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Exponential stabilization of the linearized Navier–Stokes equation by pointwise feedback noise controllers

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We address the problem of exponential stabilization in probability of the linearized Navier–Stokes equations in an equilibrium point. This is done by designing a linear stochastic feedback controller with support in a point or on a discrete set of points of the domain. This controller consists of steady-state impulse component with support in a finite set of points modulated by an unsteady feedback noise controller.

1 Introduction

In this paper, we address the problem of linear stabilization of steady-state solutions X_e to Navier-Stokes equations by mean of a noise internal controller with support in discrete set of points of the domains. More precisely, the stabilizable feedback controller proposed here is of the form

$$u = \left(\sum_{k=1}^{M} \mu_k \delta(\xi_k) \right) \sum_{j=1}^{N} \left\langle X - X_e, \varphi_j^* \right\rangle \dot{\beta}_j, \tag{1}$$

where $\mu_k \subset \mathbb{C}$, $\delta(\xi_k)$ is the Dirac measure concentrated in the point $\xi_k \in \mathcal{O} \subset R^d$, $d = 2, 3$, $\{\beta_j\}$ is a system of independent Brownian motions and $\{\varphi_j^*\}_{j=1}^N$ are eigenfunctions to dual Stokes–Oseen operator corresponding to eigenvalues $\{\lambda_j;\ \mathrm{Re}\,\lambda_j \leq \gamma\}$, $j = 1, ..., N$.

The main result amounts to saying that the feedback controller (1) exponentially stabilizes in probability the linearized Navier–Stokes system in a certain weak sense to be discussed below.

Stabilizing stochastic feedback controllers for Navier–Stokes equations but with support in an arbitrary open subset $\mathcal{O}_0 \subset \mathcal{O}$ were designed by Barbu [8] and Barbu & Da Prato [11]. The feedback controller (1) we propose here is concentrated on a finite set of spatial points $\xi_k \in \mathcal{O}$, which practically can be arbitrarily chosen. As we shall see from the construction below, this controller is also robust to structural perturbations of the system. A noise controller of similar form was designed in Barbu [10] for the stabilization of the equilibrium profile of a periodic fluid flow in a 2-D channel. The normal boundary controller designed in this latter case is concentrated on the wall $y = 1$ of the channel $(-\infty, \infty) \times (0, 1)$.

It should be said that, compared with the performances of deterministic Riccati-based stabilization controllers developed by Barbu & Triggiani [12], Barbu, Lasiecka & Triggiani [9, 13], Fursikov [20], Raymond [24], the noise stabilizing controller is easy to implement and avoids large numerical computations, which are practically untractable for Navier–Stokes equations. On the other hand, it is no analogy of the feedback law (1) in the framework of the deterministic controller and perhaps the only way to design a stabilizing and robust feedback controller with discrete support is to represent it in a stochastic form. It should be emphasized, however, that the stabilization occurs in a weak topology that is in distributional sense. For other literature on stabilization by noise but considered in a different context we refer to Arnold, Craul & Wihstutz [3], Deng, Krstic & Williams [18], Caraballo et al. [16]. In particular, the last two works are concerned with stabilization by noise of PDEs.

2 Problem statement and control design

The dimensionless Navier–Stokes equations for incompressible flow in an open, bounded domain $\mathcal{O} \subset R^d$, $d = 2, 3$, are given by

$$
(2) \qquad \begin{cases}
X_t - \nu\Delta X + (X \cdot \nabla)X = \nabla p + f_e & \text{in } (0,\infty) \times \mathcal{O} \\
\nabla \cdot X = 0 & \text{in } (0,\infty) \times \mathcal{O} \\
X = 0 & \text{on } (0,\infty) \times \partial\mathcal{O} \\
X(0,\xi) = X_0(\xi), & \xi \in \mathcal{O}.
\end{cases}
$$

Here, $f_e \equiv f_e(\xi)$, $\xi \in \mathcal{O}$, is a smooth function and $\partial\mathcal{O}$ is the boundary of $\mathcal{O}$.

If $X_e \equiv X_e(\xi)$ is a stationary solution to (2), i.e.,

$$
\begin{aligned}
&-\nu\Delta X_e + (X_e \cdot \nabla)X_e = \nabla p_e + f_e \text{ in } \mathcal{O} \\
&\nabla \cdot X_e = 0, \;\; X_e = 0 \text{ on } \partial\mathcal{O},
\end{aligned}
$$

then, defining $y = X - X_e$, we get for the error y the Stokes–Oseen equation

$$
(3) \qquad \begin{cases}
y_t - \nu\Delta y + (X_e \cdot \nabla)y + (y \cdot \nabla)X_e = \nabla p \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,\infty) \times \mathcal{O} \\
\nabla \cdot y = 0 \quad \text{in } (0,\infty) \times \mathcal{O} \\
y = 0 \qquad\;\; \text{on } (0,\infty) \times \partial\mathcal{O} \\
y(0,\xi) = x(\xi) = X_0(\xi) - X_e(\xi), \quad \xi \in \mathcal{O}.
\end{cases}
$$

Our objective here is to design an internal feedback controller u with support in a finite numbers of points $\{\xi_k\}_{k=1}^M \subset \mathcal{O}$, which exponentially stabilizes system (3). This is a fundamental problem in the linear theory of fluid dynamics (Joseph [21]) and can be viewed as a first step to the stabilization of the stationary solution X_e to (2). To this aim, let us first introduce a few notations and functional spaces used in the theory of the Navier–Stokes equations (Temam [28]).

Everywhere in the following, $L^2(\mathcal{O})$ is the space of square integrable functions on $\mathcal{O} \subset R^d$, $d = 2, 3$, and $H^k(\mathcal{O})$, $k = 1, 2$, $H_0^1(\mathcal{O})$ are standard Sobolev spaces on $\mathcal{O}$ (Adams [2]). Here, $\mathcal{O}$ is a bounded and open subset of R^d, $d = 2, 3$, with smooth boundary $\partial\mathcal{O}$.

Denote by H the space of all free divergence tangential functions on $\mathcal{O}$, i.e.,

$$H = \{y \in (L^2(\mathcal{O}))^d;\ \nabla \cdot y = 0,\ y \cdot n = 0 \ \text{ on } \partial\mathcal{O}\},$$

when n is the normal vector to $\partial\mathcal{O}$.

Denote by $P : (L^2(\mathcal{O}))^d \to H$ the Leray projector on H and by A the operator

$$Ay = -P(\Delta y),\ \forall y \in D(A) = (H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}))^d \cap H.$$

Consider the linear operator

$$\mathcal{A} = \nu A + A_0, \quad D(\mathcal{A}) = D(A),$$

where $A_0 y = P((X_e \cdot \nabla)y + (y \cdot \nabla)X_e)$.

In the following, it is convenient to work on the complexified space $\widetilde{H} = H + iH$ and we shall denote again $\mathcal{A}$ the extension to the operator $\mathcal{A}$ on this space. It is well known that the resolvent $(\lambda I - \mathcal{A})^{-1}$ is compact for each $\lambda \in \rho(\mathcal{A})$ and the spectrum $\sigma(\mathcal{A})$

is of the form $\{\lambda_j\}_{j=1}^{\infty}$. We fix $\gamma > 0$ and note that there is a finite number of eigenvalues $\{\lambda_j\}_{j=1}^{N}$ such that

$$\text{(4)} \qquad \operatorname{Re}\lambda_j \leq \gamma, \quad j = 1, ..., N.$$

(For simplicity, we shall call such eigenvalues "unstable", though only that with $\operatorname{Re}\lambda_j \leq 0$ are in this category.)

For each λ_j consider the corresponding eigenfunction φ_j, each λ_j being repeated according to its (algebraic) multiplicity, m_j. The adjoint operator $\mathcal{A}^*$ with $D(\mathcal{A}^*) = D(\mathcal{A})$ has the eigenvalues $\overline{\lambda}_j$ with corresponding eigenfunction φ_j^*.

Next, we shall impose for simplicity the following hypothesis

(H1) *Each* λ_j, $j = 1, ..., N$, *is semi-simple.*

This means that for each λ_j the algebraic multiplicity coincides with the geometric multiplicity and so

$$\text{(5)} \qquad \mathcal{A}\varphi_j = \lambda_j \varphi_j, \ \mathcal{A}^*\varphi^* = \overline{\lambda}_j \varphi_j^*, \quad j = 1, ..., N.$$

An immediate consequence of this hypothesis, which is generically satisfied for "almost all" X_e, is that $\{\varphi_j\}$, $\{\varphi_j^*\}$ can be chosen in such a way that

$$\text{(6)} \qquad \left\langle \varphi_i, \varphi_j^* \right\rangle = \delta_{i,j}, \quad i, j = 1, ..., N.$$

Here, $\langle \cdot, \cdot \rangle$ is the scalar product in $\widetilde{H}$. We note also that $-\mathcal{A}$ generates a C_0-analytic semigroup $e^{-\mathcal{A}t}$ on $\widetilde{H}$.

Consider now a probability space $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}_{t>0}$ and a system of independent complex Brownian motion $\{\beta_j = \beta_j^1 + i\beta_j^2\}_{j=1}^{N}$ in this probability space. We shall use the standard notations for spaces of adapted $\widetilde{H}$-valued processes.

In particular, $C_W([0, T]; L^2(\Omega, \widetilde{H}))$ is the space of adapted $\widetilde{H}$-valued continuous processes on $[0, T]$. (We refer to Da Prato &

Zabczyk [17] for notations and basic results on infinite dimensional stochastic differential equations.)

We note that in these terms the controlled system (3) can be rewritten as a state system

$$\begin{aligned} &\frac{dy}{dt} + \mathcal{A}y = 0, \ \ \forall t \geq 0, \\ &y(0) = x \end{aligned} \tag{7}$$

where $y : [0, \infty) \to \widetilde{H}$. (We take $x \in H$.)

In the following, we shall denote by the same symbol $|\cdot|$ the norm in H, $\widetilde{H}$ and in $\mathbb{C}$.

Now, we fix $\{\xi_k\}_{k=1}^M \subset \mathcal{O}$ and $\{\mu_k\}_{k=1}^M \subset \mathbb{C}$ such that

$$\left| \sum_{k=1}^{M} \mu_k \varphi_i^*(\xi_k) \right| > 0, \quad \forall i = 1, 2, ..., N. \tag{8}$$

Theorem 1 below is the main result.

