Applications of Compact Superharmonic Functions: Path Regularity and Tightness of Capacities

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The talk includes results from joint works with **Nicu Boboc** and **Michael Röckner**

Classical case

E : bounded open set in $\mathbb{R}^d$.

u : bounded strictly positive potential, that is a positive superharmonic function for which every harmonic minorant is negative

Property : for every increasing sequence $(D_n)_n$ of relatively compact open subsets of E with $\bigcup_n D_n = E$, the sequence $(H_u^{D_n})_n$ decreases to zero, or equivalently

$$\inf_n H_u^{D_n} = 0 \quad \lambda\text{-a.e.},$$

where $H_u^{D_n}$ is the Perron-Wiener-Brelot solution of the Dirichlet problem on D_n with boundary data u.

A **harmonic space** possessing such a potential is called $\mathfrak{P}$-harmonic.

[C.Constantinescu, A. Cornea, Springer 72]

General frame

E : a Lusin topological space with Borel σ-algebra $\mathcal{B}$

$\mathcal{L}$: the generator of a right Markov process
$X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ with state space E:

– $(\Omega, \mathcal{F})$ is a measurable space, P^x is a probability measure on $(\Omega, \mathcal{F})$ for every $x \in E$

– $(\mathcal{F}_t)_{t \geq 0}$: filtration on Ω

– The mapping $[0, \infty) \times \Omega \ni t \longmapsto X_t(\omega) \in E$ is $\mathcal{B}([0, \infty)) \times \mathcal{F}$-measurable

– X_t is $\mathcal{F}_t/\mathcal{B}$–measurable for all t

– There exists a semigroup of kernels $(P_t)_{t\geq 0}$ on $(E, \mathcal{B})$, called **transition function** of X, such that for all $t \geq 0$, $x \in E$ and $A \in \mathcal{B}$ one has

$$P^x(X_t \in A) = P_t(x, A)$$

[If $f \in p\mathcal{B}$ then $E^x(f \circ X_t) = P_t f(x)$]

– For each $\omega \in \Omega$ the mapping

$$[0, \infty) \ni t \longmapsto X_t(\omega) \in E$$

is right continuous

Path regularity: quasi-left continuity, standardness

Let λ be a finite measure on E. The right process X is called λ-**standard** if:

– X has **càdlàg trajectories** P^{λ}-a.e., i.e., it possesses

left limits in E P^{λ}-a.e. on $[0, \zeta)$; ζ is the life time of X

– X is **quasi-left continuous up to** ζ P^{λ}-a.e., i.e., for every increasing

sequence $(T_n)_n$ of stopping times with $T_n \nearrow T$ we have

$$X_{T_n} \longrightarrow X_T \; P^{\lambda}\text{-a.e. on } [T < \zeta]$$

The right process X is called **standard** if it is λ-standard for every measure λ.

The resolvent of the process X

$\mathcal{U} = (U_q)_{q>0}$,

$$U_q f(x) := E^x \int_0^\infty e^{-qt} f \circ X_t dt = \int_0^\infty e^{-qt} P_t f(x) dt \ ,$$

$$x \in E, q > 0, f \in p\mathcal{B}$$

$\mathcal{L}$ is the infinitesimal generator of $(U_q)_{q>0}$, $\left[U_q = (q - \mathcal{L})^{-1} \right]$

$\mathcal{L}$-superharmonic function

The following properties are equivalent for a function $v : E \longrightarrow \overline{\mathbb{R}}_+$:

(i) v is $(\mathcal{L} - q)$-superharmonic

(ii) There exists a sequence $(f_n)_n$ of positive, bounded, Borel measurable functions on E such that $U_q f_n \nearrow v$

(iii) $\alpha U_{q+\alpha} v \leq v$ for all $\alpha > 0$ and $\alpha U_{q+\alpha} v \nearrow v$

$\mathcal{S}(\mathcal{L} - q)$: the set of all $(\mathcal{L} - q)$-superharmonic functions

Reduced function

If $M \in \mathcal{B}$, $q > 0$, and $u \in \mathcal{S}(\mathcal{L} - q)$ then the **reduced function of** u **on** M (with respect to $\mathcal{L} - q$) is the function $R_d^M u$ defined by

$$R_q^M u := \inf\left\{ v \in \mathcal{S}(\mathcal{L} - q) : \; v \geq u \text{ on } M \right\} .$$

- The reduced function $R_q^M u$ is universally $\mathcal{B}$-measurable.

- The functional $M \longmapsto c_\lambda(M)$, $M \subset E$, defined by

$$c_\lambda(M) := \inf\{ \int_E R_q^G 1 d\lambda : \; G \text{ open} , \; M \subset G\}$$

is a Choquet capacity on E.

- $R_q^M f(x) = E^x(f(X_{D_M}))$ [Hunt's Theorem]

Tightness of capacity – càdlàg property of the paths

An increasing sequence $(F_n)_n \subset \mathcal{B}$ is called λ**-nest** provided that

$$\inf_n R_q^{E \setminus F_n} 1 = 0 \quad \lambda - \text{a.e.}$$

- If X has **càdlàg trajectories** P^λ-a.e. then the capacity c_λ is **tight**, i.e., there exists a λ-nest of compact sets.

[T. Lyons & M. Röckner, *Bull. London. Math. Soc.* 1992]

[L.B. & N. Boboc, *Bull. London. Math. Soc.* 2005]

- The following assertions are equivalent for a finite measure λ on E.

(i) The capacity c_λ is tight.

