Weyl-Pedersen Calculus on Coadjoint Orbits of Nilpotent Lie Groups

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Contents

Introduction

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- I. Beltiță, D. Beltiță, Modulation spaces of symbols for representations of nilpotent Lie groups. J. Fourier Anal. Appl. (to appear).
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Coadjoint orbits, preduals, and Fourier transform

• G connected, simply connected, nilpotent Lie group, Lie algebra $\mathfrak{g}$ ($\exp_G\colon \mathfrak{g} \to G$ is a diffeom.) , $\mathfrak{g}^*$ linear dual space to $\mathfrak{g}$

$\mathrm{Ad}_g(X) = (\mathrm{d}/\mathrm{d}t)(g\exp(tX)g^{-1})|_{t=0}$, $X \in \mathfrak{g}$, $g \in G$

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• The mapping $f \to \hat{f}(X) = \int_{\mathcal{O}} \mathrm{e}^{-\mathrm{i}\langle \xi, X \rangle} f(\xi) \mathrm{d}\xi$ is invertible, and extends to unitary $L^2(\mathcal{O}) \to L^2(\mathfrak{g}_e)$.

Irreducible representations

$\pi\colon G \mapsto \mathcal{B}(\mathcal{H})$ irreducible representation associated to $\mathcal{O}$.
• $\mathcal{H}_\infty$ space of smooth vectors for π – nuclear Fréchet space.
• $\mathcal{H}_{-\infty}$ continuous antilinear functionals on $\mathcal{H}_\infty$.

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• $\mathcal{B}(\mathcal{H})_\infty$ space of smooth vectors for the unitary representation
$\pi \otimes \bar{\pi}\colon G \times G \to \mathcal{B}(\mathfrak{S}_2(\mathcal{H})), \quad (\pi \otimes \bar{\pi})(m_1, m_2)T = \pi(m_1)T\pi(m_2)^{-1}$.

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- $\mathcal{B}(\mathcal{H})_\infty \simeq \mathcal{H}_\infty \widehat{\otimes} \mathcal{H}_\infty$

Ambiguity function and Wigner distribution

- *Ambiguity function*
- $f \in \mathcal{H}$, $\phi \in \mathcal{H}$ or $f \in \mathcal{H}_{-\infty}$ and $\phi \in \mathcal{H}_{\infty}$:

$$\mathcal{A}^{\pi}_{\phi} f \colon \mathfrak{g}_e \to \mathbb{C},\ \mathcal{A}^{\pi}_{\phi} f(X) = (f \mid \pi(\exp_G X)\phi)$$

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- Nuclearity $\Rightarrow$ $\mathcal{W}^\pi$ extends cont. $\mathcal{H}_\infty \hat{\otimes} \overline{\mathcal{H}}_\infty \cong \mathcal{B}(\mathcal{H})_\infty \to \mathcal{S}(\mathfrak{g}_e)$.

Weyl-Pedersen calculus: Definition and first properties

N.V. Pedersen (1994)

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- $\mathsf{Tr}(\mathsf{Op}^\pi(a)) = \int_{\mathcal{O}} a(\xi)\mathrm{d}\xi$
- $\mathsf{Op}^\pi(\bar{a}) = \mathsf{Op}^\pi(a)^*$
- $\mathsf{Tr}(\mathsf{Op}^\pi(a)\,\mathsf{Op}^\pi(b)) = \int_{\mathcal{O}} a(\xi)b(\xi)\mathrm{d}\xi$
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- $a\#b$ the symbol of $\mathsf{Op}^\pi(a)\,\mathsf{Op}^\pi(b)$

Weyl-Pedersen calculus: Further properties

- $(\mathrm{Op}(a)f \mid g) = \langle a, \overline{\mathcal{W}(g,f)}\rangle$, $a \in \mathcal{S}'(\mathcal{O})$, $f, g \in \mathcal{H}_\infty$.

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- $\mathrm{Op}(\mathcal{W}^\pi(f_1, f_2)) = f_1 \otimes \bar{f_2} = (\cdot \mid f_2)f_1$
- In particular, when $\mathcal{H} = L^2(\mathbb{R}^d)$ for some d,

$$\mathrm{Op}(\mathcal{W}^\pi(K)) = K \quad \forall\, K \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$$

Group square

• $G \ltimes G$ semi-direct product defined by the action of G on itself by inner automorphisms.

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- $\exp_{G \ltimes G}\colon \mathfrak{g} \ltimes \mathfrak{g} \to G \ltimes G, \quad (X, Y) \mapsto (\exp_G X, \exp_G(-X)\exp_G(X+Y))$.

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• Continuous unitary representation

$$\pi^{\ltimes}\colon G \ltimes G \to \mathcal{B}(\mathfrak{S}_2(\mathcal{H})), \quad \pi^{\ltimes} = (\pi \otimes \bar{\pi}) \circ \mu$$

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Group square

• $G \ltimes G$ semi-direct product defined by the action of G on itself by inner automorphisms.

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• $\mu\colon G \ltimes G \to G \times G, \quad (g_1, g_2) \mapsto (g_1 g_2, g_1)$ isomorphism of Lie groups with tangent map $\mathbf{L}(\mu)\colon \mathfrak{g} \ltimes \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}, \quad (X, Y) \mapsto (X + Y, X)$.