Theorem 1 *Under assumptions* (8) *for* $|\eta|$ *sufficiently large, the feedback noise controller*

$$u(t) = \eta \sum_{j=1}^{N} \left\langle y(t), \varphi_j^* \right\rangle \dot{\beta}_j(t) \sum_{k=1}^{M} \mu_k \delta(\xi_k) \tag{9}$$

weakly exponentially stabilizes in probability the state system (7)*. More precisely, the solution* y *to the closed-loop system*

$$\begin{aligned} &dy(t) + \mathcal{A}y(t)dt = \eta \sum_{j=1}^{N} \left\langle y(t), \varphi_j^*(t) \right\rangle d\beta_j(t) \sum_{k=1}^{M} \mu_k \delta(\xi_k) \\ &y(0) = x \end{aligned} \tag{10}$$

satisfies

$$\begin{aligned} &\mathbb{P}\left[\lim_{t\to\infty} \langle X(t), \psi\rangle = 0\right] = 1, \\ &\qquad \forall \psi \in (H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}))^d \cap H. \end{aligned} \tag{11}$$

Equation (10) is taken in Ito's sense in the dual space $((H^2(\mathcal{O})\cap H_0^1(\mathcal{O}))^d \cap H)'$ (Da Prato & Zabczyk [17]). More precisely, the solution y to (10) is in the following "mild" sense

$$y(t)=e^{-\mathcal{A}t}x+\eta\sum_{j=1}^{N}\sum_{k=1}^{M}\int_0^t\left\langle y(s),\varphi_j^*\right\rangle e^{-\mathcal{A}(t-s)}(\delta(\xi_k))d\beta_j(s), \tag{12}$$

where $e^{-\mathcal{A}t}\delta(\xi_k) \in ((H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}))^d \cap H)'$ is defined by

$$e^{-\mathcal{A}t}\delta(\xi_k)(\psi) = (e^{-\mathcal{A}t}\psi)(\xi_k),$$
$$\forall \psi \in (H^2(\mathcal{O}) \cap H_0^2(\mathcal{O}))^d \cap H) = D(A).$$

(Here $'$ stands for the dual space.)

Since $e^{-\mathcal{A}t}\psi \in H^2(\mathcal{O}) \subset C(\overline{\mathcal{O}})$, the latter makes sense and so (10) has a solution $y \in C_W([0,T]; L^2(\Omega,(D(A))'))$ on each interval $[0,T]$. More generally, if $\mu \in (\mathcal{M}(\mathcal{O}))^d$ is a bounded measure on $\mathcal{O}$ such that

$$\mu(\varphi_i^*) \neq 0, \quad \forall i = 1, ..., N, \tag{13}$$

we have

Theorem 2 *For $|\eta|$ large enough, the feedback law*

$$u = \eta\mu\sum_{j=1}^{N}\left\langle y,\varphi_j^*\right\rangle \dot{\beta}_j \tag{14}$$

stabilizes system (7) in the sense of (11).

For instance, one might take $\mu\in((H^2(\mathcal{O})\cap H_0^1(\mathcal{O}))^d\cap H)'$ of the form

$$\mu(\psi) = \int_\Gamma h(\xi)\psi(\xi)d\sigma_\xi, \ \forall \psi \in (H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}))^d \cap H,$$

where Γ is a smooth surface (or manifold) of $\mathcal{O}$ and h is a continuous function on $\overline{\mathcal{O}}$.

In particular, by Theorem 1, if $\xi_0 \in \mathcal{O}$ is such that

$$|\varphi_i^*(\xi_0)| \neq 0, \quad \forall i = 1, ..., N,$$

then the feedback law

$$u = \eta\delta(\xi_0) \sum_{j=1}^{N} \left\langle y, \varphi_j^* \right\rangle \dot{\beta}_j \tag{15}$$

stabilizes for $|\eta|$ large enough system (7) in the sense of (11).

Since, by the unique continuation property of the eigenfunctions to Stokes–Oseen operator $\mathcal{A}^*$, each φ_j^* is not identically zero on any open subset of $\mathcal{O}$, we may conclude therefore that for almost all $\xi_0 \in \mathcal{O}$ there is a noise controller of the form (15) which weakly stabilizes in probability system (3). It should be emphasized that the feedback controller (9) uses only a discrete set of points ξ_k, $k = 1, ..., M$, for actuation. This means that the controlled velocity field will consist of a steady-state impulse component

$$\widetilde{\mu} = \sum_{k=1}^{M} \mu_k \delta(\xi_k)$$

modulated by the unsteady feedback noise controller

$$u_0(t) = \sum_{j=1}^{N} \left\langle y(t), \varphi_j^* \right\rangle \dot{\beta}_j(t). \tag{16}$$

Since the steady-state component of the controller is singular (in fact, it is a measure), the stabilization is in the weak topology only, i.e., in the sense of distributions on $\mathcal{O}$. However, as we shall see later, this controller is robust with respect to small perturbations of the system.

3 Proofs

3.1 Proof of Theorem 1

We consider the spaces

$$X_1 = \text{lin span}\{\varphi_j\}_{j=1}^N = P_N(\widetilde{H}),\ X_2 = (I - P_N)\widetilde{H}.$$

We shall denote by X_1^* and X_2^* the spaces

$$\text{lin span}\{\varphi_j^*\}_{j=1}^N \quad \text{and} \quad (I - P_N^*)\widetilde{H},$$

respectively. Here, P_N is the algebraic projection on X_1, and P_N^* is its dual. (See Kato [22].)

The operator $\mathcal{A}$ leaves invariant both spaces X_1, X_2 (Kato [22]) and we set

$$\mathcal{A}_1 = \mathcal{A}|_{X_1}, \qquad \mathcal{A}_2 = \mathcal{A}|_{X_2}.$$

Notice that $\sigma(\mathcal{A}_1) = \{\lambda_j\}_{j=1}^N$, $\sigma(\mathcal{A}_2) = \{\lambda_j\}_{j=N+1}^\infty$. Moreover, the infinite dimensional operator $\mathcal{A}_2 : D(\mathcal{A}_2) \subset X_2 \to X_2$ generates a C_0-analytic semigroup and since $\sigma(\mathcal{A}_2) \subset \{\lambda;\ \text{Re}\,\lambda_j > \gamma\}$, it follows by the logarithmic spectral growth property that, for some $\varepsilon > 0$,

$$\|e^{-\mathcal{A}_2 t}\|_{L(\widetilde{H},\widetilde{H})} \le Ce^{-(\gamma+\varepsilon)t}, \quad \forall t > 0. \tag{17}$$

(See Bensoussan et al. [14].)

Next, we decompose system (10) as

$$\begin{aligned} & y = y^1 + y^2, \quad y^1 = \sum_{j=1}^N y_j\varphi_j, \\ & dy^1 + \mathcal{A}_1 y^1 dt = \eta P_N \widetilde{\mu} \sum_{j=1}^N y_j d\beta_j, \\ & y^1(0) = P_N x. \end{aligned} \tag{18}$$

$$
\begin{aligned}
&dy^2 + \mathcal{A}_2 y^2 dt = \eta (I - P_N) \widetilde{\mu} \sum_{j=1}^{N} y_j d\beta_j \\
&y^2(0) = (I - P_N)x,
\end{aligned} \tag{19}
$$

where

$$
\widetilde{\mu} = \sum_{k=1}^{M} \mu_k \delta(\xi_k).
$$

The solution y^2 to (19) is taken in the "mild" sense (12), that is

$$
\begin{aligned}
y^2(t) &= e^{-\mathcal{A}_2 t}(I - P_N)x \\
&+ \eta \sum_{j=1}^{N} \int_0^t y_j(s) e^{-\mathcal{A}_2 (t-s)} (I - P_N) \widetilde{\mu} d\beta_j(s),
\end{aligned} \tag{20}
$$

where $e^{-\mathcal{A}t}(I - P_N)\widetilde{\mu} \in ((H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}))^d \cap H)'$ is given by

$$
\begin{aligned}
(e^{-\mathcal{A}t}(I - P_N)\widetilde{\mu})(\psi) = \sum_{k=1}^{N} \mu_k (I - P_N^*) e^{-\mathcal{A}_2^* t} \psi(\xi_k) \\
\forall \psi \in (H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}))^d \cap H.
\end{aligned} \tag{21}
$$

Now, in virtue of (6), system (18) can be rewritten as

$$
\begin{aligned}
&dy_i + \lambda_i y_i dt = \eta \sum_{j=1}^{N} y_j \zeta_i d\beta_j, \ i = 1, ..., N, \\
&y_i(0) = y_i^0 = \langle P_N x, \varphi_i^* \rangle,
\end{aligned} \tag{22}
$$

where

$$
\zeta_i = \sum_{k=1}^{M} \mu_k \varphi_i^*(\xi_k), \ i = 1, ..., N.
$$

We set $z_i = e^{\widetilde{\gamma} t} y_i$, where $\widetilde{\gamma} = \gamma + \varepsilon$ is such that

$$
\operatorname{Re} \lambda_j > \gamma + \varepsilon \text{ for } j > N
$$

and rewrite (22) as

$$dz_i + (\lambda_j - \widetilde{\gamma})z_i\,dt = \eta\zeta_i \sum_{j=1}^{N} z_j d\beta_j$$
$$z_i(0) = y_i^0.$$

We apply Ito's formula and obtain

$$\begin{aligned}
&\frac{1}{2}\,d|z_i|^2 + (\operatorname{Re}\lambda_j - \widetilde{\gamma})|z_i|^2 dt = \frac{1}{2}\,\eta^2|\zeta_i|^2 \sum_{j=1}^{N} |z_j|^2 dt \\
&+\eta \sum_{j=1}^{N} (\operatorname{Re}(\zeta_i z_i)\operatorname{Re} z_j + \operatorname{Im}(\zeta_i z_i)\operatorname{Im} z_j)d\beta_j^1 \qquad (23)\\
&+\eta \sum_{j=1}^{N} (\operatorname{Re}(\zeta_i z_i)\operatorname{Im} z_j - \operatorname{Im}(\zeta_i z_i)\operatorname{Re} z_j)d\beta_j^2,\ i = 1, ..., N.
\end{aligned}$$