(ii) For every increasing sequence $(D_n)_n$ of open sets with $\bigcup_n D_n = E$ we have

$$\inf_n R_q^{E \setminus D_n} 1 = 0 \quad \lambda - \text{a.e.}$$

- Assume that the space E is endowed with a **Ray topology** (a topology generated by a cone of bounded $\mathcal{L}$-superharmonic functions).

If the capacity c_λ is tight, then the process X has càdlàg trajectories P^λ-a.e. Moreover X is λ-standard (i.e., it is quasi left continuous).

Compact Lyapunov function

A $(\mathcal{L} - q)$-superharmonic function v is called **compact Lyapunov function** provided that it is finite λ-a.e. and the set $[v \leq \alpha]$ is relatively compact for all $\alpha > 0$.

- The following assertions are equivalent.

(a) The capacity c_λ is tight, i.e., there exists a λ-nest of compact sets.

(b) There exists a compact Lyapunov function.

Existence of a Markov process

Theorem

We endow E with a Lusin topology $\mathcal{T}(\mathcal{C})$ generated by a vector lattice $\mathcal{C}$ of bounded, $\mathcal{B}$-measurable, real-valued functions on E such that $1 \in \mathcal{C}$, there exists a countable subset of $\mathcal{C}_+$, separating the points of E.
Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a Markovian resolvent of kernels on $(E, \mathcal{B})$ such that the following conditions (a) and (b) and (c) hold:
(a) $\quad U_\alpha(\mathcal{C}) \subset \mathcal{C}$ *for all* $\alpha > 0$;
(b) $\quad \lim_{\alpha\to\infty} \alpha U_\alpha f(x) = f(x)$ *for all* $f \in \mathcal{C}$ *and* $x \in E$.
(c) *There exists a compact Lyapunov function (which is finite λ-a.e., where λ be a finite measure on E).*

Then there exists a standard process with state space E such that its resolvent equals $\mathcal{U}$ λ-quasi everywhere.

A1. Construction of martingale solutions for stochastic PDE on Hilbert spaces

[L.B., N. Boboc, & M. Röckner, *J. Evol. Eq.* 2006]

A2. Explicit construction of compact Lyapunov functions for Lévy processes on Hilbert spaces

[L.B., A. Cornea, & M. Röckner, *RIMS Proceedings* 2008]
[L.B., A. Cornea, & M. Röckner (preprint 2010)]

A3. Construction of measure valued branching processes associated to some nonlinear PDE

[L.B., *J. Euro. Math. Soc.* (to appear)]

[L.B. & M. Röckner, *Complex Analysis and Op. Th.* 2011]

A1. Construction of martingale solutions for stochastic PDE on Hilbert spaces

Consider SPDE on a Hilbert space H (with inner product $\langle \, , \, \rangle$ and norm $|\cdot|$) of type

$$(1) \qquad dX(t) = \big[AX(t) + F_0\big(X(t)\big)\big]dt + \sqrt{C}dW(t) \; ;$$

– $W(t)$, $t \geq 0$, is a cylindrical Brownian motion on H;

– C is a positive definite self-adjoint linear operator on H;

– $A : \mathcal{D}(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup on H.

– Furthermore,

$$F_0(x) := y_0 \ , \quad x \in \mathcal{D}(F),$$

where $y_0 \in F(x)$ such that $|y_0| = \min_{y \in F(x)} |y|$, and $F : \mathcal{D}(F) \subset H \to 2^H$ is an m-dissipative map. This means that $\mathcal{D}(F)$ is a Borel set in H and $\langle u - v, x - y \rangle \leq 0$ for all $x, y \in \mathcal{D}(F)$, $u \in F(x)$, $v \in F(y)$, and

$$\text{Range}(I - F) := \bigcup_{x \in \mathcal{D}(F)} \big(x - F(x)\big) = H.$$

– Since for any $x \in \mathcal{D}(F)$ the set $F(x)$ is closed, non-empty and convex, F_0 is well-defined.

Remark. Such equations have been studied in [G. Da Prato, M. Röckner, PTRF 02], the main novelty being that F_0 has no continuity properties.
However, a martingale solution to (1) was only constructed under the assumption that the inverse C^{-1} of C exists and is bounded and that $A = A^*$ where $\big(A^*, \mathcal{D}(A^*)\big)$ denotes the adjoint of $\big(A, \mathcal{D}(A)\big)$. Hence, in particular, the case, where C is trace class, was not covered in the final result.

The underlying Kolmogorov operator

A heuristic application of Itô's formula to a solution of (1) implies that the Kolmogorov operator on test functions

$$\varphi \in \mathcal{E}_A(H) := \text{lin. span}\big\{\sin\langle h, x\rangle,\ \cos\langle h, x\rangle \;\big|\; h \in \mathcal{D}(A^*)\big\}$$

has the following form:

$$\mathcal{L}_0\varphi(x) = \frac{1}{2}\cdot \mathrm{Tr}\big[CD^2\varphi(x)\big] + \big\langle x, A^* D\varphi(x)\big\rangle + \big\langle F_0(x), D\varphi(x)\big\rangle\ ,\ x \in H,$$

where $D\varphi(x)$, $D^2\varphi(x)$ denote the first and second Fréchet derivatives of φ at $x \in H$ considered as an element in H and as an operator on H, respectively.

– By the chain rule we have $D\varphi(x) \in \mathcal{D}(A^*)$ for all $\varphi \in \mathcal{E}_A(H)$, $x \in H$.

