Blow-up for NLS on surfaces

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Outline

1. Focusing NLS on $\mathbb{R}^2$.
2. Focusing NLS on surfaces.
3. Inhomogeneous focusing cubic NLS with potential on $\mathbb{R}^2$.

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Global existence

We consider the focusing NLS:

$$\begin{cases} i\partial_t u + \Delta u + |u|^p u = 0, \quad (t,x) \in (T_-, T_+) \times \mathbb{R}^2, \\ u_{\restriction t=0} = u_0 \in H^1(\mathbb{R}^2). \end{cases}$$

The conservation of mass $\|u(t)\|_{L^2}$ and energy

$$E(u) = \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 - \frac{1}{p+2}\|u(t)\|_{L^{p+2}}^{p+2},$$

combined with Gagliardo-Nirenberg's inequality yields

$$E(u_0) \geq \|\nabla u(t)\|_2^2 \left(\frac{1}{2} - \frac{C_{p+2}}{p+2}\|u_0\|_{L^2}^2 \|\nabla u(t)\|_{L^2}^{p-2} \right).$$

$\rightsquigarrow$ for $p < 2$ or for $p = 2$ and $\|u_0\|_{L^2}^2 < \frac{2}{C_4} = \|Q\|_{L^2}^2$, the solution is global in time (Weinstein '83).
(Q is the minimizer of G-N's inequality, solution of $\Delta Q + Q^3 = Q$)

Blow-up solutions

- Glassey's criterium '77:

$$\partial_t^2 \int |x|^2 |u(t,x)|^2 dx = 16E(u_0) - \frac{4(2p-4)}{p+2} \|u(t)\|_{L^{p+2}}^{p+2} dx,$$

$\rightsquigarrow$ for $p \geq 2$, $xu_0(x) \in L^2$, $E(u_0) < 0$, the solution blows-up.

- For the critical power $p = 2$, the pseudo-conformal transformation of $e^{it} Q(x)$ is the solution

$$S(t) = \frac{e^{i\frac{|x|^2}{4t}}}{t} e^{\frac{i}{t}} Q\left(\frac{x}{t}\right),$$

$\rightsquigarrow$ $\|Q\|_{L^2}$ is the critical mass for blow-up (Weinstein '83).

- $S(t)$-type solutions are the only blow-up solutions of critical mass (Merle '93). The rate of blow-up is $1/t$.
- $\exists$ many solutions $S(t-T,x) + w(t,x)$, with w smooth at $t = T$ (Bourgain-Wang '97, Krieger-Schlag '09), of $1/t$-type.

$\rightsquigarrow$The pseudo-conformal blow-up is unstable.

Blow-up solutions

- There exist an open subset of initial data such that the solutions blow up, and at the blow-up rate is $\sqrt{\frac{\log|\log|t||}{|t|}}$ (Perelman '01, Merle-Raphaël '04)
  $\rightsquigarrow$ The "log-log" blow-up is stable.
- The log-log blow-up is also structurally stable (Planchon-Raphaël '01 on bounded domains, Burq-Gérard-Raphaël on manifolds).
- On star-shaped domains Glassey's principle is still valid (Kavian '87).
- $\exists$ S(t)-type solutions on domains and more generally at flat points of a surface (Ogawa-Tsutsumi '90, Burq-Gérard-Tzvetkov '03)

Question

Q : what happens with the frontier between global existence and blow-up when the geometry varies ? do $1/t$ blow-up solutions survive ?

Geometry is known to change things :

- The regularity s of the threshold local wellposedness/instability in H^s for the defocusing cubic NLS varies :
  $s = 0$ for $\mathbb{R}^2$ (Cazenave-Weissler '90, Christ-Colliander-Tao '03),
  $s = \frac{1}{4}$ for $\mathbb{S}^2$ (Burq-Gérard-Tzvetkov '02,'05).
  This is not an issue of the compactness of $\mathbb{S}^2$:
  $s = 0$ on $\mathbb{T}^2$ (Bourgain '93, Burq-Gérard-Tzvetkov '02).
- Dispersive properties are weaker for positive curvature (on the sphere), and stronger for negative curvature, enough to lower the short/long-range threshold power for scattering properties of the defocusing NLS (Banica, Pierfelice, Banica-Carles-Staffilani, Banica-Duyckaerts, Banica-Carles-Duyckaerts, Anker-Pierfelice, Ionescu-Staffilani '05-'10 on hyperbolic space, Damek-Ricci spaces, rotationally symmetric manifolds)

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Global existence/Blow-up

Theorem (B. '07)

The solutions of NLS on $\mathbb{H}^2$ are global for $p < 2$ or for $p = 2$ and small initial data. Blow-up occurs if $p \geq 2$, U_0 radial, $d_{\mathbb{H}^2}(0,x)U_0(x) \in L^2(\mathbb{H}^2)$, $E(U_0) < c\,\|U_0\|^2_{L^2(\mathbb{H}^2)}$, for some positive geometric constant c.