Now, we apply in (23) Ito's formula to the function $\varphi(r) = r^\delta$, $r \in (0, \infty)$, $0 < \delta < \frac{1}{2}$. (This function is not of class C^2 but, arguing as in Barbu [8], i.e., replacing φ b y $\varphi_\varepsilon(r) = (r^2 + \varepsilon)^{\frac{\delta}{2}}$, $\varepsilon > 0$, and letting ε tend to zero, the argument below can be made rigorous.) We have

$$\varphi'(r) = \delta r^{\delta - 1},\ \varphi''(r) = \delta(\delta - 1)r^{(\delta - 1)}$$

and so (23) yields via Ito's formula

$$\begin{aligned}
&d|z_i|^{2\delta} + 2\delta(\operatorname{Re}\lambda_i - \widetilde{\gamma})|z_i|^{2\delta} dt \\
&= (2\delta - 1)\delta\eta^2|\zeta_i|^2|z_i|^{2(\delta-1)} \sum_{j=1}^{N} |z_j|^2 dt \qquad (24)\\
&+2\delta|z_i|^{2(\delta-1)}\operatorname{Re}\left(\sum_{j=1}^{N} (\zeta_i z_i \bar{z}_j)d\beta_j\right),\ \mathbb{P}\text{-a.s.,} \\
&\qquad i = 1, ..., N.
\end{aligned}$$

We set

$$|z|^{2\delta} = \sum_{j=1}^{N} |z_i|^{2\delta}.$$

Then (24) yields

$$(25) \qquad |z(t)|^{2\delta} + \int_0^t H(s)ds = |y^0|^{2\delta} + M(t), \mathbb{P}\text{-a.s.},$$

where

$$H = \delta \sum_{i=1}^{N} ((1-2\delta)\eta^2 |\zeta_i|^2 |z_i|^{2(\delta-1)} |z|^2 - 2(\operatorname{Re}\lambda_i - \gamma)|z_i|^{2\delta})$$

$$M(t) = 2\delta \operatorname{Re} \int_0^t \sum_{i=1}^{N} \sum_{j=1}^{N} |z_i|^{2(\delta-1)} (\zeta_i z_i \bar{z}_j) d\beta_j.$$

By assumption (8) we have that, for $0 < \delta < \frac{1}{2}$ and $|\eta|$ sufficiently large,

$$(26) \qquad H(t) \geq \rho |z(t)|^{2\delta}, \ \mathbb{P}\text{-a.s.}, \ \forall t \geq 0,$$

where $\rho > 0$.

We note that M is a local martingale, while $t \to \int_0^t H(s)ds$ is an increasing process and $t \to |z(t)|^{2\delta}$ is a semimartingale.

Then, by the martingale convergence theorem (see, for instance, Lemma 2.1 in Barbu [8]), it follows by (25) and (26) that

$$\lim_{t\to\infty} |z(t)|^{2\delta} < \infty, \ \mathbb{P}\text{-a.s.}$$

and

$$\int_0^\infty E|z(t)|^{2\delta} dt < \infty.$$

Hence,

$$(27) \qquad \lim_{t\to\infty} |z(t)| = \lim_{t\to\infty} |y^1(t)| e^{\widetilde{\gamma} t} = 0, \ \mathbb{P}\text{-a.s.}$$

and

$$\int_0^\infty e^{2\widetilde{\gamma}t}|y^1(t)|^2dt < \infty,\ \mathbb{P}\text{-a.s.} \tag{28}$$

Now, we come back to system (19). It can be equivalently written as

$$d(y^2e^{\gamma t}) + (\mathcal{A}_2 - \gamma)(y^2e^{\gamma t})dt = \eta(I - P_N)\widetilde{\eta}\sum_{j=1}^{N} e^{\gamma t}y_j d\beta_j(t).$$

Then, for each $\psi \in D(A) = (H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}))^d \cap H$, we have (see (20))

$$\begin{aligned}\left\langle y^2(t), \psi\right\rangle e^{\gamma t} &= e^{-(\mathcal{A}_2-\gamma)t}\left\langle (I-P_N)x, \psi\right\rangle \\ &+\eta\sum_{j=1}^{N}\sum_{k=1}^{M}\mu_k\int_0^t e^{\gamma s}y_j(s)(I-P_N^*)e^{-(\mathcal{A}_2^*-\gamma)(t-s)}\psi(\xi_k)d\beta_j(s).\end{aligned}$$

Since, as seen earlier, we have for $\widetilde{\gamma} = \gamma + \varepsilon$,

$$\|e^{-\mathcal{A}_2t}(I - P_N)\|_{L(\widetilde{H},\widetilde{H})} \le Ce^{-\widetilde{\gamma}t},\quad \forall t \ge 0, \tag{29}$$

it remains to estimate the integral term

$$Z(t) = \eta\sum_{j=1}^{N}\sum_{k=1}^{M}\mu_k\int_0^t e^{\gamma s}y_j(s)(I-P_N^*)e^{-(\mathcal{A}_2^*-\gamma)(t-s)}\psi(\xi_k)d\beta_j(s)$$
$$\forall \psi \in (H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}))^d \cap H.$$

Let $z(t)$ be the solution to the stochastic differential equation

$$\begin{aligned}&dz(t) + (\mathcal{A}_2^* - \gamma)z(t) = \eta A\psi\sum_{j=1}^{N} e^{\gamma t}y_j(t)d\beta_j(t),\\ &z(0) = 0.\end{aligned} \tag{30}$$

Then we have

$$Z(t)=\sum_{k=1}^{M}\mu_k A^{-1}z(t)(\xi_k),\ \mathbb{P}\text{-a.s.},\ t>0. \tag{31}$$

Since, by (29), $e^{-(\mathcal{A}_2^*-\gamma)t}$ is exponentially stable in X_2^*, it follows by Lyapunov's theorem that there is a self-adjoint, continuous and positive definite operator Q on X_2^* (i.e., $\langle Qz,z\rangle>0$, $\forall z\neq 0$) such that

$$\operatorname{Re}\langle(\mathcal{A}_2^*-\gamma)z,Qz\rangle=\frac{1}{2}\,|z|^2,\ \forall z\in X_2^*. \tag{32}$$

Then, applying in (30) Ito's formula to the function $z\to\frac{1}{2}\,\langle Qz,z\rangle$ we obtain by (32) that

$$\begin{aligned}
&\frac{1}{2}\,\langle Qz(t),z(t)\rangle+\frac{1}{2}\int_0^t|z(s)|^2ds=\frac{1}{2}\,\langle Qz(0),z(0)\rangle+\\
&+\frac{1}{2}\,\eta^2\sum_{j=1}^{N}\int_0^t e^{2\gamma s}|y_j(s)|^2\,\langle Q\psi,z(s)\rangle\,ds\\
&+\eta\sum_{j=1}^{N}\operatorname{Re}\int_0^t e^{\gamma s}y_j(s)\,\langle Q\psi,z(s)\rangle\,d\beta_j(s).
\end{aligned} \tag{33}$$

By (28) and (32), it follows once again by Lemma 3.1 in Barbu [8] that there exists

$$\lim_{t\to\infty}\langle Qz(t),z(t)\rangle<\infty,\ \mathbb{P}\text{-a.s.}\quad\text{and}\quad\int_0^\infty|z(s)|^2ds<\infty,\ \mathbb{P}\text{-a.s.}$$

Hence $\lim_{t\to\infty}|z(t)|^2=0$, $\mathbb{P}$-a.s. Now, taking into account that $|A^{-1}z(t)|\le C|z(t)|_{C(\overline{Q})}$, $\forall t\ge 0$, where $C(\overline{\mathcal{O}})$ is the space of continuous functions on $\overline{\mathcal{O}}$, we infer by (31) that

$$\lim_{t\to\infty}|Z(t)|=\lim_{t\to\infty}(|\,\langle y(t),\psi\rangle\,|e^{\gamma t})=0,\ \mathbb{P}\text{-a.s.}$$

and this implies (11), as claimed.

3.2 Proof of Theorem 2

It is identical with that of Theorem 1 and so we omit it. We note only that, in this case, system (22) reduces to

$$dy_i + \lambda_i y_i\, dt = \eta\mu(\varphi_i^*) \sum_{j=1}^{N} y_j\, d\beta_j,\ \ i = 1, ..., N,$$

and so the proof continues on the same lines.

4 The robustness of the noise feedback controller

We shall show here that the feedback controller (9) is robust to small structural perturbation of system (3). Indeed, if condition (8) is hold, then it still remains true for small perturbations of system (3) and, more precisely, of its eigenfunctions system. In fact, by the spectral stability of Stokes–Oseen operator $\mathcal{A}$ (Kato [22], p. 240), a small variation of magnitude ε in X_e will lead to a new eigenfunction system $\{\varphi_{j,\varepsilon}^*\}_{j=1}^N$ for which we still have

$$\sup_{0\le\varepsilon\le\varepsilon_0} \left| \sum_{k=1}^{M} \mu_k \varphi_{i,\varepsilon}^*(\xi_k) \right| > \rho > 0,\ \ \forall i = 1, ..., N. \tag{34}$$

This implies that at the level of unstable modes the system has gain stability margin independent of ε. In other words, the solution $y^1 = y_\varepsilon^1$ to corresponding system (18) satisfies (28) uniformly in ε and so (30), (32) imply that

$$\lim_{t\to\infty} \langle y_\varepsilon(t), \psi\rangle\, e^{\gamma t} = 0,\ \ \mathbb{P}\text{-a.s.},\quad \forall \psi \in (H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}))^d \cap H,$$

uniformly for $0 < \varepsilon < \varepsilon_0$. (Here, y_ε is the solution to perturbed system (10).)

The main conclusion from this brief analysis is that the noise controller has a robust stabilizing effect which is not always the case with the deterministic stabilizing feedback controllers of the form (9) (Barbu [8]).

Remark 1 In absence of assumption (H1), Theorem 1 (as well as Theorem 2) remains true for a feedback controller u of the form

$$u(t) = \eta \sum_{j=1}^{N} \langle y(t), \phi_j \rangle \, \dot{\beta}_j(t) \sum_{k=1}^{M} \delta(\xi_k). \tag{35}$$

Here, $\{\phi_j\}_{j=1}^{N}$ is obtained by $\{\varphi_j\}$ by Schmidt's orthogonalization algorithm.

Then, $X_1 = \text{lin span}\{\phi_j\}_{j=1}^{N}$ and, if one assumes that

$$\left| \sum_{j=1}^{N} \mu_k \phi_i(\xi_k) \right| < 0, \quad \forall i = 1, ..., N, \tag{36}$$

then it follows by the same argument that u is weakly stabilizable in the sense of Theorem 1. The details are omitted.