– $\mathcal{L}_0$ is well-defined for all φ of the form $\varphi(x) = f\big(\langle h_1, x\rangle, \dots, \langle h_M, x\rangle\big)$, $x \in H$, with $f \in C^2(\mathbb{R}^M)$, $M \in \mathbb{N}$, $h_1, \dots, h_M \in \mathcal{D}(A^*)$.

Assumptions (as in [G. Da Prato, M. Röckner, PTRF 02]):

$(H1)$ (i) A is the infinitesimal generator of a strongly continuous semigroup e^{tA}, $t \geq 0$, on H, and there exists a constant $\omega > 0$ such that $\langle Ax, x\rangle \leq -\omega\, |x|^2$ for all $x \in \mathcal{D}(A)$.

(ii) C is self-adjoint, nonnegative definite and such that $\mathrm{Tr}\, Q < \infty$, where $Qx := \int_0^\infty e^{tA}\, C\, e^{tA^*}\, x dt$, $x \in H$.

$(H2)$ There exists a probability measure μ on the Borel σ-algebra $\mathcal{B}(H)$ of H such that

$$(i) \int_{\mathcal{D}(F)} \left(|x|^{2p} + |F_0(x)|^p + |x|^{2p} \cdot |F_0(x)|^p\right) \mu(\mathrm{d}x) < \infty.$$

(ii) For all $\varphi \in \mathcal{E}_A(H)$ we have $\mathcal{L}_0\varphi \in L^p(H, \mu)$ and $\int \mathcal{L}_0\varphi d\mu = 0$ ('infinitesimal invariance').
(iii) $\mu\big(\mathcal{D}(F)\big) = 1$.

Construction of the semigroup; cf. [G. Da Prato, M. Röckner, PTRF 02]

For simplicity, we shall treat only the case $p = 2$.
By assumption $(H2)$ (ii) one can prove that $\big(\mathcal{L}_0, \mathcal{E}_A(H)\big)$ is dissipative on $L^2(H, \mu)$, hence closable. Let $\big(\mathcal{L}, \mathcal{D}(\mathcal{L})\big)$ denote its closure.

Assumptions $(H1)$ and $(H2)$ imply that $\big(\mathcal{L}, \mathcal{D}(\mathcal{L})\big)$ is m-dissipative, hence generates a C_0-semigroup $P_t := e^{t\mathcal{L}}$, $t \geq 0$, on $L^2(H, \mu)$ which is Markovian, i.e. positivity preserving and $P_t 1 = 1$ for all $t \geq 0$.

The associated resolvent of contractions on $L^2(E, \mu)$

Clearly, μ is invariant for $(P_t)_{t\geq 0}$, i.e. $\int P_t f d\mu = \int f d\mu$ for all $t \geq 0$, $f \in L^2(H, \mu)$. For $f \in L^2(H, \mu)$ and $\alpha > 0$ we define

$$V_\alpha f := \int_0^\infty e^{-\alpha t} P_t f \, dt.$$

Then $(V_\alpha)_{\alpha > 0}$ is a strongly continuous Markovian contraction resolvent.

Additional assumption

$(H3)$ (i) There exists an orthonormal basis $\{e_j \mid j \in \mathbb{N}\}$ of H so that $\bigcup_{N\in\mathbb{N}} E_N$ with $E_N := \text{lin. span}\{e_j \mid 1 \le j \le N\}$ is dense in $\mathcal{D}(A^*)$ with respect to $|\cdot|_{A^*}$ and such that for the orthogonal projection P_N onto E_N in H we have that the function $H \ni x \mapsto \langle P_N x, A^* P_N x\rangle$ converges in $L^1(H, \mu)$ to $H \ni x \mapsto \langle x, A^* x\rangle$ (defined to be $+\infty$ if $x \in H \setminus \mathcal{D}(A^*)$).

(ii) There exist two increasing functions $\varrho_1, \varrho_2 : [1, \infty) \to (0, \infty)$ such that

$$\big|F_0(x)\big|^2 \le \varrho_1\big(|x|\big) + \varrho_2\big(|x|\big)\,\big|\langle x, A^* x\rangle\big|$$

for all $x \in H$, and the function on the right hand side is in $L^1(H, \mu)$.

Compact Lyapunov function constructed by approximation

Let $u, g : H \longrightarrow \mathbb{R}_+$, $u, g \in L^p(H, \mu)$ such that u has compact level sets.

Assume that there exist two sequences, $(u_N)_N \in \mathcal{D}(\mathcal{L})$ and $(g_N)_N \in L^p(H, \mu)$,such that:

• $(\beta - \mathcal{L})u_N \leq g_N$ for all $N \in \mathbb{N}$;

• $(u_N)_N$ converges μ-a.e to u and $(g_N)_N$ converges in $L^p(H, \mu)$ to g as $N \to \infty$.

Then $v := V_\beta g$ is a compact Lyapunov function
(because $u_N \leq V_\beta g_N$ for all N and passing to the limit we get $u \leq v$).

