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ON SAINT VENANT'S PRINCIPLE

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joint work with

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Consider a linear elasticity problem in a domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$). The physical state of the body is described by the solution u of the problem

$$\begin{cases} u \in H^1(\Omega, \mathbb{R}^n) \\ -div(Ce(u)) = f \quad \text{in } \Omega \\ u = u_D \quad \text{on } \Gamma_D \\ Ce(u)n = g \quad \text{on } \Gamma_N \end{cases}$$

Consider a functional J, for instance

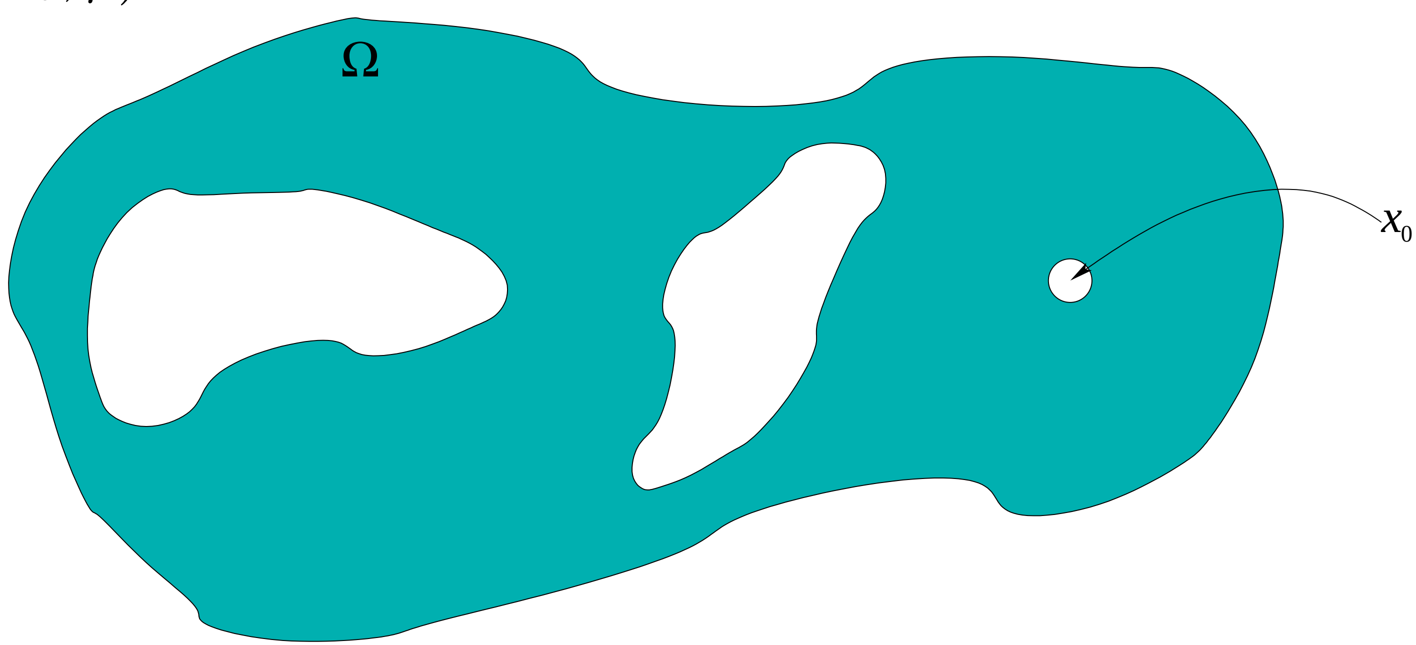
$$J(u) = \int_\Omega Ce(u)e(u)$$

(many other examples can be considered).

The goal of topology optimization is to describe the behaviour of an objective function $J(u)$ when the domain Ω is perturbed by introducing a microscopic hole at a specific location.

Consider a point x_0 in Ω, assume $f = 0$ in a neighbourhood of x_0.

