Periodic solutions of pendulum-like perturbations of singular and bounded ϕ-Laplacians

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The classical case

Consider the forced pendulum equation with periodic boundary conditions

$$u'' + \mu \sin u = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \qquad (1)$$

where $\mu > 0$ and h is continuous on $[0, T]$. We denote by $\overline{h}$ the mean value of h over $[0, T]$. If $h(t) = \nu \sin t$, then the corresponding action integral

$$\mathcal{A}(u) := \int_0^T \left[\frac{u'^2(t)}{2} + \mu \cos u(t) + \nu u(t) \sin u(t) \right] dt$$

has a minimum over the space of T-periodic C^1-functions on $[0, T]$, which is a solution of (1). The arguments easily extends to the case where h is such that $\overline{h} = 0$.

• *G. Hamel*, Ueber erzwungene Schingungen bei endlischen Amplituden, Math. Ann. 86 (1922), 1-13.

“The description of the set $\mathcal{P}$ of h for which (1) has at least one solution seems to remain a *terra incognita*."

• *S. Fucik*, Solvability of Nonlinear Equations and Boundary Value Problems, Reidel, Dordrecht, 1980.

Motivated by this remark, but also unaware of the existence of Hamel’s paper, Dancer and Willem, independently, reintroduced in the early nineteen eighties the use of the direct method of the calculus of variations, in the setting of Sobolev spaces.

• *E.N. Dancer*, On the use of asymptotics in nonlinear boundary value problems, Ann. Mat. Pura Appl. 131 (1982), 167-185.
• *M. Willem*, Oscillations forcées de l'équation du pendule, Pub. IRMA Lille, 3 (1981), V-1-V-3.

The existence of a second solution is proved by Mawhin and Willem using Brezis-Coron-Nirenberg variant of the Ambrosetti-Rabinowitz mountain pass lemma.
• *J. Mawhin, M. Willem*, Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations, J. Differential Equations 52 (1984), 264-287.

$$|\overline{h}| < \mu, \quad 2\pi\mu + ||\widetilde{h}||_1 < 3,$$
$$\arcsin\left(\frac{|\overline{h}|}{\mu}\right) < \frac{\pi}{2} - \frac{\pi}{6}\left(2\pi\mu + ||\widetilde{h}||_1\right).$$

• *J. Mawhin*, Periodic oscillations of forced pendulum-like equations, Lecture Notes Math. 964 (1982), 458-476.

$$|\overline{h}| < \mu,$$
$$\arcsin\left(\frac{|\overline{h}|}{\mu}\right) < \frac{\pi}{2} - \frac{\pi}{6}\left(4\pi\min\{\mu - \overline{h}, \mu + \overline{h}\} + ||\widetilde{h}||_1\right).$$

• *R. Kannan, R. Ortega*, Periodic solutions of pendulum-type equations, J. Diff. Eq. 59 (1985), 123-144.

Consider the forced pendulum equation with friction and periodic boundary conditions

$$u'' + cu' + \mu \sin u = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \quad (2)$$

where $c, \mu > 0$ and h is continuous on $[0, T]$.

Given positive constants c, μ and T, there exists h with mean value zero such that (2) has no solutions.
• *R. Ortega, E. Serra, M. Tarallo*, Non-continuation of the periodic oscillations of a forced pendulum in the presence of friction, Proc. Amer. Math. Soc. 128 (2000), 2659-2665.

If

$$||h||_\infty < \mu,$$

then (2) has at least two solutions, and if

$$||h||_\infty = \mu,$$

then (2) has at least one solution.

• *J. Mawhin, M. Willem*, Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations, J. Differential Equations 52 (1984), 264-287.

The ϕ-Laplacian case

Theorem 1 *Let* $0 < a < +\infty$, $\phi : (-a, a) \to \mathbb{R}$ *be an increasing homeomorphism such that* $\phi(0) = 0$, $f : \mathbb{R} \to \mathbb{R}$ *be continuous,* $h \in C$ *and* $\mu > 0$. *If*

$$aT < \pi\sqrt{3}, \quad |\overline{h}| < \mu \cos\left(\frac{aT}{2\sqrt{3}}\right),$$

then the periodic problem

$$(\phi(u'))' + f(u)u' + \mu \sin u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \quad (3$$

has at least two solution u_1, u_2 *such that* $-\frac{\pi}{2} < \overline{u}_1 < \frac{\pi}{2} < \overline{u}_2 < \frac{3\pi}{2}$.

Remark 1 If in Theorem 1 one assumes

$$aT = \pi\sqrt{3},$$

then problem (3) has at least one solution for any $h \in C$ with $\overline{h} = 0.$

Remark 2 Using Schauder fixed point theorem P.J. Torres has proved Theorem 1 under the more restrictive assumptions

$$aT < 2\sqrt{3}, \quad |\overline{h}| < \mu\left(1 - \frac{aT}{2\sqrt{3}}\right).$$

Assume that $f = 0$ and $\phi = \Phi'$, where Φ is continuous on $[-a, a]$ of class C^1 on $(-a, a)$ and strictly convex. Then, (3) has at least one solution.

• *H. Brezis, J. Mawhin*, Periodic solutions of the forced relativistic pendulum, Differential Integral Eq. 23 (2010), 801-810.

Problem

$$(\phi(u'))' - u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T),$$

has a unique solution for any $h \in C$.

• *C.B., J. Mawhin*, Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian, J. Differential Equations 243 (2007), 536–557.

Theorem 2 *Let $\mu > 0$ and assume that $h \in C$ satisfies*

$$||h||_\infty \leq \mu.$$

Then the periodic problem

$$(\phi(u'))' + f(u)u' + \mu \sin u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \quad \text{(4)}$$

has at least one solution. Moreover, if

$$||h||_\infty < \mu,$$

then (4) has at least two solutions not differing by a multiple of 2π.

Corollary 1 *Assume that* $\psi : \mathbb{R} \to (-c, c)$ $(0 < c \leq \infty)$ *is an increasing homeomorphism such that* $\psi(0) = 0$ *and*

$$||[h - \mu]^{-}||_1 < c/2.$$

If $||h||_\infty \leq \mu$ *then*

$$(\psi(u'))' + \mu \sin u = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{5}$$

has at least one solution. Moreover, if $||h||_\infty < \mu$, *then* (5) *has at least two solutions not differing by a multiple of* 2π.

Example 1 If $h \in C$ is such that

$$\overline{h} = 0, \quad ||h||_\infty < \mu < 1/2T,$$

then the periodic problem

$$\Big(\frac{u'}{\sqrt{1+u'^2}}\Big)' + \mu \sin u = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least two solutions not differing by a multiple of 2π.

• *C.B., P. Jebelean, J. Mawhin*, Periodic solutions of pendulum-like perturbations of singular and bounded ϕ-Laplacians, J. Dynamics Differential Equations, to appear.