem

Variables : $u\,,\ \psi\,,\ T$ and φ .

Equations of motion and energy equation :

$$F' + \rho_0 \mathcal{F}_t = \rho_0\, \ddot{u}\,, \qquad H' + \rho_0 \mathcal{L}_t = (I_1 + I_2)\, \ddot{\psi}\,,$$

$$h' - g + \rho_0 p = \rho_0 \varkappa \ddot{\varphi}, \qquad q' + \rho_0 S = \rho_0\, \theta_0\, \dot{\eta}\,.$$

Constitutive equations :

$$F = A_3\, u' + \sigma' B_0\, \psi' + K_4\, \varphi + K_6\, \varphi' + G_1\, T,$$

$$H = \sigma' B_0\, u' + C_3\, \psi'\,, \qquad q = K\, T'\,,$$

$$g = K_1\, \varphi + K_3\, \varphi' + K_4\, u' + G_3\, T,$$

$$h = K_2\, \varphi' + K_3\, \varphi + K_6\, u' + G_4\, T\,,$$

$$\rho_0 \eta = -G\, T - G_1\, u' - G_3\, \varphi - G_4\, \varphi'\,.$$

Bending - shear problem

Variables : $\boldsymbol{w}$ and $\boldsymbol{\vartheta}$.

Equations of motion :

$$\boldsymbol{Q}' + \rho_0 \boldsymbol{\mathcal{F}}_n = \rho_0 \ddot{\boldsymbol{w}} \, ,$$

$$\boldsymbol{L}' + \boldsymbol{Q} - \rho_0 \, \boldsymbol{t} \times \boldsymbol{\mathcal{L}}_n = \big(I_2 \, \boldsymbol{d}_1 \otimes \boldsymbol{d}_1 + I_1 \, \boldsymbol{d}_2 \otimes \boldsymbol{d}_2 \big) \cdot \ddot{\boldsymbol{\vartheta}} .$$

Constitutive equations :

$$\boldsymbol{Q} = \big(A_1 \, \boldsymbol{d}_1 \otimes \boldsymbol{d}_1 + A_2 \, \boldsymbol{d}_2 \otimes \boldsymbol{d}_2 \big) \cdot \big(\boldsymbol{w}' - \boldsymbol{\vartheta} \big) \; ,$$

$$\boldsymbol{L} = \big(C_2 \, \boldsymbol{d}_1 \otimes \boldsymbol{d}_1 + C_1 \, \boldsymbol{d}_2 \otimes \boldsymbol{d}_2 \big) \cdot \boldsymbol{\vartheta}' \, .$$

Straight rods without natural twisting

In this case : $\sigma(s) = 0, \quad \boldsymbol{d}_\alpha(s) = \boldsymbol{e}_\alpha\,, \quad \boldsymbol{t} = \boldsymbol{e}_3$.

The constitutive tensors simplify in the form :

$$\begin{gathered} \boldsymbol{G}_1 = G_1\,\boldsymbol{t}, \quad \boldsymbol{G}_2 = \boldsymbol{0}, \quad G_4 = 0, \\ K_3 = 0, \qquad \boldsymbol{K}_5 = \boldsymbol{K}_6 = \boldsymbol{K}_7 = \boldsymbol{0}, \\ \boldsymbol{K}_4 = K_4\,\boldsymbol{t}, \quad \boldsymbol{A} = A_1\boldsymbol{d}_1\otimes\boldsymbol{d}_1 + A_2\boldsymbol{d}_2\otimes\boldsymbol{d}_2 + A_3\boldsymbol{t}\otimes\boldsymbol{t}, \\ \boldsymbol{B} = \boldsymbol{0}, \quad \boldsymbol{C} = C_1\boldsymbol{d}_1\otimes\boldsymbol{d}_1 + C_2\boldsymbol{d}_2\otimes\boldsymbol{d}_2 + C_3\boldsymbol{t}\otimes\boldsymbol{t}, \end{gathered}$$

and the extension - torsion problem decouples.