Consider a small parameter $\rho > 0$ and denote by Ω_ρ the perforated domain $\Omega \setminus \overline{B(x_0, \rho)}$.



Denote by u_ρ the solution of the problem :

$$\begin{cases} u_\rho \in H^1(\Omega_\rho, \mathbb{R}^n) \\ -div(Ce(u_\rho)) = f \quad \text{in } \Omega_\rho \\ u_\rho = u_D \quad \text{on } \Gamma_D \\ Ce(u_\rho)n = g \quad \text{on } \Gamma_N \\ Ce(u_\rho)n = 0 \quad \text{on } \partial B(x_0, \rho) \end{cases}$$

Define

$$J_\rho(u_\rho) = \int_{\Omega_\rho} Ce(u_\rho)e(u_\rho)$$

In topology optimization, the main issue is to study the variation of the objective functional $J_\rho(u_\rho) - J(u)$.

The topological derivative λ of the functional J is defined by

$$\lambda = \lim_{\rho \to 0} \frac{J_\rho(u_\rho) - J(u)}{\rho^n}$$

In several papers (see Masmoudi & al. 2001, Lewiński & Sokołowski 2003, etc.), an asymptotic development is deduced for J_ρ in the form

$$J_\rho(u_\rho) = J(u) + \lambda \rho^n + o(\rho^n)$$

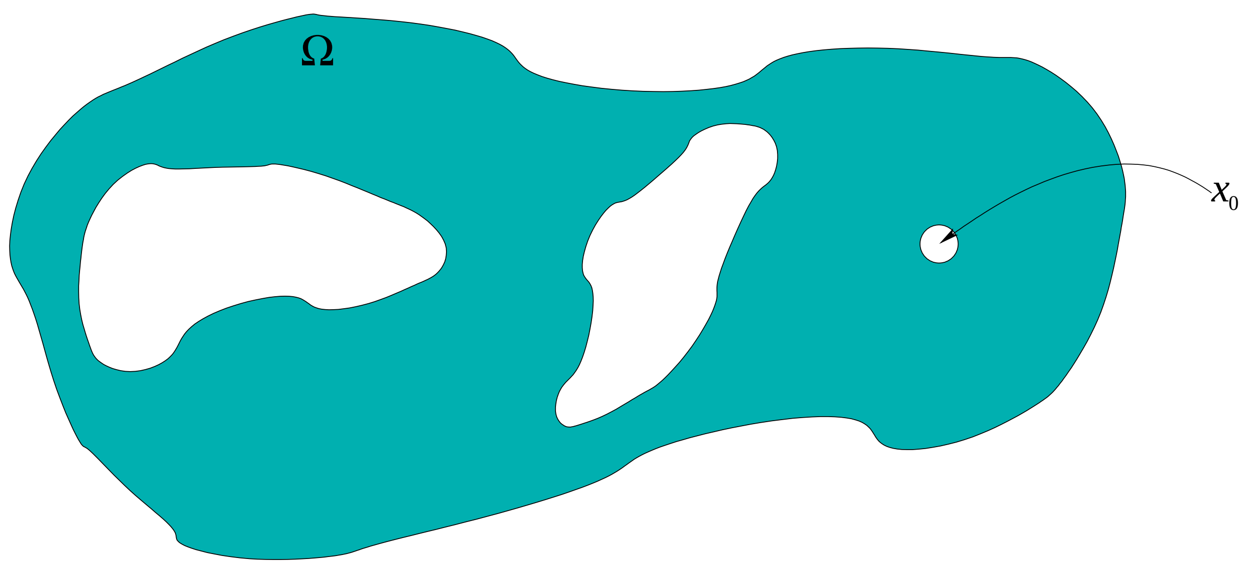
where n is the space dimension.

It turns out that the topological derivative λ depends on the (unperturbed) state u and on the location x_0 of the hole.