Application

Take:

$$u(x) := |x|^2 \ , \ x \in H,$$

$$u_N := u \circ P_N \ , \ N \in \mathbb{N};$$

$$g(x) := 2\,|x|^2 + \big(2 + \varrho_2(|x|)\big)\,\big|\langle x, A^* x\rangle\big| + \varrho_1\big(|x|\big),\ x \in H,$$

$$g_N := g \circ P_N,\ N \in \mathbb{N};$$

- Assumptions $(H2)$ and $(H3)$ imply that $u_N \longrightarrow u$ μ-a.e. and $g_N \longrightarrow g$ in $L^1(H, \mu)$ as $N \to \infty$

Theorem

Assume $(H1) - (H3)$*, let* $(V_\alpha)_{\alpha>0}$ *be as defined above and* $\mathcal{T}$ *be the weak topology on* H*. Then there exists a right process with state space* H*, associated with* $(V_\alpha)_{\alpha>0}$*, which is a martingale solution of* (1)*.*

A2. Explicit construction of compact Lyapunov functions for Lévy processes on Hilbert spaces

– E : a real separable Banach space with topological dual E'

– $\mathcal{L}$: the generator of a resolvent $(U_\alpha)_{\alpha>0}$, e.g. a closed linear operator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ on $C_u(E)$ (the space of all bounded uniformly continuous real valued functions on E) topological space E) such that

$$(\alpha - \mathcal{L})^{-1} = U_\alpha, \quad \alpha > 0$$

– $\mathcal{L}$ is initially only known for "nice" functions $u : E \longrightarrow \mathbb{R}$; one can check that the given $\mathcal{L}$ is defined on the set $\mathcal{P}$ of polynomials of elements in E', i.e., function $u : E \longrightarrow \mathbb{R}$ of type

$$u(x) = p(l_1(x), l_2(x), \dots, l_m(x)), \quad x \in E$$

with $m \in \mathbb{N}$ arbitrary and $l_1, \dots l_m \in E'$, p a polynomial in m variables.

– Assume that $\mathcal{L}$ is a diffusion operator, that is it satisfies the Leibniz rule and that $\mathcal{L}l = 0$ for all $l \in E'$. This is e.g. the case when L is a differential operator only involving second derivatives. More precisely, for $u \in \mathcal{P}$, $u = p(l_1, l_2, \ldots, l_m)$,

$$(1) \qquad \mathcal{L}u(x) = \sum_{i,j=1}^{\infty} {}_{E'}\langle l_j, A(x) l_i \rangle_E \partial_i \partial_j \, p(l_1, \ldots, l_m)$$

where ∂_i denotes derivative with respect to the i-th variable and for $x \in E$, $A(x) : E' \longrightarrow E$, linear, bounded and nonnegative, continuous in x. Note that the sum in (1) is finite.

– Let $(P_t)_{t\geq 0}$ be the corresponding semigroup:

$$P_t u - u = \int_0^t P_s \mathcal{L} u \, ds, \quad t \geq 0.$$

In particular for $l \in E'$, $t \geq 0$,

$$P_t l^2 - l^2 = \int_0^t P_s(L l^2) ds = \int_0^t P_s(2 _{E'}\langle l, A(x) l\rangle_E) ds \geq 0.$$

It follows that

$$P_t l^2 \geq l^2, \text{ for all } t > 0 \text{ and } l \in E'.$$

– *Conclusion:* The function l^2 is $\mathcal{L}$-subharmonic.

– Construct a Lyapunov function in terms of $l \in E'$ and obtain, in particular, that infinite dimensional Lévy processes are quasi-left continuous: the function

$$v = U_q l^2$$

is $\mathcal{L}$-superharmonic and has compact level sets because

$$qv = qU_q l^2 \geq l^2.$$

A3. Nonlinear evolution equation

$$(*) \quad \begin{cases} \frac{d}{dt} v_t(x) = \mathcal{L} v_t(x) + \Phi(x, v_t(x)) \\ \\ v_0 = f, \end{cases}$$

where $f \in pb\mathcal{B}$.

Aim: *To give a probabilistic treatment of the equation* $(*)$.

- $\mathcal{L}$ is the infinitesimal generator of a right Markov process with state space E, called **spatial motion.**

Branching mechanism

A function $\Phi : E \times [0, \infty) \longrightarrow \mathbb{R}$ of the form

$$\Phi(x, \lambda) = -b(x)\lambda - c(x)\lambda^2 + \int_0^\infty (1 - e^{-\lambda s} - \lambda s) N(x, ds)$$

- $c \geq 0$ and b are bounded $\mathcal{B}$-measurable functions
- $N : p\mathcal{B}((0, \infty)) \longrightarrow p\mathcal{B}(E)$ is a kernel such that

$N(u \wedge u^2) \in bp\mathcal{B}$

Examples of branching mechanisms

$\Phi(\lambda) = -\lambda^\alpha$ if $1 < \alpha \leq 2$
$\Phi(\lambda) = \lambda^\alpha$ if $0 < \alpha < 1$

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Construction of the nonlinear semigroup ([Fitzsimmons 88])

The equation

$$(*) \quad \begin{cases} \frac{d}{dt} v_t(x) = \mathcal{L} v_t(x) + \Phi(x, v_t(x)) \\ \\ v_0 = f, \end{cases}$$

is formally equivalent with

$$(**) \quad v_t(x) = P_t f(x) + \int_0^t P_s(x, \Phi(\cdot, v_{t-s}))ds,$$

$$t \geq 0, x \in E$$

The following assertions hold.

i) For every $f \in bp\mathcal{B}$ the equation $(**)$ has a unique solution $(t, x) \longmapsto V_t f(x)$ jointly measurable in (t, x) such that $\sup_{0 \le s \le t} ||V_s f||_\infty < \infty$, for all $t > 0$.

ii) For all $t \ge 0$ and $x \in E$ we have $0 \le V_t f(x) \le e^{\beta t} ||f||_\infty$.

iii) If $t \longmapsto P_t f(x)$ is right continuous on $[0, \infty)$ for all $x \in E$ then so is $t \longmapsto V_t f(x)$.

iv) The mappings $f \longmapsto V_t f$ form a nonlinear semigroup of operators on $bp\mathcal{B}$.