Gagliardo-Niremberg inequalities are valid with constant larger or equal than the one on $\mathbb{R}^2$

$\rightsquigarrow$ global existence for $p < 2$ or for $p = 2$ and $\|U_0\|^2_{L^2(\mathbb{H}^2)} < \frac{2}{C_4^{hyp}} \leq \frac{2}{C_4}$.

In the radial case, by taking $U(r) = \sqrt{\frac{r}{\sinh r}}u(r)$, if U satisfies NLS on $\mathbb{H}^2$ then $i\partial_t u + \Delta u - \left(\frac{1}{2} - \frac{1}{4}\left(\frac{\cosh^2}{\sinh^2} - \frac{1}{r^2}\right)\right)u + \left(\frac{r}{\sinh r}\right)^{\frac{p}{2}}|u|^p u = 0.$

$\rightsquigarrow$ global existence for $p = 2$ and $\|U_0\|^2_{L^2(\mathbb{H}^2)} = \|u_0\|^2_{L^2(\mathbb{R}^2)} < \frac{2}{C_4} = \|Q\|^2_{L^2}$.

Global existence/Blow-up

The second derivative of $\|U(t,x)d_{\mathbb{H}^2}(0,x)\|^2_{L^2(\mathbb{H}^2)}$ is, for radial solutions U,

$$16E(u) - \int_{\mathbb{H}^2} |U|^2 \Delta^2_{\mathbb{H}^2} r^2 - \frac{4p}{p+2} \int_{\mathbb{H}^2} |U|^{p+2} \left(\frac{\cosh r}{\sinh r} r - 1 \right) - 8\frac{p-2}{p+2} \int_{\mathbb{H}^2} |U|^{p+2}.$$

$\rightsquigarrow$ for $p \geq 2$, $r\,U_0(r) \in L^2$, $E(U_0) < \frac{\inf \Delta^2_{\mathbb{H}^2} r}{16} \|U_0\|^2_{L^2(\mathbb{H}^2)}$, $U(t)$ blows-up.

Remarks :

- $p = 2$ is the critical power for blow-up,
- null-energy initial data leads to blow-up, contrary to $\mathbb{R}^2$,
- the ground state on $\mathbb{H}^2$ is of positive energy and mass smaller than Q,
- the non-radial calculation does not allow to conclude the same,
- this argument relies also on the fact that there is only one chart,
- for the sphere, a Glassey criterium was given for radial initial data symmetric with respect to the equator (Ma-Zhao '07).

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Passage to the inhomogeneous NLS with potential on $\mathbb{R}^2$

We choose as a more general toymodel rotationnaly symmetric surfaces M given by metrics of the form $ds^2 = dr^2 + \phi^2(r)d\omega^2$, so that

$$\Delta_M = \partial_r^2 + \frac{\phi'}{\phi}\partial_r + \frac{1}{\phi^2}\Delta_{\mathbb{S}^2}.$$

Examples : $\mathbb{R}^2$ ($\phi(r) = r$), $\mathbb{H}^2$ ($\phi(r) = \sinh r$), intermediate manifolds (e.g. $\phi(r) = r + r^5$), compact perturbations of $\mathbb{R}^2$ or $\mathbb{H}^2$.
For radial solutions U of the cubic NLS on M,

$$i\partial_t u + \Delta u - \underbrace{\left(\frac{\phi''}{2\phi} - \frac{1}{4}\left(\frac{\phi'^2}{\phi^2} - \frac{1}{r^2}\right)\right)}_{V} u + \underbrace{\frac{r}{\phi}}_{g} |u|^2 u = 0,$$

with $U(t,r) = \sqrt{\frac{r}{\phi(r)}}\, u(t,r)$.
We shall get as a result that if V and g are bounded at infinity, and $\phi^{(3)}(0) = 0$, then there exists a “pseudo-conformal” blow-up solution.

Pseudo-conformal blow-up for the inhomogeneous NLS with potential

Theorem (B.-Carles-Duyckaerts '09)

We consider

$$i\partial_t u + \Delta u - V(x)u + g(x)|u|^2 u = 0.$$

Assume $V(x), g(x) \in \mathbb{R}$, $V(0) = 0$, $g(0) = 1$, *all their derivatives bounded, and*

$$Dg(0) = 0, \quad D^2 g(0) = 0.$$

Then there exists $T > 0$ *and a solution* $\tilde{u} \in C^0((0, T), \Sigma)$ *such that*

$$\left\| \tilde{u}(t) - \widetilde{S}(t) \right\|_{\Sigma} \underset{t \to 0^+}{\longrightarrow} 0,$$

$$\widetilde{S}(t,x) := \frac{e^{i\frac{|x|^2}{4t} - i\theta\left(\frac{1}{t}\right)}}{t} Q\left(\frac{x}{t}\right), \quad \theta(\tau) \sim \tau, \ \tau \to +\infty.$$