Remark 2 In terms of state X to the linearized system associated with (2), the stabilizing feedback controller (5) has the form

$$u(t) = \eta \sum_{j=1}^{N} \left\langle X(t) - X_e, \varphi_j^* \right\rangle \dot{\beta}_j(t) \sum_{k=1}^{M} \mu_k \delta(\xi_k).$$

Of course, taking the real and imaginary parts of X, we can represent this controller in real terms. As a matter of fact, if all λ_j with $j = 1, ..., N$ are real valued, then the controller (9) is real.

5 Stabilization of a plane periodic channel flow by noise wall normal controllers

Consider a laminar flow in a two-dimensional channel with the walls located at $y = 0, 1$. We shall assume that the velocity field $(u(t, x, y), v(t, x, y))$ and the pressure $p(t, x, y)$ are 2π periodic in $x \in (-\infty, \infty)$. (See, e.g., [27] for a discussion on periodic flows.)

The dynamic of flow is governed by the incompressible $2 - D$ Navier-Stokes equation

$$
\begin{aligned}
& u_t - \nu\Delta u + uu_x + vu_y = p_x,\ \ x \in R,\ \ y \in (0,1) \\
& v_t - \nu\Delta v + uv_x + vv_y = p_y,\ \ x \in R,\ \ y \in (0,1) \\
& u_x + v_y = 0 \\
& u(t,x,0) = u(t,x,1) = 0, \\
& v(t,x,0) = 0,\ \ v(t,x,1) = v^*,\quad \forall x \in R \\
& u(t,x+2\pi,y) \equiv u(t,x,y), \\
& v(t,x+2\pi,y) \equiv v(t,x,y),\quad y \in (0,1).
\end{aligned} \tag{37}
$$

Consider a steady-state flow governed by (37) with zero vertical velocity component, i.e., $(U(x, y), 0)$. (This is a stationary flow sustained by a pressure gradient in the x direction.) Since the flow is freely divergent, we have $U_x \equiv 0$ and so $U(x, y) \equiv U(y)$. Alternatively, substituting into (37) gives

$$-\nu U''(y) = p_x(x,y),\ \ p_y(x,y) \equiv 0.$$

Hence $p \equiv p(x)$ and $U''' \equiv 0$, i.e., $U(y) = C(y^2 - y)$, $\forall y \in (0, 1)$, where $C < 0$. In the following, we take $C = -\frac{a}{2\nu}$, where $a \in R^+$.

We recall that the stability property of the stationary flow $(U, 0)$ varies with the Reynolds number $\frac{1}{\nu}$; there is $\nu_0 > 0$ such that for $\nu > \nu_0$ the flow is stable while for $\nu < \nu_0$ it is unstable.

Our aim here is the stabilization of this flow profile by a boundary controller $v(t, x, 1) = v^*(t, x)$, $t \geq 0$, $x \in R$, that is, only the normal velocity v is controlled on the wall $y = 1$.

The linearization of (37) around steady-state flow $(U(y), 0)$ leads to the following system

$$
\begin{aligned}
&u_t - \nu\Delta u + u_x U + vU' = p_x,\ y \in (0,1),\ x,t \in R,\\
&v_t - \nu\Delta v + v_x U = p_y,\\
&u_x + v_y = 0,\ u(t,x,0) = u(t,x,1) = 0,\\
&v(t,x,0) = 0,\ v(t,x,1) = v^*(t,x),\\
&u(t,x+2\pi,y) \equiv u(t,x,y),\\
&v(t,x+2\pi,y) \equiv v(t,x,y).
\end{aligned} \tag{38}
$$

Here the actuator v^* is a normal velocity boundary controller on the wall $y = 1$. However, there is no actuation in $x = 0$ for steamwise or inside the channel.

The main result here (see Theorem 5 below) is that the exponential stability with probability 1 can be achieved using a finite number M of Fourier modes and a stochastic feedback controller

$$
\begin{aligned}
v^*(t,x) &= \sum_{|k|\le M} v_k^*(t)e^{ikx},\quad t \ge 0,\ x \in R,\\
v_k^*(t) &= -\eta \sum_{j=1}^{N} \left(\int_0^{2\pi}\int_0^1 (v_{yy}(t,x,y)\right.\\
&\qquad \left. - k^2 v(t,x,y))e^{-ikx}(\varphi_j^k)^*(y)dx\,dy \right) \dot{\beta}_j.
\end{aligned} \tag{39}
$$

Here $\{(\varphi_j^k)^*\}_{j=1}^N$ is a system of functions in $L^2(0,1)$ defined in formula (59) below and $\beta_j(t) = \beta_j^1(t) + i\beta_j^2(t)$ are independent complex Brownian motions in a probability space $\{\Omega, \mathbb{P}, \mathcal{F}, \mathcal{F}_t\}$. The feedback stabilization of Navier-Stokes equation using noise controllers was already obtained by the author in [8] with internal and tangential boundary controllers. (For other literature on stochastic stabilization, we refer to [3, 16, 18, 23].)

The boundary stabilization of Navier-Stokes equations with deterministic tangential feedback controllers was studied in [9, 19, 20, 24, 25]. However there are few results on boundary stabilization with normal controllers and almost all refer to periodic flows in $2-D$ channels ([1, 4, 7, 28, 29, 30]). Most of these stabilization results are established for small Reynold number with exception of [7, 28] and [29], where the tangential stabilizable feedback controller is constructed without any apriori condition on ν. It should be said, however, that the present stochastic approach is completely new into this context and provides a simple stabilizable normal controller of the form (39) which is easy to implement into the system.

5.1 The Fourier functional setting

Let $L^2_\pi(Q)$, $Q = (0, 2\pi) \times (0, 1)$ be the space of all functions $u \in L^2_{\text{loc}}(R \times (0, 1))$ which are 2π-periodic in x. These functions are characterized by their Fourier series

$$\begin{aligned} & u(x, y) = \sum_k a_k(y) e^{ikx}, \; a_k = \bar{a}_{-k}, \; a_0 = 0, \\ & \sum_k \int_0^1 |a_k|^2 dy < \infty. \end{aligned}$$

We set

$$\begin{aligned} H_\pi = \{(u, v) \in (L^2_\pi(Q))^2; \; & u_x + v_y = 0, \\ & v(x, 0) = v(x, 1) = 0\}. \end{aligned}$$

(If $u_x + v_y = 0$, then the trace of (u, v) at $y = 0, 1$ is well defined as an element of $H^{-1}(0, 2\pi) \times H^{-1}(0, 2\pi)$ (see, e.g., [27])).

We have

$$H_\pi = \left\{ u = \sum_{k \neq 0} u_k(y) e^{ikx}, \right.$$
$$v = \sum_{k \neq 0} v_k(y) e^{ikx},\ v_k(0) = v_k(1) = 0,$$
$$\sum_{k \neq 0} \int_0^1 (|u_k|^2 + |v_k|^2) dy < \infty,$$
$$\left. iku_k(y) + v'_k(y) = 0,\ \text{a.e. } y \in (0,1),\ k \in R \right\}.$$

We now return to system (38) and rewrite it in terms of the Fourier coefficients $\{u_k\}_{k=-\infty}^{\infty}$, $\{v_k\}_{k=-\infty}^{\infty}$. We have

$$(40) \qquad \begin{aligned} & (u_k)_t - \nu u''_k + (\nu k^2 + ikU) u_k + U' v_k = ikp_k, \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. in } (0,1) \\ & (v_k)_t - \nu v''_k + (\nu k^2 + ikU) v_k = p'_k \\ & iku_k + v'_k = 0,\ \text{a.e. on } (0,1),\ k \neq 0 \\ & u_k(t,0) = u_k(t,1) = 0, \\ & v_k(t,0) = 0,\ v_k(t,1) = v_k^*(t). \end{aligned}$$

Here

$$p = \sum_{k \neq 0} p_k(t,y) e^{ikx},\ u = \sum_{k \neq 0} u_k(t,y) e^{ikx},$$
$$v = \sum_{k \neq 0} v_k(t,y) e^{ikx}.$$

This yields

$$\begin{aligned} & ik(v_k)_t - ik\nu v''_k + ik^2(\nu k + iU) v_k - (u'_k)_t + \nu u'''_k \\ & \quad -k(\nu k + iU) u'_k - ikU' u_k - U' v'_k - U'' v_k = 0. \end{aligned}$$

Taking $u_k = -\frac{1}{ik}\, v_k'$, we obtain that

$$\begin{aligned} & ik(v_k)_t - ik\nu v_k'' + ik^2(\nu k + iU)v_k + \frac{1}{ik}\,(v_k'')_t \\ & -\frac{\nu}{ik}\, v_k^{\mathrm{iv}} + \frac{1}{i}\,(\nu k + iU)v_k'' - U''v_k = 0, \\ & \qquad\qquad\qquad\qquad t \geq 0,\ y \in (0,1). \end{aligned}$$

Finally,

$$\begin{aligned} & (v_k'' - k^2 v_k)_t - \nu v_k^{\mathrm{iv}} + (2\nu k^2 + ikU)v_k'' \\ & \qquad -k(\nu k^3 + ik^2 U + iU'')v_k = 0, \\ & \qquad\qquad\qquad t \geq 0,\ y \in (0,1) \\ & v_k'(t,0) = v_k'(t,1) = 0, \\ & v_k(t,0) = 0,\ v_k(t,1) = v_k^*(t). \end{aligned} \tag{41}$$

In the following we shall denote by H the complexified space $L^2(0,1)$ with the norm $|\cdot|$ and product scalar denoted by $\langle\cdot,\cdot\rangle$. We shall denote by $H^m(0,1)$, $m = 1,2,3$, the standard Sobolev spaces on $(0,1)$ and

$$\begin{aligned} & H_0^1(0,1) = \{v \in H^1(0,1);\ v(0) = v(1) = 0\} \\ & H_0^2(0,1) = \{v \in H^2(0,1) \cap H_0^1(0,1); \\ & \qquad\qquad\qquad v'(0) = v'(1) = 0\}. \end{aligned}$$

We set $\mathcal{H}{=}H^4(0,1) \cap H_0^2(0,1)$ and denote by $\mathcal{H}'$ the dual of $\mathcal{H}$ in the pairing with pivot space H, that is $\mathcal{H}{\subset}H{\subset}\mathcal{H}'$ algebraically and topologically. Denote by $(H^2(0,1))'$ the dual of $H^2(0,1)$ and by $H^{-1}(0,1)$ the dual of $H_0^1(Q)$ with the norm denoted $\|\cdot\|_{-1}$. Denote also by $H_\pi^{-1}(Q)$ the space $L^2(0,2\pi;H^{-1}(0,1))$ with the norm $\|\cdot\|_{H_\pi^{-1}(Q)}$.