Space of measures

$M(E)$: the space of all positive finite measures on $(E, \mathcal{B})$ endowed with the weak topology.

For a function $f \in bp\mathcal{B}$ consider the mappings

$l_f : M(E) \longrightarrow \mathbb{R}$,

$$l_f(\mu) := \langle \mu, f \rangle := \int f d\mu, \; \mu \in M(E),$$

$e_f : M(E) \longrightarrow [0, 1]$

$$e_f := exp(-l_f).$$

$\mathcal{M}(E)$:= the σ-algebra on $M(E)$ generated by $\{l_f | \; f \in bp\mathcal{B}\}$, the Borel σ-algebra on $M(E)$

The transition semigroup on the space of measures

Let $(V_t)_{t\geq 0}$ *be the nonlinear semigroup of operators on* $bp\mathcal{B}$*. Then there exists a unique Markovian semigroup of kernels* $(Q_t)_{t\geq 0}$ *on* $(M(E), \mathcal{M}(E))$ *such that for all* $f \in bp\mathcal{B}$ *and* $t > 0$ *we have*

$$Q_t(e_f) = e_{V_t f}.$$

The infinitesimal generator of the forthcoming branching process

If $\overline{\mathcal{L}}$ is the infinitesimal generator of the semigroup $(Q_t)_{t\geq 0}$ on $M(E)$ and

$$F = e_f$$

with $f \in bp\mathcal{B}$, then

$$\overline{\mathcal{L}}F(\mu) = \int_E \mu(dx)c(x)F''(\mu, x) +$$
$$\int_E \mu(dx)[\mathcal{L}F'(\mu, \cdot)(x) - b(x)F'(\mu, x)] +$$
$$\int_E \mu(dx) \int_0^\infty N(x, ds)[F(\mu + s\delta_x) - F(\mu) - sF'(\mu, x)]$$

where $F'(\mu, x)$ and $F''(\mu, x)$ are the first and second variational derivatives of F $[F'(\mu, x) = \lim_{t\to 0} \frac{1}{t}(F(\mu + t\delta_x) - F(\mu))]$.

Linear and exponential type superharmonic functions for the branching process

Let

$$\beta := ||b^-||_\infty,$$

$\beta' \geq \beta$ and

$$b' := b + \beta'.$$

Then $b' \geq 0$ and let $(P_t^{b'})_{t \geq 0}$ be the transition function of the right Markov process (which is transient if $\beta' > \beta$), having $\mathcal{L} - b'$ as infinitesimal generator.

If $u \in bp\mathcal{B}$ then the following assertions are equivalent.

i) $u \in \mathcal{S}(\mathcal{L} - b')$

ii) $l_u \in \mathcal{S}(\overline{\mathcal{L}} - \beta')$

iii) For every $\alpha > 0$ we have $1 - e_{\alpha u} \in \mathcal{S}(\overline{\mathcal{L}} - \beta')$.

Existence of Lyapunov functions for the superprocess

Assume that the spatial motion X is a Hunt process (i.e., it is quasi-left-continuous on $[0, \infty)$).

Then for every $\lambda \in M(E)$ there exists a compact Lyapunov function F with respect to the (X, Φ)-superprocess, such that $F(\lambda) < \infty$.

Sketch of the proof.

Since the spatial motion X has càdlàg trajectories it follows that there exists a λ-nest of compact sets of E

$\Longrightarrow$

there exists a Lyapunov function $v \in L^1(E, \lambda) \cap \mathcal{S}(\mathcal{L} - b')$.

$\Longrightarrow$

$F := l_v \in \mathcal{S}(\overline{\mathcal{L}} - \beta')$ and it has compact level sets

(cf. [V. Bogachev, Springer 2007]).

Remark.

(i) The existence of the compact Lyapunov functions is the main step for the proof of the càdlàg property of the paths of the measure-valued (X, Φ)-superprocess.

(ii) Zenghu Li proved (manuscript, 2009) that in order to get the quasi left continuity of the branching process, the hypothesis "X is a Hunt process" is necessary.

Theorem (The measure-valued branching process)

Assume that one of the following two conditions holds:
• b, c and N do not depend on $x \in E$;
• $(P_t)_{t \geq 0}$ is a Feller semigroup (on the locally compact space E) and $V_t(C_0(E)) \subset C_0(E)$ for every $t \geq 0$.
Then the following assertions hold.

i) There exists a right Markov branching process (called (X, Φ)-superprocess) with state space $M(E)$, having $(Q_t)_{t \geq 0}$ as transition function.

ii) If in addition X is a Hunt process (i.e., it is quasi-left continuous and has a.s. left limits in E) then the (X, Φ)-superprocess is also a Hunt process.

Negative definite functions defined on the convex cone of bounded $\mathcal{L}$-superharmonic functions

$$\mathcal{S} := b\mathcal{S}(\mathcal{L})$$

A function $\varphi : \mathcal{S} \longrightarrow \mathbb{R}$ is named **positive definite** if for all $n \geq 1$, $\{v_1, v_2, ..., v_n\} \subset \mathcal{S}$ and $\{a_1, a_2, ..., a_n\} \subset \mathbb{R}$ we have

$$\sum_{i,j} a_i a_j \varphi(v_i + v_j) \geq 0.$$

A function $\varphi : \mathcal{S} \longrightarrow \mathbb{R}$ is termed **negative definite** provided that for all $n \geq 2$, $\{v_1, v_2, ..., v_n\} \subset \mathcal{S}$ and $\{a_1, a_2, ..., a_n\} \subset \mathbb{R}$ with $\sum_{i=1}^{n} a_i = 0$ we have

$$\sum_{i,j} a_i a_j \varphi(v_i + v_j) \leq 0.$$

Considering $\mathcal{S}$ as an Abelian semigroup, a bounded **semicharacter** of $\mathcal{S}$ is a function $\rho : \mathcal{S} \longrightarrow [-1, 1]$ such that $\rho(0) = 1$ and $\rho(u + v) = \rho(u)\rho(v)$ for all $u, v \in \mathcal{S}$.
The set $\widehat{\mathcal{S}}$ of all bounded semicharacters of $\mathcal{S}$ is an Abelian semigroup (under the pointwise multiplication, with neutral element the constant semicharacter 1) and it is a compact Hausdorff topological semigroup endowed with the topology of pointwise convergence.