For homogeneous materials, we can solve analytically the problems of extension, torsion and bending–shear, which reduce to ODEs.

Equations for 3D orthotropic rods

Consider a 3D rod which occupies the domain

$$\mathcal{B} = \{ \ (x_1, x_2, x_3) \mid (x_1, x_2) \in \Sigma, \ \ x_3 \in [0, l] \ \} \ .$$

The 3D equations of motion are :

$$t^*_{ji,j} + \rho^* f^*_i = \rho^* \ddot{u}^*_i \ , \qquad h^*_{i,i} - g^* + \rho^* p^* = \rho^* \varkappa^* \ddot{\varphi}^* ,$$
$$q^*_{i,i} + \rho^* S^* = \rho^* \theta^*_0 \dot{\eta}^* \ .$$

Denote the integration over the cross-section :

$$\langle \ f \ \rangle = \int_\Sigma f \ \mathrm{d}x_1 \mathrm{d}x_2 \ , \qquad \forall f \ .$$

The constitutive equations for orthotropic thermoelastic materials with voids are :

$$t^*_{11} = c_{11}e^*_{11} + c_{12}e^*_{22} + c_{13}e^*_{33} + \beta_1\varphi^* - b_1T^*,$$

$$t^*_{22} = c_{12}e^*_{11} + c_{22}e^*_{22} + c_{23}e^*_{33} + \beta_2\varphi^* - b_2T^*,$$

$$t^*_{33} = c_{13}e^*_{11} + c_{23}e^*_{22} + c_{33}e^*_{33} + \beta_3\varphi^* - b_3T^*,$$

$$t^*_{12} = 2c_{66}e^*_{12}, \qquad t^*_{23} = 2c_{44}e^*_{23}, \qquad t^*_{31} = 2c_{55}e^*_{31},$$

$$h^*_1 = \alpha_1\varphi^*_{,1}, \qquad h^*_2 = \alpha_2\varphi^*_{,2}, \qquad h^*_3 = \alpha_3\varphi^*_{,3},$$

$$g^* = \beta_1e^*_{11} + \beta_2e^*_{22} + \beta_3e^*_{33} + \xi\varphi^* - mT^*,$$

$$\rho^*\eta^* = aT^* + b_1e^*_{11} + b_2e^*_{22} + b_3e^*_{33} + m\varphi^*,$$

$$q^*_i = K^*_i T^*_{,i},$$

where $e^*_{ij} = \frac{1}{2}(u^*_{i,j} + u^*_{j,i})$ is the 3D strain tensor.

Determination of constitutive coefficients

Consider straight porous rods made of an orthotropic and homogeneous material.

We determine the constitutive coefficients:

$$A_i\,,\quad C_i\,,\quad K_1\,,\quad K_2,\quad K_4,\quad G_1,\quad G_3\quad \text{and}\quad G$$

by comparison of simple exact solutions for directed curves with the results from 3D theory.

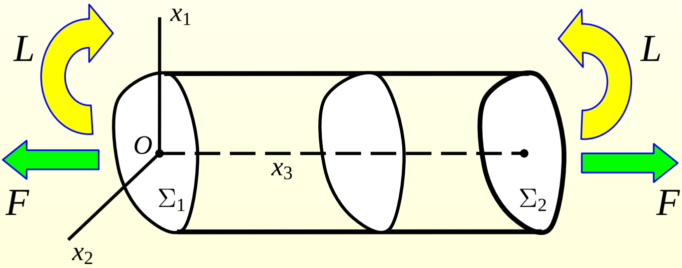
Use the notations : $$E_0 = \frac{\det(c_{ij})_{3\times 3}}{c_{11}c_{22} - c_{12}^2}\,,$$

$$\nu_1 = \frac{c_{13}c_{22} - c_{23}c_{12}}{c_{11}c_{22} - c_{12}^2}\,,\qquad \nu_2 = \frac{c_{23}c_{11} - c_{13}c_{12}}{c_{11}c_{22} - c_{12}^2}\,.$$