For each $k \in R$ we denote by $L_k{:}D(L_k) \subset H{\rightarrow}H$ and $F_k{:}D(F_k) \subset H{\rightarrow}H$ the operators

$$L_k v = -v'' + k^2 v,\ v \in D(L_k) = H_0^1(0,1) \cap H^2(0,1) \tag{42}$$

$$
(43) \qquad \begin{aligned} F_k v = \nu v^{\mathrm{iv}} - (2\nu k^2 + ikU)v'' + k(\nu k^3 + ik^2U + iU'')v, \\ \forall v \in D(F_k) = H^4(0,1) \cap H_0^2(0,1). \end{aligned}
$$

We set

$$
\mathcal{F}_k v = \nu v^{\mathrm{iv}} - (2\nu k^2 + ikU)v'' + k(\nu k^3 + ik^2U + iU'')v
$$

and consider the solution V_k of the equation

$$
(44) \qquad \begin{aligned} &\theta V_k + \mathcal{F}_k V_k = 0,\ y \in (0,1), \\ &V_k'(0) = V_k'(1) = 0,\ V_k(0) = 0,\ V_k(1) = v_k^*(t). \end{aligned}
$$

(As easily seen, for θ positive and sufficiently large, there is a unique solution V_k to (44).) Then, subtracting (41) and (44), we obtain that

$$
(L_k v_k)_t + F_k(v_k - V_k) - \theta V_k = 0,\ t \geq 0.
$$

Equivalently,

$$
(45) \qquad \begin{aligned} (L_k(v_k - V_k))_t + F_k(v_k - V_k) = \theta V_k - (L_k V_k)_t, \\ v_k - V_k \in D(F_k). \end{aligned}
$$

(The meaning of $L_k V_k$, which is a distribution on $(0,1)$, will be explained later on.)

In order to represent equation (45) as an abstract boundary control system, we consider the operator $\mathcal{A}_k : D(\mathcal{A}_k) \subset H \to H$ defined by

$$
(46) \qquad \begin{aligned} &\mathcal{A}_k = F_k L_k^{-1}, \\ &D(\mathcal{A}_k) = \{u \in H;\ L_k^{-1} u \in D(F_k)\}. \end{aligned}
$$

We have

Lemma 3 *The operator $-\mathcal{A}_k$ generates a C_0-analytic semigroup on H and for each $\lambda \in \rho(-\mathcal{A}_k)$ (the resolvent set of $-\mathcal{A}_k$), $(\lambda I + \mathcal{A}_k)^{-1}$ is compact. Moreover, one has for each $\gamma > 0$*

$$\begin{aligned} &\sigma(-\mathcal{A}_k) \subset \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda \leq -\gamma\}, \\ &\forall |k| \geq M = \frac{1}{\sqrt{2\nu}}\left(\gamma + 1 + \frac{a}{\sqrt{2\nu}}\right)^{\frac{1}{2}}, \end{aligned} \tag{47}$$

where $\sigma(-\mathcal{A}_k)$ is the spectrum of $-\mathcal{A}_k$.

Proof. For $\lambda \in \mathbb{C}$ and $f \in H = L^2(0,1)$ consider the equation

$$\lambda u + \mathcal{A}_k u = f$$

or, equivalently,

$$\lambda L_k v + F_k v = f. \tag{48}$$

Taking into account (42), (43) yields

$$\begin{aligned} &\operatorname{Re}\lambda \int_0^1 (|v'|^2 + k^2|v|^2)dy + \nu \int_0^1 |v''|^2 dy \\ &+2\nu k^2 \int_0^1 |v'|^2 dy + k^4\nu \int_0^1 |v|^2 dy \\ &+k\int_0^1 U'(\operatorname{Re} v' \operatorname{Im} v - \operatorname{Im} v' \operatorname{Re} v)dy \\ &= \operatorname{Re}\langle f, v\rangle \end{aligned} \tag{49}$$

$$\begin{aligned} &\operatorname{Im}\lambda \int_0^1 (|v'|^2 + k^2|v|^2)dy + k \int_0^1 |v'|^2 dy \\ &+k \int_0^1 \left(k^2 U + \frac{1}{2}U''\right)|v|^2 = \operatorname{Im}\langle f, v\rangle. \end{aligned} \tag{50}$$

Taking into account that $\|u\|_{L^2(0,1)} \leq C\|u\|_{H_0^1(0,1)}$, we see by (49), (50) that, for some $a > 0$,

$$|(\lambda I + \mathcal{A}_k)^{-1} f| \leq \frac{C}{|\lambda| - a}\, |f| \text{ for } |\lambda| > a,$$

which implies that $-\mathcal{A}_k$ is infinitesimal generator of C_0-analytic semigroup, $e^{-\mathcal{A}_k t}$ on H.

Moreover, by (49), (50) see that $(\lambda I + \mathcal{A}_k)^{-1}$ is compact in H and it follows also that all the eigenvalues λ of $-\mathcal{A}_k$ satisfy the estimate

$$\begin{aligned}
&\operatorname{Re}\lambda \int_0^1 (|v_k'|^2 + k^2|v_k|^2)dy + 2\nu k^2 \int_0^1 |v_k'|^2 dy \\
&\quad +\nu \int_0^1 |v_k''|^2 dy + \nu k^4 \int_0^1 |v_k|^2 dy \\
&\quad \leq -k \int_0^1 U'(\operatorname{Re} v_k' \operatorname{Im} v_k - \operatorname{Im} v_k' \operatorname{Re} v_k) dy \\
&\quad \leq 2\nu k^2 \int_0^1 |v_k'|^2 dy + \frac{1}{2\nu} \int_0^1 |U'|^2 |v_k|^2 dy \\
&\quad \leq 2\nu k^2 \int_0^1 |v_k'|^2 dy + \frac{a^2}{8\nu^3} \int_0^1 |v_k|^2 dy \\
&\mathcal{A}_k v_k = -\lambda v_k.
\end{aligned}$$

Let $\gamma > 0$ be arbitrary but fixed. Then, by the above estimate we see that

$$\operatorname{Re}\lambda \leq -\gamma \;\text{ if } |k| \geq \frac{1}{\sqrt{2\nu}} \left(\gamma + 1 + \frac{a}{\sqrt{2}\,\nu}\right)^{\frac{1}{2}}.$$

This implies (47), as claimed. ■

In particular, it follows by Lemma 3 that, for $|k| \geq M$, we have

$$\|e^{-\mathcal{A}_k t}\|_{L(H,H)} \leq Ce^{-\gamma t}, \quad \forall t \geq 0.$$

More precisely, we have by (41) that

$$\begin{aligned}
&\frac{1}{2}\ \frac{d}{dt}\ \left(|v_k'(t)|^2_{L^2(0,1)} + k^2|v_k(t)|^2_{L^2(0,1)}\right)\\
&+\nu\left(|v_k''(t)|^2_{L^2(0,1)} + k^2|v_k'(t)|^2_{L^2(0,1)}\right)\\
&+\nu k^4|v_k(t)|^2_{L^2(0,1)} = kUI_m\left(\int_0^1 v_k(t)v_k''(t)dt\right)
\end{aligned}$$

and this yields

$$\begin{aligned}
&\int_0^1 (|v_k'(t,y)|^2 + k^2|v_k(t,y)|^2)dy\\
&\leq Ce^{-\nu k^2 t}\int_0^1 (|v_k'(0,y)|^2 + k^2|v_k(0,y)|^2)dy,\ t \geq 0,
\end{aligned} \tag{51}$$

for $|k| \geq M$. This implies that it suffices to stabilize system (40) (equivalently (41)) for $|k| \leq M$ only.

Now, coming back to system (45), we set

$$\tilde{z}_k(t) = L_k(v_k(t) - V_k(t)) \tag{52}$$

and write it as

$$\begin{aligned}
\tilde{z}_k(t) &= e^{-\mathcal{A}_k t}\tilde{z}_k(0)\\
&\quad + \int_0^t e^{-\mathcal{A}_k(t-s)}(\theta V_k(s) - (L_k V_k(s))_s)ds\\
&= e^{-\mathcal{A}_k t}z_k(0) - L_k V_k(t) + e^{-\mathcal{A}_k t}L_k V_k(0)\\
&\quad + \int_0^t e^{-\mathcal{A}_k(t-s)}(\theta V_k(s) + \widetilde{F}_k V_k(s))ds,
\end{aligned} \tag{53}$$

where $\widetilde{F}_k : H \to \mathcal{H}'$ is the extension of F_k to all of H defined by

$${}_{\mathcal{H}'}\left\langle \widetilde{F}_k v, \psi \right\rangle_{\mathcal{H}} = \int_0^1 v(y)F_k^*\psi(y)dy, \forall \psi \in D(F_k^*).$$

Here, F_k^* is the dual of F_k, that is,

$$F_k^* = \nu\psi^{\rm iv} - ((\nu k^2 - ik) \cup \psi)'' + (k - ik^2U - iU'')\psi,$$
$$D(F_k^*) = H^4(0,1) \cap H_0^2(0,1).$$

Define similarly $\widetilde{\mathcal{A}}_k$, the extension of $\mathcal{A}_k$ from H to $(D(\mathcal{A}_k^*))'$. Likewise $\mathcal{A}_k$, the operator $\widetilde{\mathcal{A}}_k$ generates a C_0-analytic semigroup on $(D(\mathcal{A}_k))' = (H^2(0,1))'$.

In the same way is defined the extension of L_k given by (42) to an operator from H to $(H_0^1(0,1) \cap H^2(0,1))'$ again denoted L_k.