(1) Let $\Psi : \mathcal{S} \longmapsto \mathbb{R}$. Then Ψ is negative definite if and only if $e^{-t\Psi}$ is positive definite for all $t > 0$.

(2) Let $\varphi : \mathcal{S} \longmapsto \mathbb{R}$ be a bounded positive function with $\varphi(0) \geq 0$. Then there exists a unique positive Radon measure ν on $\widehat{\mathcal{S}}$ such that

$$\varphi(v) = \int_{\widehat{\mathcal{S}}} \rho(v)\nu(d\rho), \quad \text{for all } v \in \mathcal{S}.$$

[C. Berg, J.P.R. Christensen & P. Ressel:
Harmonic analysis on semigroups,Springer 1984]

Compact Lyapunov functions for Lévy processes on Hilbert space

Let $(H, \langle , \rangle)$ be a real Hilbert space with corresponding norm $\|\cdot\|$ and Borel σ-algebra $\mathcal{B}(H)$.

Let $\lambda : H \longrightarrow \mathbb{C}$ be a continuous negative definite function such that $\lambda(0) = 0$.

By Bochner's Theorem there exists a finitely additive measure ν_t, $t > 0$, on $(H, \mathcal{B}(H))$ such that for its Fourier transform we have

$$\widehat{\nu}_t(\xi) := \int_H e^{i\langle \xi, h\rangle} \nu_t(dh) = e^{-t\lambda(\xi)}, \quad \xi \in H.$$

Let E be a Hilbert space such that $H \subset E$ continuously and densely, with inner product $\langle , \rangle_E$ and norm $\|\cdot\|$.

- Identifying H with its dual H' we have

$$E' \subset H \subset E$$

continuously and densely, and

$$_{E'}\langle \xi, h \rangle_E = \langle \xi, h \rangle,$$

for all $\xi \in E'$, $h \in H$.
For simplicity we, therefore, write for the dualization $_{E'}\langle \ , \ \rangle_E$ between E' and E also $\langle \ , \ \rangle$.

- We assume, that $H \subset E$ is Hilbert-Schmidt. (Such a space E always exists.)

By the Bochner-Minols Theorem each ν_t extends to a measure on $(E, \mathcal{B}(E))$, which we denote again by ν_t, such that

$$\widehat{\nu}_t(\xi) = \int_E e^{i\langle \xi, z\rangle} \nu_t(dz) \quad \forall \xi \in E'.$$

- λ restricted to E' is Sazonov continuous, i.e. continuous with respect to the topology generated by all Hilbert-Schmidt operators on E'.

Hence by Levy's continuity theorem on Hilbert spaces (cf. [N.N. Vakhania, V.I. Tarieladze, and S.A. Chobanyan,87]), $\nu_t \to \delta_0$ weakly as $t \to 0$. Here δ_0 denotes Dirac measure on $(E, \mathcal{B}(E))$ concentrated at $0 \in E$.

• By the Levy-Khintchine Theorem on Hilbert space (see e.g. [K.R. Parthasarathy, 67]) we have for all $\xi \in E'$

$$\lambda(\xi) = -i\langle \xi, b\rangle + \frac{1}{2}\langle \xi, R\xi\rangle - \int_E \left(e^{i\langle \xi, z\rangle - 1} - \frac{i\langle \xi, z\rangle}{1 + \|z\|^2} \right) M(dz),$$

where $b \in E$, $R : E' \longrightarrow E$ is linear such that its composition with the Riesz isomorphism $i_R : E \to E'$ is a non-negative symmetric trace class operator, and M is a Levy measure on $(E, \mathcal{B}(E))$, i.e. a positive measure on $(E, \mathcal{B}(E))$ such that

$$M(\{0\}) = 0, \quad \int_E (1 \wedge \|z\|^2) M(dz) < \infty.$$

Defining the probability measure

$$p_t(x, A) := \nu_t(A - x),\ t > 0,\ x \in E, A \in \mathcal{B}(E),$$

we obtain a semigroup of Markovian kernels $(P_t)_{t \geq 0}$ on $(E, \mathcal{B}(E))$.

There exists a conservative Markov process $\mathbb{M}$ with transition function $(P_t)_{t \geq 0}$ which has càdlàg paths (cf. [M. Fuhrman, M. Röckner, *Potential Analysis* 00]). $\mathbb{M}$ is just an infinite dimensional version of a classical Lévy process.
Each P_t maps $C_b(E)$ into $C_b(E)$, hence so does its associated resolvent $U_\beta = \int_0^\infty e^{-t\beta} P_t dt$, $\beta > 0$. In addition, $P_t f(z) \to f(z)$ as $t \to 0$, hence $\beta U_\beta f(z) \to f(z)$ as $\beta \to \infty$ for all $f \in C_b(E)$, $z \in E$. Hence $\mathbb{M}$ is also quasi-left continuous, and thus a standard process.