It is meaningful to study the difference $u_\rho - u$. It satisfies

$$\begin{cases} u_\rho - u \in H^1(\Omega_\rho), \\ -div(Ce(u_\rho - u)) = 0 \text{ in } \Omega_\rho \\ u_\rho - u = 0 \text{ on } \Gamma_D \\ Ce(u_\rho - u)n = 0 \text{ on } \Gamma_N \end{cases}$$

$$Ce(u_\rho - u)n = -Ce(u)n \text{ on } \partial B(x_0, \rho)$$



Recall that $f = 0$ in some fixed neighbourhood of x_0. Then the forces $-Ce(u)n$ appearing at the interface $\partial B(x_0, \rho)$ have zero resultant and zero momentum:

$$\int_{\partial B(x_0,\rho)} Ce(u)n = \int_{B(x_0,\rho)} div(Ce(u)) = \int_{B(x_0,\rho)} f = 0$$

$$\int_{\partial B(x_0,\rho)} Ce(u)n \wedge x = \int_{B(x_0,\rho)} f \wedge x = 0$$

Saint Venant's principle according to Love - 1927

... the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part.

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C. Barbarosie, A.-M. Toader, Saint-Venant's principle and its connections to shape and topology optimization, Journal of Applied Mathematics and Mechanics (ZAMM), vol. 88 (1), p. 23–32, 2008

Saint Venant's principle in mathematical form

Let $\Omega \subset \mathbb{R}^n$ be a domain having Lipschitz boundary, and x_0 a fixed arbitrary point in Ω. Let $a \in L^\infty(\Omega, [\alpha, \beta])$ be a function such that a is constant in a ball $B_n(x_0, R)$ of fixed radius R. Let $\rho > 0$ be a small parameter and consider a function $U_\rho \in H^1(\Omega_\rho)$ verifying

$$-div(a\nabla U_\rho) = 0 \quad \text{in} \quad \Omega_\rho = \Omega \setminus \overline{B_n(x_0, \rho)}$$

$$U_\rho = 0 \quad \text{on} \quad \Gamma_D$$

$$a\nabla U_\rho n = 0 \quad \text{on} \quad \Gamma_N$$

where $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. Suppose U_ρ satisfies

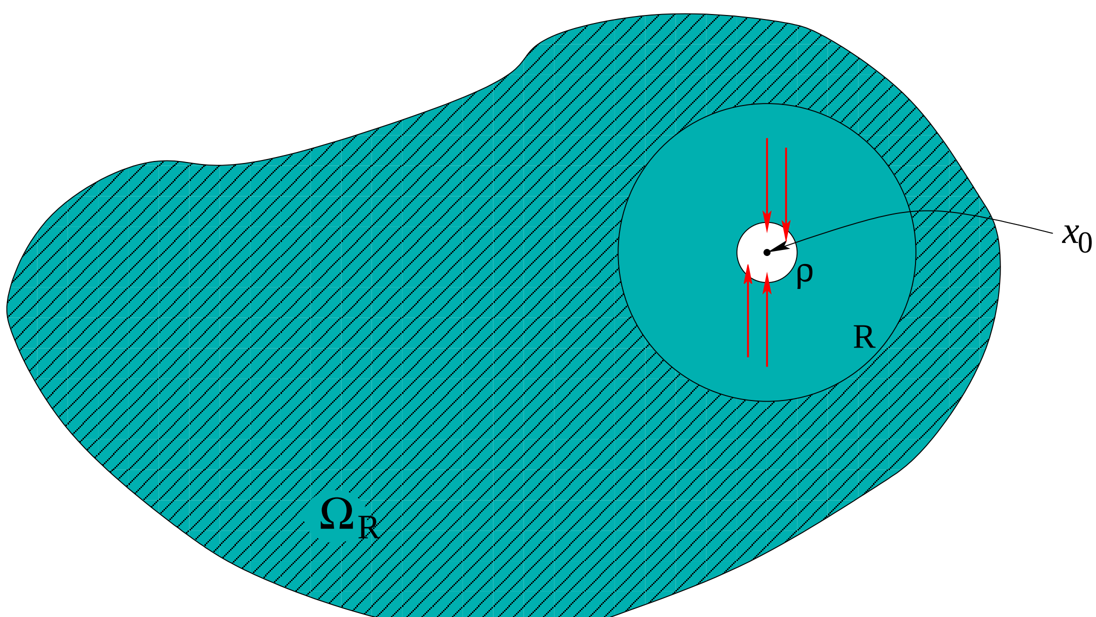
$$\int_{\partial B(x_0,\rho)} a\nabla U_\rho n = 0$$