Then, (53) can be rewritten as

$$\frac{d}{dt}\, z_k(t) + \widetilde{\mathcal{A}}_k z_k(t) = (\theta + \widetilde{F}_k)V_k(t),\ t \geq 0. \tag{54}$$

For each $u \in R$, we denote by $V = D_k u \in H^4(0,1)$ the solution to the equation (see (44))

$$\begin{aligned} &\theta V + \mathcal{F}_k V = 0,\ \forall y \in (0,1) \\ &V'(0) = V'(1) = 0,\ V(0) = 0,\ V(1) = u. \end{aligned} \tag{55}$$

The operator D_k is called the Dirichlet map associated with $\theta + \mathcal{F}_k$ and it is easily seen that the dual $((\theta + \widetilde{F}_k)D_k)^*$ is given by

$$((\theta + \widetilde{F}_k)D_k)^*\varphi = \nu\varphi'''(1),\ \forall \varphi \in D(F_k). \tag{56}$$

We note also that, in virtue of (44), we have $V_k = D_k v_k^*(t)$ and so (54) can be rewritten as

$$\frac{d}{dt}\, z_k(t) + \widetilde{\mathcal{A}}_k z_k(t) = (\theta + \widetilde{F}_k)D_k v_k^*(t),\ \forall t \geq 0, \tag{57}$$

where z_k is given by (52).

5.2 Feedback stabilization

Let $\gamma > 0$ and let $k \in R$, $|k| \leq M$ given by (47). Then, the operator $-\mathcal{A}_k$ has a finite number $N = N_k$ of the eigenvalues $\lambda_j = \lambda_j^k$ with $\operatorname{Re} \lambda_j \geq -\gamma$. (In the following we shall omit the index k from $\mathcal{A}_k$ and λ_j^k.)

We shall denote by $\{\varphi_j^k\}_{j=1}^N$ the corresponding eigenfunctions and repeat each λ_j^k according to its algebraic multiplicity m_j. We have

$$\mathcal{A}_k \varphi_j^k = -\lambda_j^k \varphi_j^k, \quad j = 1, ..., N \tag{58}$$

and recall that the geometric multiplicity of λ_j^k is the dimension of eigenfunction space corresponding to λ_j^k. The algebraic multiplicity of λ_j^k is the dimension of the range of the projection operator

$$P_j = -\frac{1}{2\pi i} \int_{\Gamma_j} (\lambda I + \mathcal{A}_k)^{-1} d\lambda,$$

where Γ_j is a smooth closed curve encircling λ_j^k. In general, the algebraic multiplicity m_j^a is greater than the geometric multiplicity m_j^g and, if $m_j^a = m_j^g$, the eigenvalue λ_j^k is called semi-simple (see [22]). In the following, we shall often omit the index k from λ_j^k and φ_j^k, respectively. In general, (58) holds in the generalized sense, i.e., $(\mathcal{A}_k - \lambda_j)^\ell \varphi_j = 0$ for $\ell = 1, 2, ..., m_j^a$. Here we shall assume that the following assumption holds.

(A$_1$) *All the eigenvalues $\lambda_j = \lambda_j^k$ with $1 \leq j \leq N$ are semi-simple.*

Such a condition can be checked in each case taking into account that λ_j are eigenvalues λ of the boundary value problem

$$\begin{aligned} &\lambda(-v'' + k^2 v) + \nu v^{\text{iv}} - (2\nu k^2 + ikU)v'' \\ &\qquad + k(\nu k^3 + ik^2 U + iU'')v = 0, \ \ y \in (0,1), \\ &v(0) = v(1) = 0, \ \ v'(0) = v'(1) = 0, \end{aligned}$$

for $|k| \leq M$.

As we shall see in the following this assumption is not absolutely necessary for the construction of the stabilizing controller, but it simplifies however the argument.

If we denote by $(\varphi_j^k)^*$ the eigenfunction to the dual operator $-\mathcal{A}_k^*$, i.e.,

$$\mathcal{A}_k^*(\varphi_j^k)^* = -\overline{\lambda}_j^k(\varphi_j^k)^*, \quad j = 1, ..., N, \tag{59}$$

it follows by assumption (A_1) that the system $(\varphi_j^k)^*$ can be chosen in such a way that

$$\left\langle \varphi_\ell^k, (\varphi_j^k)^* \right\rangle = \delta_{\ell j}, \quad \ell, j = 1, ..., N. \tag{60}$$

We denote by X_N^u the space generated by $\{\varphi_j^k\}_{j=1}^N$ and X_N^s that generated by $\{\varphi_j^k\}_{j=N+1}^\infty$.

We have $H = X_N^u \oplus X_N^s$ and denote by P_N the (algebraic) projection of H onto X_N^u.

We consider in (57) (equivalently (54)) the feedback controller

$$v_k^*(t) = \eta \sum_{\ell=1}^N \left\langle L_k v_k(t), (\varphi_j^k)^* \right\rangle \dot{\beta}_j(t), \ t \geq 0, \tag{61}$$

where $\{\beta_j\}_{j=1}^N$ are independent complex Brownian and $\dot{\beta}_j$ is the white noise associated with β_j. More precisely, we take $\beta_j = \beta_j^1 + i\beta_j^2$, where $\{\beta_j^\ell\}_{j=1}^N$ are independent real Brownian motions.

Then we are lead to the stochastic closed loop system

$$\begin{aligned}
&d(L_k v_k(t)) + \widetilde{\mathcal{A}}_k(L_k v_k(t))dt \\
&\quad = \eta \sum_{j=1}^N (\theta + \widetilde{F}_k) D_k \left\langle L_k v_k(t), (\varphi_j^k)^* \right\rangle d\beta_j \\
&(L_k v_k)(0) = L_k v_k^0,
\end{aligned} \tag{62}$$

which represents the abstract and rigorous formulation of the boundary closed loop stochastic system (see (41))

$$
\begin{aligned}
&d(L_k v_k(t)) + \mathcal{F}_k v_k(t)dt = 0,\ t \geq 0\\
&v_k(t,1) = \eta \sum_{j=1}^{N} \left\langle L_k v_k(t), (\varphi_j^k)^* \right\rangle \dot{\beta}_j(t),\ |k| \leq M,\\
&v_k(t,1) = 0 \text{ for } |k| > M.
\end{aligned} \tag{63}
$$

The feedback controller (61) can be equivalently expressed in term of normal velocity v as (see (39))

$$
\begin{aligned}
v_k(t,1) = -\eta \sum_{j=1}^{N} \bigg(\int_0^{2\pi}\!\!\int_0^1 (v_{yy}(t,x,y) - k^2 v(t,x,y)) e^{-ikx}&\\
(\varphi_j^k)^*(y) dx\, dy \bigg) \dot{\beta}_j(t),\ |k| \leq M&\\
v_k(t,1) = 0 \ \text{ for } |k| > M.&
\end{aligned} \tag{64}
$$

Equation (62) should be viewed in the following mild sense,

$$
\begin{aligned}
v_k(t) &= L_k^{-1} e^{-\mathcal{A}_k t}(L_k v_k^0)\\
&+\eta \sum_{j=1}^{N} L_k^{-1} \int_0^t e^{-\mathcal{A}_k(t-s)}(\widetilde{F}_k + \theta) D_k \left\langle L_k v_k(s), (\varphi_j^k)^* \right\rangle d\beta_j(s)
\end{aligned}
$$

and has a unique solution $v_k \in C([0,\infty); L^2(\Omega), H_0^1(0,1))$ which is adapted to the filtration $\mathcal{F}_t$ (see, e.g., [17], p. 244).

We have

Theorem 4 *For $|k| \leq M$ and $|\eta|$ sufficiently large, we have*

$$
\begin{aligned}
&\mathbb{P}\left[\lim_{t \to \infty} e^{\gamma t} \|v_k(t)\|_{-1} = 0 \right] = 1,\\
&E \int_0^\infty e^{2\gamma\delta t} |v_k(t)|_{-1}^2 dt \leq C |v_k(0)|_{-1}^2,
\end{aligned} \tag{65}
$$

where $0 < \delta < \frac{1}{2}$ and E is the expectation.

Taking into account (40) and (51), which imply the exponential decay of v_k and u_k as $k \to \infty$, we obtain by Theorem 4 the exponential stabilization of system (38) with the feedback controller (64).

Theorem 5 *Under assumptions of Theorem 4, the solution*

$$
\begin{aligned}
& u(t,x,y) = \sum_{k\neq 0} u_k(t,y)e^{ikx}, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in R,\ y \in (0,1) \\
& v(t,x,y) = \sum_{k\neq 0} v_k(t,y)e^{ikx},
\end{aligned}
\tag{66}
$$

to equation (38) *with the boundary feedback controller* (64) *is exponentially stable with probability one. Namely, one has*

$$
\begin{aligned}
& \mathbb{P}\left[\lim_{t\to\infty} e^{\gamma t}(\|u(t)\|_{H_\pi^{-1}(Q)} + \|v(t)\|_{H_\pi^{-1}(Q)}) = 0\right] = 1, \\
& E\int_0^\infty e^{2\gamma\delta t}(\|u(t)\|^{2\delta}_{H_\pi^{-1}(Q)} + \|v(t)\|^{2\delta}_{H_\pi^{-1}(Q)})dt \\
& \qquad\qquad\qquad \leq C(\|u(0)\|^{2\delta}_{H_\pi^{-1}(Q)} + \|v(0)\|^{2\delta}_{H_\pi^{-1}(Q)}).
\end{aligned}
\tag{67}
$$

5.3 Proof of Theorem 4

We shall proceed as in [8].