- Because $H \subset E$ is Hilbert-Schmidt we can find $e_n \in E'$, $n \in \mathbb{N}$, which form a total set in E' and an orthonormal basis in H, and $\lambda_n \geq 0$, $n \in \mathbb{N}$, such that $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\bar{e}_n := \lambda_n^{-\frac{1}{2}} e_n$ form on orthonormal basis in E. Furthermore,

$$\lambda_n \langle e_n, z \rangle = \langle e_n, z \rangle_E \quad \forall n \in \mathbb{N},\, z \in E$$

and thus

$$H = \{z \in E : \sum_{n=1}^{\infty} \lambda_n^{-1} \langle \bar{e}_n, z \rangle_E^2 < \infty\}.$$

For details see, e.g., [S. Albeverio, M. Röckner, *PTRF* 89].

Assumption (which is always fulfilled if λ is sufficiently regular)

(H) There exists $C > 0$ such that for all $n \in \mathbb{N}$

$$\int \langle e_n, z \rangle^2 \nu_t(dz) \leq C(1 + t^2), \quad t > 0.$$

Remark

If λ is sufficiently regular, one can deduce that for every $\xi \in E'$

$$\int \langle \xi, z \rangle^2 \nu_t(dz) = -\frac{d^2}{d\varepsilon^2} e^{-t\lambda(\varepsilon\xi)}\Big|_{\varepsilon=0}$$

$$= t^2 \left(\langle \xi, b \rangle + \int_E \langle \xi, z \rangle \frac{\|z\|^2}{1 + \|z\|^2} M(dz) \right)^2$$

$$+ t \left(\langle \xi, R\xi \rangle + \int_E \langle \xi, z \rangle^2 M(dz) \right)$$

where we assume that ξ is such that $\int_E \langle \xi, z \rangle^2 M(dx) < \infty$. Hence in this case (H) holds provided $\sup\{\int_E \langle \xi, z \rangle^2 M(dz) : |\xi| \leq 1\} < \infty$ and M is symmetric or finite.

Compact Lyapunov functions for Lévy processes

Let $q_n \in [1, \infty)$ such that $q_n \to \infty$ as $n \to \infty$ and

$$\sum_{n=1}^{\infty} q_n \lambda_n < \infty$$

and define $q : E \to \overline{\mathbb{R}}_+$ by

$$q(z) := \left(\sum_{n=1}^{\infty} q_n \langle \bar{e}_n, z \rangle_E^2 \right)^{\frac{1}{2}} . \tag{0.1}$$

Then q has compact level sets in E.

Define

$$E_0 := \{z \in E : q(z) < \infty\}.$$

Theorem

Let $v_0 := U_1 q^2$ and for every $z \in E$, $v_z := v_0 \circ T_z^{-1}$. Then v_z is a compact Lyapunov function such that

$$z + E_0 = [v_z < \infty] =: E_z$$

and each E_z is invariant with respect to $(P_t)_{t \geq 0}$.

Measure representation of the positive definite functions defined on the convex cone of all bounded $\mathcal{L}$-superharmonic functions

Assume that $\beta U_\beta 1 = 1$. *Let* $\varphi : \mathcal{S} \longrightarrow [0, 1]$ *be a positive definite function having the following two order continuity properties:*

i) If $v \in \mathcal{S}$ *then* $\varphi(\frac{1}{n}v) \nearrow \varphi(0)$;

ii) If $(v_n)_n \subset \mathcal{S}$ *is pointwise increasing to* $v \in \mathcal{S}$ *then* $\varphi(v_n) \searrow \varphi(v)$.

Then there exists a unique finite measure $\overline{P}$ *on* $(M(E), \mathcal{M}(E))$ *such that*

$$\varphi(v) = \overline{P}(e_v), \ \text{for all } v \in \mathcal{S}.$$

Corollary ([Fitzsimmons 88])

Let $\varphi : bp\mathcal{B} \longrightarrow [0,1]$ *be positive definite such that* $\varphi(f_n) \nearrow \varphi(0)$ *whenever* $(f_n)_n \subset bp\mathcal{B}$ *and* $f_n \searrow 0$ *pointwise.*

Then there exists a unique finite measure $\overline{P}$ *on* $(M(E), \mathcal{M}(E))$ *such that*

$$\varphi(f) = \overline{P}(e_f), \quad \text{for all } f \in bp\mathcal{B}.$$

The Gaussian case – Brownian motion on an abstract Wiener space

Let (E, H, μ) be an *abstract Wiener space*

- $\big(H, \langle\ ,\ \rangle\big)$ is a separable real Hilbert space with corresponding norm $|\cdot|$, which is continuously and densely embedded into a Banach space $\big(E, \|\cdot\|\big)$, which is hence also separable;

- μ is a Gaussian measure on $\mathcal{B}$ ($=$ the Borel σ-algebra of E), that is, each $l \in E'$, the dual space of E, is normally distributed with mean zero and variance $|l|^2$.