For each $r \in [\rho, R]$, define the energy in $\Omega_r = \Omega \setminus \overline{B_n(x_0, r)}$

$$E(r, U_\rho) = \int_{\Omega_r} a|\nabla U_\rho|^2.$$

Then the following estimate holds, for fixed R, where $c = 2/(n-1)$,

$$E(R, U_\rho) \leq E(\rho, U_\rho)\, R^{-c} \rho^{c}\,.$$



• in two dimensions $c = 2$ and

$$E(R, U_\rho) \leq E(\rho, U_\rho)\, R^{-2} \rho^2 \,,$$

• in three dimensions $c = 1$ and

$$E(R, U_\rho) \leq E(\rho, U_\rho)\, R^{-1} \rho \,,$$

Provided $E(\rho, U_\rho)$ is bounded, the energy $E(R, U_\rho)$ is small (of order ρ^c) outside a fixed ball $B_n(x_0, R)$. Therefore $\|\nabla U_\rho\|^2_{L^2(\Omega_R)}$ is of order ρ^c.

In the case when Γ_D is not negligible in $\partial\Omega$, Poincaré inequality implies that $\|U_\rho\|^2_{L^2(\Omega_R)}$ is also small (of order ρ^c).

The assumption that $E(\rho, U_\rho)$ is bounded must be checked separately for each particular case. For instace, U_ρ bounded in $H^1(\Omega)$ is sufficient.

Note that U_ρ may be defined in the entire domain Ω. In this case, the equilibrium condition

$$\int_{\partial B(x_0,\rho)} a\nabla U_\rho n = 0$$

is equivalent to

$$\int_{B(x_0,\rho)} div(a\nabla U_\rho) = 0$$

Proof ingredients :

Poincaré-Wirtinger inequality Let Ω be a bounded and connected domain in $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that, for any $u \in H^1(\Omega)$ with $\displaystyle\int_\Omega u = 0$, one has

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}.$$

Poincaré-Wirtinger inequality for functions on a sphere Let u be a function in $H^1(S_{n-1}(0,r))$, where $S_{n-1}(0,r)$ is the sphere centered at 0 and of radius r. If $\displaystyle\int_{S_{n-1}(0,r)} u = 0$, then

$$\|u\|_{L^2(S_{n-1}(0,r))} \leq r\,(n-1)\,\|\nabla u\|_{L^2(S_{n-1}(0,r))}.$$

Suppose that in Saint-Venant's principle we obtained an exponential decay of the energy.

Note that the difference $u_\rho - u$ satisfies all hypothesis in our statement of Saint-Venant's principle.

This means that, for a fixed radius R, the norm of $u_\rho - u$ in $L^2(\Omega \setminus B_n(x_0, R))$ goes to zero exponentially as $\rho \to 0$ (in particular, it goes to zero faster than any power of ρ). Consider now the case when $f = 0$ in some neighbourhood of x_0 (e.g. in the ball $B_n(x_0, R)$). Recall that

$$J_\rho(u_\rho) - J(u) = \int_\Omega f(u_\rho - u) = \int_{\Omega \setminus B_n(x_0,R)} f(u_\rho - u)$$

This value goes to zero (as $\rho \to 0$) faster than any power of ρ, which implies that the topological derivative, denoted by λ in the asymptotic expansion, must be zero. Thus, the topological

This contradicts mechanical common sense – there are many meaningful examples with forces applied only on small parts of the body – and also contradicts results obtained in the literature.

This means that exponential decay of the energy is not to be expected in Saint Venant’s principle for domains of arbitrary shape.

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Thank you !