Namely, we set $y = L_k v_k$ and represent it as

$$y = P_N y + (I - P_N)y,\ P_N y = \sum_{j=1}^N y_j \varphi_j.$$

Then, in virtue of (60), system (62) can be rewritten as

$$
\begin{aligned}
& dy_\ell + \lambda_\ell y_\ell dt = \eta((\widetilde{F}_k + \theta)D_k)^*(\varphi_\ell^k)^* \sum_{j=1}^N y_j d\beta_j, \\
& y_\ell(0) = \left\langle P_N L_k v_k(0), (\varphi_\ell^k)^* \right\rangle = y_\ell^0,\ \ell = 1, ..., N.
\end{aligned}
\tag{68}
$$

and

$$
\begin{aligned}
(69) \qquad & dy^s + \widetilde{\mathcal{A}}_k^s y^s dt = \eta(I - P_N) \sum_{j=1}^{N} (\widetilde{F}_k + \theta) D_k(y_j) d\beta_j, \\
& y^s(0) = (I - P_N) L_k v_k(0),
\end{aligned}
$$

where $y^s = (I - P_N)y$, $\mathcal{A}_k^s = (I - P_N)\mathcal{A}_k$ and $\widetilde{\mathcal{A}}_k^s$ is the extension of $\mathcal{A}_k^s$ to all of H. (When there is no danger of confusion, we shall omit $\sim$.) Taking into account that

$$
(70) \qquad \overline{\lambda}_\ell L_k(\varphi_\ell^k)^* + F_k^*(\varphi_\ell^k)^* = 0,\ \ \ell = 1, ..., N,
$$

and therefore $(\varphi_\ell^k)^* \in D(F_k^*) = D(F_k)$ we see by (56) that

$$
\zeta_\ell = ((\widetilde{F}_k + \theta) D_k)^* (\varphi_\ell^k)^* = \nu(((\varphi_\ell^k)^*)'''(1),\ \ \ell = 1, ..., N.
$$

Then, in virtue of (70), we have that

$$
(71) \qquad |\zeta_\ell| \geq \rho > 0, \quad \forall \ell = 1, 2, ..., N.
$$

Indeed, by (70), we see that

$$
((\varphi_\ell^k)^*)'''(1)\varphi(1) - ((\varphi_\ell^k)^*)''(1)\varphi'(1) = 0,
$$

for all the solutions $\varphi \in H^4(0,1)$ to the equation

$$
\mathcal{F}_k \varphi + \lambda(-\varphi'' + k^2\varphi) = 0, \quad \varphi(0) = \varphi'(0) = 0
$$

and taking into account that

$$
\begin{aligned}
&(\varphi_\ell^k)^*(0) = (\varphi_\ell^k)^*(1) = 0, \\
&((\varphi_\ell^k)^*)'(0) = ((\varphi_\ell^k)^*)'(1) = 0,
\end{aligned}
$$

it follows that $((\varphi_\ell^k)^*)''(1) = 0$ and, therefore, $((\varphi_\ell^k)^*)'''(1) \neq 0$ unless $(\varphi_\ell^k)^* \equiv 0$.

We rewrite (68) as

$$dy_\ell+\lambda_\ell y_\ell dt = \eta\zeta_\ell \sum_{j=1}^N y_j d\beta_j,\ \ \ell = 1, ..., N,$$
$$y_\ell(0) = y_\ell^0. \tag{72}$$

Lemma 6 *For $|\eta|$ sufficiently large, we have*

$$\lim_{t\to\infty}\left(\sum_{\ell=1}^N |y_\ell(t)|e^{\gamma t}\right) = 0,\ \ \mathbb{P}\text{-}a.s., \tag{73}$$

$$\int_0^\infty e^{2\gamma t}\sum_{\ell=1}^N |y_\ell(t)|^2 dt < \infty,\ \ \mathbb{P}\text{-}a.s., \tag{74}$$

$$E\int_0^\infty e^{2\gamma\delta t}\sum_{\ell=1}^N |y_\ell(t)|^{2\delta} dt \le C\sum_{\ell=1}^N |y_\ell(0)|^{2\delta}, \tag{75}$$

where $0 < \delta < \frac{1}{2}$.

Proof. We shall prove (73) and (75) for $\gamma = 0$ because the general case follows from this by substituting y_ℓ into (72) by $y_\ell e^{\gamma t}$ taking into account that $\mathrm{Re}\,\lambda_\ell \le -\gamma$ for $\ell = 1, ..., N$. We apply in (72) Ito's formula to the function $y \to \frac{1}{2}\,|y|^2$. We get

$$\frac{1}{2}\,d|y_\ell(t)|^2 + \mathrm{Re}\ \lambda_\ell|y_\ell(t)|^2 dt = \frac{1}{2}\eta^2|\zeta_\ell|^2\sum_{j=1}^N |y_j|^2 dt$$
$$+\eta\sum_{j=1}^N(\mathrm{Re}(\zeta_\ell y_\ell)\mathrm{Re}\,y_j + \mathrm{Im}(\zeta_\ell y_\ell)\mathrm{Im}\,y_j)d\beta_j^1$$
$$+\eta\sum_{j=1}^N(\mathrm{Re}(\zeta_\ell y_\ell)\mathrm{Im}\,y_j - \mathrm{Im}(\zeta_\ell y_\ell)\mathrm{Re}\,y_j)d\beta_j^2,\ \ \ell = 1, ..., N.$$

Equivalently,

$$
\begin{aligned}
dz_\ell(t) + 2\ \mathrm{Re}\ \lambda_\ell z_\ell(t)dt &= \eta^2|\zeta_\ell|^2 \sum_{j=1}^N z_j dt \\
&+2\eta \sum_{j=1}^N (\mathrm{Re}(\zeta_\ell y_\ell)\mathrm{Re}\, y_j + \mathrm{Im}(\zeta_\ell y_\ell)\mathrm{Im}\, y_j) d\beta_j^1 \\
&+\eta \sum_{j=1}^N (\mathrm{Re}(\zeta_\ell y_\ell)\mathrm{Im}\, y_j - \mathrm{Im}(\zeta_\ell y_\ell)\mathrm{Re}\, y_j) d\beta_j^2,
\end{aligned} \tag{76}
$$

where $z_\ell = |y_\ell|^2$.

Here, we have used the following version of Ito's formula for the complex stochastic equation $dy + \lambda y\, dt = \sum_{j=1}^N \eta_j d(\beta_j^1 + i\beta_j^2)$. Namely,

$$
\begin{aligned}
\frac{1}{2} d|y(t)|^2 + \mathrm{Re}\,\lambda|y(t)|^2 &= \frac{1}{2}\sum_{j=1}^N |\eta_j(t)|^2 dt \\
+\sum_{j=1}^N (\mathrm{Re}\, y(t)\mathrm{Re}\,\eta_j(t) &+ \mathrm{Im}\, y(t)\mathrm{Im}\,\eta_j(t)) d\beta_j^1(t) \\
+\sum_{j=1}^N (\mathrm{Im}\, y(t)\mathrm{Re}\,\eta_j(t) &- \mathrm{Re}\, y(t)\mathrm{Im}\,\eta_j(t)) d\beta_j^2(t).
\end{aligned}
$$

In (76) we apply once again Ito's formula to the function $\psi(z) = z^\delta$, where $0 < \delta < \frac{1}{2}$. We have $\psi'(z) = \delta z^{\delta-1},\ \psi''(z) = \delta(\delta-1)z^{\delta-2}$ and so, we obtain that

$$
\begin{aligned}
dz_\ell^\delta(t) + 2\delta\,\mathrm{Re}\,\lambda_\ell z_\ell^\delta(t)dt &= \delta\eta^2|\zeta_\ell|^2 \sum_{j=1}^N z_j(t) z_\ell^{\delta-1}(t)dt \\
+\delta(\delta-1)\eta^2|\zeta_\ell|^2 &\sum_{j=1}^N z_j(t) z_\ell^{\delta-1}(t)dt \\
+2\eta\delta z_\ell^{\delta-1} &\sum_{j=1}^M (M_{j\ell}^1(t)d\beta_j^1(t) + M_{j\ell}^2(t)d\beta_j^2(t)),\ \ell=1,\dots,N,
\end{aligned} \tag{77}
$$

where

$$M^1_{j\ell} = \mathrm{Re}(\zeta_\ell y_\ell)\mathrm{Re}\, y_j + \mathrm{Im}(\zeta_\ell y_\ell)\mathrm{Im}\, y_j,$$
$$M^2_{j\ell} = \mathrm{Re}(\zeta_\ell y_\ell)\mathrm{Im}\, y_j - \mathrm{Im}(\zeta_\ell y_\ell)\mathrm{Re}\, y_j.$$

(The previous calculation is apparently formal, because ψ is not of class C^2. However, arguing as in [8], it can be made rigorous if we replace ψ by $\psi_\varepsilon(z) = (\varepsilon + z)^\delta$ and let $\varepsilon \to 0$.) Then (77) yields for all $\ell = 1, ..., N$,

$$\begin{aligned} &d|y_\ell(t)|^{2\delta} + 2\delta\,\mathrm{Re}\,\lambda_\ell |y_\ell(t)|^{2\delta}dt \\ &\quad + \delta\eta^2|\zeta_\ell|^2(1-2\delta)\sum_{j=1}^N |y_j(t)|^2|y_\ell(t)|^{2(\delta-1)}dt \\ &\quad = 2\eta\delta\sum_{j=1}^N (M^1_{j\ell}(t)d\beta^1_j(t) + M^2_{j\ell}(t)d\beta^2_j(t))|y_\ell(t)|^{2(\delta-1)}. \end{aligned}$$

Finally,

$$\begin{aligned} &d\sum_{\ell=1}^N |y_\ell(t)|^{2\delta} + 2\delta\sum_{\ell=1}^N \mathrm{Re}\,\lambda_\ell|y_\ell(t)|^{2\delta}dt \\ &\quad + \delta\eta^2(1-2\delta)\sum_{j=1}^N |y_j(t)|^2\sum_{\ell=1}^N |\zeta_\ell|^2|y_\ell(t)|^{2(\delta-1)}dt \\ &\quad = 2\eta\delta\sum_{\ell=1}^N\sum_{j=1}^N (M^1_{j\ell}(t)d\beta^1_j(t) + M^2_{j\ell}(t)d\beta^2_j(t))|y_\ell(t)|^{2(\delta-1)}. \end{aligned} \tag{78}$$

Recalling (71) and that $0<\delta<\frac{1}{2}$, we see by (78) that

$$E\sum_{\ell=1}^N |y_\ell(t)|^{2\delta}+2\delta E\sum_{\ell=1}^N \mathrm{Re}\,\lambda_\ell\int_0^t |y_\ell(s)|^{2\delta}ds$$
$$+\delta\eta^2(1-2\delta)\rho^2 E\int_0^t\sum_{\ell=1}^N |y_\ell(s)|^{2(\delta-1)}\sum_{j=1}^N |y_j(s)|^2 ds$$
$$\le\sum_{\ell=1}^N |y_\ell(0)|^{2\delta},\ \mathbb{P}\text{-a.s.},\ t\ge 0,$$

and therefore for $\eta^2>\frac{2}{(1-2\delta)\rho^2}\max\limits_{1\le\ell\le N}\{\mathrm{Re}\,\lambda_\ell\}$, we have

$$E\sum_{\ell=1}^N |y_\ell(t)|^{2\delta}+\gamma_0 E\int_0^t\sum_{\ell=1}^N |y_\ell(s)|^{2\delta}ds\le\sum_{\ell=1}^N |y_\ell(0)|^{2\delta} \tag{79}$$

where $\gamma_0>0$ and, therefore,

$$E\int_0^\infty\sum_{\ell=1}^N |y_\ell(s)|^{2\delta}ds\le(\gamma_0)^{-1}\sum_{\ell=1}^N |y_\ell(0)|^{2\delta}. \tag{80}$$