Bending and extension of orthotropic rods

Consider the end boundary conditions :

$$\int_{\Sigma_1} t^*_{33}\,\mathrm{d}x_1\mathrm{d}x_2 = F, \qquad \int_{\Sigma_1} x_2 t^*_{33}\,\mathrm{d}x_1\mathrm{d}x_2 = L_2\,.$$



The solutions in the two approaches (direct and 3D) coincide if and only if : $\quad (\ A = \mathrm{area}(\Sigma) \ )$

$$A_3 = A\,E_0 \ , \qquad C_1 \ = \ E_0 \int_{\Sigma} x_2^2 \, \mathrm{d}x_1 \mathrm{d}x_2 \, .$$

If we consider the end boundary conditions :

$$\int_{\Sigma_1} x_1 t_{33}^{*} \, \mathrm{d}x_1 \mathrm{d}x_2 = L_1$$

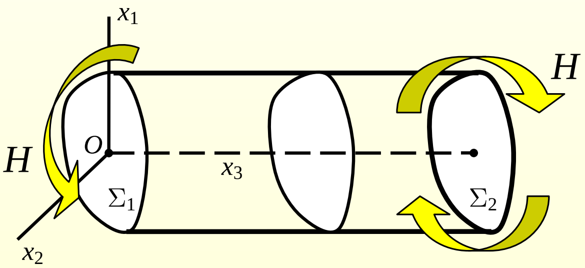
and compare the two solutions we get :

$$C_2 \ = \ E_0 \int_{\Sigma} x_1^2 \, \mathrm{d}x_1 \mathrm{d}x_2 \, .$$

Torsion of orthotropic rods

Consider the end boundary conditions :

$$\int_{\Sigma_1} (x_1 t^*_{23} - x_2 t^*_{13})\, \mathrm{d}x_1 \mathrm{d}x_2 = H\,.$$



Comparing the solutions in the two approaches (direct and 3D) we deduce that :

$$C_3 = \frac{8(c_{44}\, c_{55})^{3/2}}{(c_{44}+c_{55})^2} \int_{\Sigma^*} \phi^*(\xi_1,\xi_2)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \,,$$

where $\phi^*(\xi_1,\xi_2)$ is the solution of the problem :

$$\Delta\phi^*(\xi_1,\xi_2) = -2 \qquad \text{in } \Sigma^*,$$
$$\phi^*(\xi_1,\xi_2) = 0 \qquad \text{on } \partial\Sigma^*,$$

and

$$\Sigma^* = \{\, (\xi_1,\xi_2)\,|\,\xi_1 = x_1\sqrt{\tfrac{c_{44}+c_{55}}{2c_{55}}}\,, \xi_2 = x_2\sqrt{\tfrac{c_{44}+c_{55}}{2c_{44}}}\ \}.$$

Shear vibrations of orthotropic rods

Consider a rectangular straight rod with zero body forces, the lateral surface free of traction and the end boundary conditions :

$$u_1^* = u_2^* = 0 \quad \text{and} \quad t_{33}^* = 0 \quad \text{for} \quad x_3 = 0, l\,.$$

To determine the shear vibrations, we search :

$$\boldsymbol{u}^* = W\, e^{\,i\,\omega\,t} \sin\left(\frac{(2k+1)\pi}{a}\; x_1\right)\boldsymbol{e}_3\,, \qquad k = 0, 1, 2, \ldots$$