- We have the standard continuous and dense embeddings

$$E' \subset (H' \equiv) H \subset E \, .$$

We then have that

$${}_{E'}\langle l, h \rangle_E = \langle l, h \rangle \text{ for all } l \in E' \text{ and } h \in H.$$

- The embedding $H \subset E$ is automatically compact
- μ is H-quasi-invariant, that is for $T_h(z) := z + h$, $z, h \in E$, we have

$$\mu \circ T_h^{-1} \ll \mu \quad \text{for all } h \in H.$$

- The norm $\| \cdot \|$ is *measurable* in the sense of L. Gross (cf. Dudley-Feldman-Le Cam Theorem). Hence also the centered Gaussian measures μ_t, $t > 0$, exist on $\mathcal{B}$, whose variance are given by $t|l|^2$, $l \in E'$, $t > 0$. So,

$$\mu_1 = \mu \ .$$

Clearly, μ_t is the image measure of μ under the map $z \longmapsto \sqrt{t}z$, $z \in E$.

Gaussian semigroup on the Wiener space

For $x \in E$, the probability measure $p_t(x, \cdot\,)$ is defined by

$$p_t(x, A) := \mu_t(A - x)\ , \quad \text{for all } A \in \mathcal{B}.$$

Let $(P_t)_{t>0}$ be the associated family of Markovian kernels:

$$P_t f(x) := \int_E f(y)\, p_t(x, \mathrm{d}y) = \int_E f(x+y)\, \mu_t(\mathrm{d}y)\ , \quad f \in \mathcal{B}_+,\ x \in E.$$

- $(P_t)_{t\geq 0}$ (where $P_0 := Id_E$) induces a strongly continuous semigroup of contractions on the space $\mathcal{C}_u(E)$ of all bounded uniformly continuous real-valued functions on E.

The resolvent family

$\mathcal{U} = (U_\alpha)_{\alpha>0}$: the associated strongly continuous resolvent of contractions,

$U_\alpha = \int_0^\infty e^{-\alpha t} P_t dt$, $\alpha > 0$.

$\mathcal{L}$: the infinitesimal generator

- $\mathcal{U} = (U_\alpha)_{\alpha>0}$ induces a Markovian resolvent of kernels on $(E, \mathcal{B})$.

Aim: To construct a compact Lyapunov function,
i.e., a $(\mathcal{L} - q)$-superharmonic function v such that:

the set $\overline{[v \leq \alpha]}$ is a compact subset of $[v < \infty]$ for all $\alpha > 0$

- Let $e_n \in E'$, $n \in \mathbb{N}$, such that $\{e_n \ : \ n \in \mathbb{N}\}$ forms an orthonormal basis of H.

For each $n \in \mathbb{N}$ define $\widetilde{P}_n : E \longrightarrow H_n := span\{e_1, \ldots, e_n\} \subset E'$ by

$$\widetilde{P}_n z = \sum_{k=1}^{n} {}_{E'}\langle e_k, z\rangle_E e_k,\ z \in E,$$

and $P_n := \widetilde{P}_n|_H$, so

$$P_n h = \sum_{k=1}^{n} {}_{E'}\langle e_k, h\rangle_E e_k,\ h \in H$$

and $P_n \uparrow Id_H$ as $n \to \infty$.

Proposition

We have

$$\lim_{n\to\infty} ||\widetilde{P}_n z - z|| = 0 \text{ in } \mu\text{-measure.}$$

Let $\alpha > 1$. Passing to a subsequence if necessary, which we denote by Q_n, $n \in \mathbb{N}$, we may assume that

$$(2i) \quad ||Id_H - Q_n||_{\mathcal{L}(H,E)} \leq \alpha^{-n}$$

and

$$(2ii) \quad \mu(\{z \in E : ||z - \widetilde{Q}_n z|| > \alpha^{-n}\}) \leq \alpha^{-n}.$$

We used the compactness of the embedding $H \subset E$ for $(2i)$ and Proposition for $(2ii)$.

We define the function $q : E \longrightarrow \overline{\mathbb{R}}_+$ by

$$q_\alpha(z) := \Big(\sum_{n\geq 0} \alpha^n ||\widetilde{Q}_{n+1} z - \widetilde{Q}_n z||^2\Big)^{\frac{1}{2}}, \quad z \in E$$

where $\widetilde{Q}_0 := 0$, and let

$$E_\alpha := \{z \in E : \ \ q_\alpha(z) < \infty\}.$$

Proposition

The following assertions hold.
(i) $\mu(E_\alpha) = 1$.
(ii) *For all* $h \in H$ *we have* $q_\alpha(h) \leq \sqrt{\frac{\alpha}{\alpha-1}}|h|$. *In particular,* $H \subset E_\alpha$ *continuously, hence compactly.*
(iii) *For all* $z \in E$ *we have* $||z|| \leq \sqrt{\frac{\alpha}{\alpha-1}} q_\alpha(z)$. *In particular,* (E_α, q_α) *is complete. Furthermore,* (E_α, q_α) *is compactly embedded into* $(E, ||\cdot||)$.

Corollary

(cf. [R. Carmona, 80]) *For each* $x \in E \setminus H$ *there exists a Borel subspace* L_x *of* E *such that ,* $H \subset L_x$, $\mu(L_x) = 1$, *and* $x \notin L_x$.

Compact Lyapunov function for the Brownian motion on a Wiener space

For $z \in E$ let us put

$$E_{\alpha,z} := E_\alpha + z.$$

Theorem

(i) The function q_α is $\mathcal{L}$-subharmonic, i.e.,

$$P_t(q_\alpha^2) \geq q_\alpha^2 \text{ for all } t > 0.$$

(ii) Define $v_0 := U_1 q_\alpha^2$ and for every $z \in E$, $v_z := v_0 \circ T_z^{-1}$. Then v_z is a compact Lyapunov function such that

$$E_{\alpha,z} = [v_z < \infty]$$

and each $E_{\alpha,z}$ is invariant with respect to $(P_t)_{t \geq 0}$.