(Here E is the expectation.) We set

$$I(t)=\delta\eta^2(1-2\delta)\int_0^t\sum_{j=1}^N |y_j(s)|^2\sum_{\ell=1}^N |\zeta_\ell|^2|y_\ell(s)|^{2(\delta-1)}ds$$
$$Z(t)=\sum_{\ell=1}^N |y_\ell(t)|^{2\delta},$$
$$I_1(t)=-2\delta\sum_{\ell=1}^N \mathrm{Re}\,\lambda_\ell\int_0^t |y_\ell(s)|^{2\delta}ds$$
$$M(t)=2\eta\delta\int_0^t\sum_{\ell=1}^N\sum_{j=1}^N(M_{j\ell}^1(s)d\beta_j^1(s)+M_{j\ell}^2(s)d\beta_j^2(s))|y_\ell(s)|^{2(\delta-1)}$$

and rewrite (78) as

$$Z(t)+I(t)=Z(0)+I_1(t)+M(t),\ \mathbb{P}\text{-a.s.} \tag{81}$$

Since $I(t)$ and $I_1(t)$ are nondecreasing and $M(t)$ is a semi-martingale, it follows by the martingale convergence theorem (see Lemma 3.1 in [8]) that there is

$$\lim_{t\to\infty} Z(t)<\infty,\ \mathbb{P}\text{-a.s.}$$

Then, by (80), we see that

$$\lim_{t\to\infty} Z(t)=0,\ \mathbb{P}\text{-a.s.}$$

This completes the proof of Lemma 6. ■

Proof of Theorem 4 (continued). Coming back to (69), we note that since $\sigma(-\mathcal{A}_k^s)\subset\{\lambda;\ \operatorname{Re}\lambda\le-\gamma\}$, we have that (recall that $-\mathcal{A}_k^s$ likewise $-\mathcal{A}_k$ generates a C_0-analytic semigroup)

$$\|e^{-\mathcal{A}_k^s t}\|_{L(H,H)}\le Ce^{-\gamma t},\ \forall t\ge 0. \tag{82}$$

We have

$$\begin{aligned} y^s(t)=&e^{-\mathcal{A}_k^s t}(I-P_N)L_k v_k^0+\eta\sum_{j=1}^N\int_0^t e^{-\mathcal{A}_k^s(t-s)}(I-P_N)\\ &(\widetilde{F}_k+\theta)D_k(y_j(s)))d\beta_j(s),\ \forall t\ge 0. \end{aligned} \tag{83}$$

Recalling that $\mathcal{A}_k=F_kL_k^{-1}$, we have by (83)

$$\begin{aligned} y^s(t)=&\,e^{-\mathcal{A}_k^s t}(I-P_N)L_kv_k^0\\ &+\eta\mathcal{A}_k^s\sum_{j=1}^N\int_0^t e^{-\mathcal{A}_k^s(t-s)}(I-P_N)L_kD_k(y_j(s))d\beta_j(s)\\ &+\eta\theta\sum_{j=1}^N\int_0^t e^{-\mathcal{A}_k^s(t-s)}(I-P_N)D_k(y_j(s))d\beta_j(s). \end{aligned}$$

Hence

$$(84)\qquad \begin{aligned}(\theta{+}\mathcal{A}_k)^{-1}y^s(t)&{=}(\theta{+}\mathcal{A}_k)^{-1}e^{-\mathcal{A}_k^s t}(I{-}P_N)L_kv_k^0\\ &+\eta\sum_{j=1}^{N}\int_0^t e^{-\mathcal{A}_k^s(t-s)}(I{-}P_N)L_k(D_k(y_j(s)))d\beta_j(s).\end{aligned}$$

(We may take θ sufficiently large such that $(\theta{+}\mathcal{A}_k)^{-1}\in L(H,H)$.)

We set $X_s(t)=(\theta+\mathcal{A}_k)^{-1}y^s(t)$ and rewrite (84) as

$$(85)\qquad \begin{aligned}&dX_s(t)+\mathcal{A}_k^sX_s(t)dt=\eta(I-P_N)\sum_{j=1}^{N}L_k(D_ky_j)d\beta_j(t)\\ &X_s(0)=(\theta+\mathcal{A}_k)^{-1}(I-P_N)L_kv_k^0.\end{aligned}$$

As seen earlier, the operator $-\mathcal{A}_k^s$ generates a γ-exponentially stable C_0-semigroup on H and in the space $H^{-1}(0,1)=H^{-1}$ too, which we endow with the scalar product $\langle y,z\rangle_{-1}=\left\langle L_k^{-1}y,z\right\rangle$ and with the corresponding norm $|\cdot|_{-1}$. Then, by the Lyapunov theorem (see, e.g., [14], p.98), there is $Q{\in}L(H^{-1},H^{-1})$, $Q=Q^*\geq 0$ such that

$$\operatorname{Re}\langle Qx,\mathcal{A}_k^sx-\gamma x\rangle_{-1}=\frac{1}{2}\,|x|_{-1}^2,\quad \forall x\in D(\mathcal{A}_k^s).$$

(We note that Q which is dependent of k is positively definite in the sense that $\inf\{\langle Qx,x\rangle\,;\,|x|=1\}>0$.)

Applying Ito's formula in (84) to the function

$$\varphi(x)=\frac{1}{2}\,\langle Qx,x\rangle_{-1}$$

we obtain that

$$\begin{aligned}\frac{1}{2}\,d\,\langle QX_s(t),X_s(t)\rangle_{-1}+\tfrac{1}{2}\,|X_s(t)|_{-1}^2dt+\gamma\,\langle QX_s(t),X_s(t)\rangle_{-1}\,dt&\\ =\frac{1}{2}\,\eta^2\sum_{j=1}^{N}\langle QY_j(t),Y_j(t)\rangle_{-1}\,dt+\eta dM_0(t)&\end{aligned}$$

where

$$Y_j(t) = (I - P_N)L_k(D_k y_j(t))$$

and

$$\begin{aligned} dM_0(t) = &\sum_{j=1}^{N} \big(\langle \operatorname{Re} QX_s(t), \operatorname{Re} Y_j(t)\rangle_{-1} \\ &+ \langle \operatorname{Im} QX_s(t), \operatorname{Im} Y_j(t)\rangle_{-1}\big)\, d\beta_j^1(t) \\ &+ \sum_{j=1}^{N} \big(\langle \operatorname{Re} QX_s(t), \operatorname{Im} Y_j(t)\rangle_{-1} \\ &- \langle \operatorname{Im} QX_s(t), \operatorname{Re} Y_j(t)\rangle_{-1}\big)\, d\beta_j^2(t). \end{aligned}$$

Clearly, by (55), we have that

$$|Y_j(t)|_{-1} \le C|y_j(t)|$$

and so, by Lemma 6 it follows that

$$\int_0^\infty |Y_j(t)|_{-1}^2 e^{2\gamma t} dt < \infty, \ \mathbb{P}\text{-a.s.}$$

and

$$E \sum_{j=1}^{N} \int_0^\infty |Y_j(t)|_{-1}^{2\delta} e^{2\gamma\delta t} dt \le C \sum_{j=1}^{N} |y_j(0)|^{2\delta}. \tag{86}$$

This yields

$$\begin{aligned} &e^{2\gamma t} \langle QX_s(t), X_s(t)\rangle_{-1} + \int_0^t e^{2\gamma s}|X_s(s)|_{-1}^2 ds \\ &\quad = \langle Q(I-P_N)x, (I-P_N)x\rangle_{-1} \\ &\quad + \eta^2 \sum_{j=1}^{N} \int_0^t e^{2\gamma s} \langle QY_j(s), Y_j(s)\rangle_{-1} ds \\ &\quad + 2\eta \sum_{j=1}^{N} \int_0^t e^{2\gamma s} dM_0(s), \ t \ge 0, \ \mathbb{P}\text{-a.s.} \end{aligned}$$

Then, once again by Lemma 3.1 in [8], where

$$
\begin{aligned}
Z(t) &= e^{2\gamma t} \langle QX_s(t), X_s(t) \rangle_{-1} , \\
I(t) &= \int_0^t e^{2\gamma s} |X_s(s)|_{-1}^2 ds, \\
I_1(t) &= \eta^2 \sum_{j=1}^N \int_0^t e^{2\gamma s} \langle QY_j, Y_j \rangle_{-1} ds, \\
M(t) &= 2\eta \sum_{j=1}^N \int_0^t e^{2\gamma s} dM_0(s), \ t \geq 0, \ \mathbb{P}\text{-a.s.}
\end{aligned}
$$

we infer that

$$
\lim_{t\to\infty} e^{2\gamma t} \langle QX_s(t), X_s(t) \rangle = 0, \ \mathbb{P}\text{-a.s.}
$$

This yields

$$
\lim_{t\to\infty} (e^{\gamma t} |X_s(t)|_{-1}) = 0, \ \mathbb{P}\text{-a.s.}
$$

and, therefore,

$$
\lim_{t\to\infty} (e^{\gamma t} |(\theta + \mathcal{A}_k)^{-1} L_k v_k(t)|_{-1}) = 0, \ \mathbb{P}\text{-a.s.} \tag{87}
$$

Taking into account that, by (46), $(\theta+\mathcal{A}_k)^{-1}L_k$ is an isomorphism from $H^{-1}(0,1)$ to itself, we infer by (86), (87) that (65) holds, thereby completing the proof of Theorem 58. ∎

6 Conclusions

We have designed for the linearized Navier–Stokes equation a stochastic stabilizing feedback controller with the support in an arbitrary finite set of points ξ_k in the spatial domain $\mathcal{O}$. The design of this feedback controller involves the knowledge in ξ_k of a finite system of eigenfunctions of the dual Stokes–Oseen equation corresponding to unstable eigenvalues and it is robust to small structural perturbations of the system. This is a substantial reduction in computation over existing Riccati-based methods. The stabilization is, however, in probability and in a weak distributional sense.

A similar stochastic approach was developed for boundary stabilization of a periodic fluid in a $2-d$ channel.

This exposition is based on the following works:

a. V. Barbu, Exponential stabilization of the linearized Navier-Stokes equations by pointwise controllers, *Automatica*, 2010 (to appear).

b. V. Barbu, Stabilization of a plane periodic channel flow by noise wall normal controllers, *Systems & Control Letters*, 2010 (to appear).

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