The lowest natural frequency of shear vibrations

$$\omega = \frac{\pi}{a}\,\sqrt{\frac{c_{55}}{\rho^*}}\,.$$

Considering the same problem in the theory of rods we find the natural frequency :

$$\hat{\omega} = \frac{1}{a} \sqrt{\frac{12\, A_1}{\rho^* \, A}} \quad .$$

If we identify ω and $\hat{\omega}$, we find :

$$A_1 = k\, A\, c_{55} \ , \qquad \text{with} \quad k = \frac{\pi^2}{12} \ ,$$

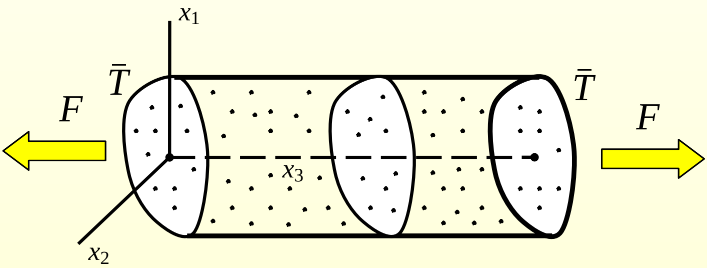
Analogously ,

$$A_2 = k\, A\, c_{44} \ , \qquad \text{with} \quad k = \frac{\pi^2}{12} \ .$$

Extension of porous thermoelastic rods

Consider the resultant axial force F and temperature $\bar{T}$ at both ends :

$$\int_{\Sigma_\alpha} t^*_{33}\,\mathrm{d}x_1\mathrm{d}x_2 = F\,, \qquad \int_{\Sigma_\alpha} T^*\,\mathrm{d}x_1\mathrm{d}x_2 = A\,\bar{T}\,.$$



The solutions in the two approaches (direct and 3D) coincide if and only if :

$$G_1 = A(b_3 - b_1\nu_1 - b_2\nu_2)\,,$$

$$G_3 = A\left(m - \frac{c_{11}b_2\beta_2 + c_{22}b_1\beta_1 - c_{12}(b_1\beta_2 + b_2\beta_1)}{\delta_1}\right),$$

$$K_1 = A\left(\xi - \frac{\beta_1^2 c_{22} + \beta_2^2 c_{11} - 2\beta_1\beta_2 c_{12}}{\delta_1}\right)\,,$$

$$K_4 = A\left(\beta_3 - \beta_1\nu_1 - \beta_2\nu_2\right)\,.$$

By comparison of constitutive equations we also identify :

$$K_2 = A\,\alpha_3\,, \quad G = A\,a\,.$$

In the case of isotropic and homogeneous materials, the constitutive coefficients become

$$c_{11} = c_{22} = c_{33} = \lambda + 2\mu\,, \quad c_{12} = c_{13} = c_{23} = \lambda\,,$$
$$c_{44} = c_{55} = c_{66} = \mu\,, \quad \alpha_i = \alpha\,, \quad \beta_i = \beta\,, \quad b_i = b,$$
$$E_0 = E, \quad \nu_1 = \nu_2 = \nu$$

We obtain by particularization the values :

$$A_1 = A_2 = k\,\mu\,A \quad \left(k = \frac{\pi^2}{12}\right), \quad A_3 = E\,A,$$

$$C_1 = E\int_\Sigma x_2^2\,\mathrm{d}x_1\mathrm{d}x_2, \quad C_2 = E\int_\Sigma x_1^2\,\mathrm{d}x_1\mathrm{d}x_2,$$

$$C_3 = 2\mu\int_\Sigma \phi^*(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2 \quad \text{with}$$

$$\Delta\phi^* = -2 \;\text{ in }\; \Sigma, \quad \phi^* = 0 \;\text{ on }\; \partial\Sigma,$$

$$G_1 = A\,\frac{\mu\,b}{\lambda+\mu}\,, \quad G_3 = A\left(m - \frac{b\beta}{\lambda+\mu}\right), \quad G = A\,a\,,$$

$$K_1 = A\left(\xi - \frac{\beta^2}{\lambda+\mu}\right), \quad K_4 = A\,\frac{\beta\,\mu}{\lambda+\mu}, \quad K_2 = A\,\alpha.$$

Conclusions

- General nonlinear theory for thermoelastic rods
- Structure of constitutive tensors
- Uniqueness of solution in the linear theory
- Decoupling of problems for straight rods
- Determination of effective stiffness values for thermoelastic orthotropic rods

Future plans :

- to consider inhomogeneous materials
- effective stiffness for functionally graded rods