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- $\phi \in \mathcal{H}_\infty \setminus 0$ window function:
$$M^{r,s}_{\phi}(\pi^{\#}) := \{f \in \mathcal{H}_{-\infty} \mid \mathcal{A}^{\pi^{\#}}_{\phi}(f) \in L^{r,s}(\mathfrak{g}_e \times \mathfrak{g}_e)\}$$

Kurbatov property

$a \in \mathcal{S}'(\mathcal{O})$, $\phi \in \mathcal{H}_\infty$,

$$K(X,Y) = \langle a, \overline{e^{\mathrm{i}\langle\cdot,Y\rangle} \# \mathcal{W}^\pi(\phi,\phi) \# e^{-\mathrm{i}\langle\cdot,X\rangle}} \rangle$$

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• $a \in M^{\infty,1}_{\mathcal{W}^\pi(\phi,\phi)}(\pi^\#)$ if and only if K satisfies the Kurbatov property:

$$Y \to \sup_{X\in\mathfrak{g}_e}|K(X,X+Y)| \in L^1(\mathfrak{g}_e)$$

Continuity results

Theorem

If $\phi \in \mathcal{H}_\infty$, then the following assertions hold:

1. *The mapping*
$$M^{\infty,1}_{\mathcal{W}^\pi(\phi,\phi)}(\pi^{\#}) \ni a \Rightarrow \mathrm{Op}(a) \in \mathcal{B}(\mathcal{H})$$
is continuous: $\|\mathrm{Op}(a)\| \le \|a\|_{M^{\infty,1}_{\mathcal{W}^\pi(\phi,\phi)}(\pi^{\#})}$.
2. $\#$ *makes the modulation space* $M^{\infty,1}_{\mathcal{W}^\pi(\phi,\phi)}(\pi^{\#})$ *into an involutive associative Banach algebra.*
3. *If* $a_0 \in 1 + M^{\infty,1}_{\mathcal{W}^\pi(\phi,\phi)}(\pi^{\#})$ *and* $\mathrm{Op}(a_0)$ *is invertible in* $\mathcal{B}(\mathcal{H})$*,*
$\Rightarrow\ \exists\, b_0 \in 1 + M^{\infty,1}_{\mathcal{W}^\pi(\phi,\phi)}(\pi^{\#})$, $\mathrm{Op}(a_0)^{-1} = \mathrm{Op}(b_0)$.

Modulation spaces of symbols: square integrable representations

• G nilpotent Lie group, Z its centre, $\mathfrak{g}$ its Lie algebra, $\mathfrak{z}$, $\mathfrak{z}^* = \mathrm{sp}\{\xi_0\}$.
• Assume $\mathcal{O} = \mathcal{O}_{\xi_0} = \xi_0 + \mathfrak{z}^\perp \Rightarrow$ corresp. representation $\pi\colon G \to \mathbb{B}(\mathcal{H})$ square integrable mod $\mathfrak{z}$, predual $\mathfrak{g}_e : \mathfrak{g}_e + \mathfrak{z} = \mathfrak{g}$.
• $\pi^\#$ is square integrable modulo centre, $\mathcal{O}^\# = \zeta^\perp \times \mathcal{O}$, $\mathfrak{g}_e \times \mathfrak{g}_e$ predual

$1 \le p, q \le \infty$, $\Phi \in \mathcal{S}(\mathcal{O})$ (any!)

$$M^{p,q}(\pi^\#) = \{F \in \mathcal{S}'(\mathcal{O}) \mid \mathcal{A}^\#_\Phi F \in L^{p,q}(\mathfrak{g}_e \times \mathfrak{g}_e)\}.$$

Example: Heisenberg group

- *Heisenberg group* $\mathbb{H}_{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, with
  Heisenberg algebra $\mathfrak{h}_{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$

$$[(q,p,t),(q',p',t')] = [(0,0,p\cdot q' - p'\cdot q)].$$

- $\xi_0 = (0,0,1)$, $\mathcal{O} = \mathbb{R}^n \times \mathbb{R}^n \times \{1\}$, $\mathfrak{g}_e = \mathbb{R}^n \times \mathbb{R}^n$.
- *Schrödinger representation:* $\pi\colon \mathbb{H}_{2n+1} \to \mathbb{B}(L^2(\mathbb{R}^n))$,

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$$(\mathrm{Op}^\pi(a)f)(q) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} a\left(\frac{q+q'}{2},p\right)\mathrm{e}^{\mathrm{i}(q-q')p}f(q')\mathrm{d}q'\mathrm{d}p'.$$

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- $M^{p,q}(\pi^{\#})$ coincides with the modulation spaces defined in time-frequency analysis (modulation space for the Heisenberg group of dimension $4n+1$). In part, $M^{\infty,1}(\pi^{\#})$ is the Sjöstrand algebra, $BC^{\infty}(\mathbb{R}^n \times \mathbb{R}^n) \subset M^{\infty,1}(\pi^{\#})$, and one recovers Sjöstrand's results.

Convolution operators on Lie groups

G be a finite-dimensional simply connected nilpotent Lie group, the inverse of the exponential map: $\log_G \colon G \to \mathfrak{g}$.

• $\mathcal{F}_G := \mathrm{span}_{\mathbb{R}}(\{\lambda_g(\xi \circ \log_G) \mid \xi \in \mathfrak{g}^*, g \in G\})$,

Then

• $\mathcal{F}_G$ is a finite dim. linear subspace of $\mathcal{C}^\infty(G)$, invariant under the left regular repres. contains the constant functions.

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When $a = a(\xi)$

$(\mathrm{Op}(a)f)(X) = \int_{\mathfrak{g}} \check{a}(Y) f((-Y) * X)) \,\mathrm{d}(X, \xi), \quad f \in L^2(\mathfrak{g})$.