Comments

- Works also in dimension 1 (the mass-critical equation is quintic).
- The result was unknown even in the case $g = 1$, i.e equation

$$i\partial_t u + \Delta u - Vu + |u|^2 u = 0.$$

  Some explicit blow-up solutions are known only in very particular cases.
- Specific to the mass-critical nonlinearity. No explicit blow-up solutions are known for other nonlinearities.
- The difficulty is to weaken the assumptions on V and g at 0. Assuming that they are very flat

$$V(x) = \mathcal{O}(|x|^7), \quad g(x) = \mathcal{O}(|x|^9) \text{ as } x \to 0,$$

  the proof reduces essentially to the argument of Bourgain and Wang.
- Bourgain-Wang solutions were also constructed for very flat perturbations of mass-critical homogeneous Hartree type nonlinearities (Krieger-Lenzmann-Raphaël '08).

Comments

- After pseudo-conformal transformation, the proof is reduced to a stability result for the Schrödinger equation around e^{itQ}. This type of results goes back to Weinstein '85 for mass-subcritical and mass-critical nonlinearity. In a supercritical context see Beceanu '09.
- Assume $\forall x, \quad g(x) \leq g(0) = 1$. Then

  $$\|u_0\|_{L^2} < \|Q\|_{L^2} \Longrightarrow u \text{ is globally defined.}$$

  By Merle '96 any minimal mass blow-up must concentrate at a maximum, and there exist blow-up solutions for all masses larger but close to $\|Q\|_{L^2}$. The blow-up solution $\tilde{u}$ is a minimal mass blow-up solution.
- The existence of a mass-critical solution blowing-up at 0 if $Dg(0) = 0$ and $D^2g(0) \neq 0$ was proven by a different method by Raphaël-Szeftel '10. They also prove uniqueness. If their methods extends to the case with potential, it yields a pseudo-conformal blow-up solution on $\mathbb{H}^2$ of $\|Q\|_{L^2}$ mass.

About the proof

We want to construct a solution u of (NLSp) defined on $]0, +\infty[$ that blows-up at $t = 0$.

1. Linearization argument (Bourgain-Wang strategy). After a pseudo-transformal conformation, include some linear terms with $-\Delta$. The new equation is $i\partial_t f - Lf = B(f)$, with L the linearized operator around $e^{it}Q$,

$$L(f) = -\Delta f + f - 2Q^2 f - Q^2 \overline{f}.$$

Works for polynomially very flat V and g.

2. Modulation. The polynomial instability of L is caused by 8 so-called secular modes. We use time dependent transformations to avoid these secular modes.

Bourgain-Wang strategy

Pseudo-conformal transformation. Let

$$v(t,x) = \mathcal{PC}(u)(t,x) = \frac{e^{\frac{i|x|^2}{4t}}}{t}\overline{u}\left(\frac{1}{t},\frac{x}{t}\right).$$

Then u is solution to (NLSp) (with $g=1$ for simplicity) if $\tilde{v}$ satisfies

$$i\partial_t v + \Delta v - \frac{1}{t^2}V\left(\frac{x}{t}\right)v + |v|^2 v = 0.$$

Linearization. Write $v = e^{it}(Q+h)$, then

$$i\partial_t h - Lh - \frac{1}{t^2}V\left(\frac{x}{t}\right)Q + \frac{1}{t^2}V\left(\frac{x}{t}\right)h + \ldots = 0.$$

To get a $1/t$ blow-up solution u it is sufficient to show the existence of a solution h such that

$$\lim_{t\to\infty} t^{1+\epsilon}\|h(t)\|_{H^1} + t^{\epsilon}\|xh(t)\|_{L^2} = 0.$$

Algebraic instabilities of the linearized operator 1

We must solve

$$h(t) = -i \int_t^{+\infty} e^{-i(\sigma - t)L} \frac{1}{\sigma^2} V\left(\frac{x}{\sigma}\right) Q(x)\, d\sigma + \dots$$

Behaviour of e^{itL}?

Theorem (Weinstein '85, Kwong '89)

$$H^1 = S \oplus M, \quad \dim S = 8,$$

where S and M are stable by e^{itL} and ($\delta_0 > 0$ is a small constant)

$$\left\| e^{itL} P_M(f) \right\|_{H^1} \leq C \|f\|_{H^1},$$

$$\left\| e^{itL} P_S(f) \right\|_{H^1} \leq C(|t| + 1)^3 \left\| f e^{-\delta_0 x} \right\|_{L^2}.$$

The fixed point argument works only if V very flat at 0.

Algebraic instabilities of the linearized operator 2

Noting $\mathcal{L} = iL$, we must solve

$$h(t,x) = -\int_t^{+\infty} e^{(\sigma-t)\mathcal{L}} \frac{i}{\sigma^2} V\left(\frac{x}{\sigma}\right) Q(x)\, d\sigma + \dots$$

The polynomial unstability of $e^{t\mathcal{L}}$ comes from the generalized kernel S:

$$\int_t^{\infty} e^{(\sigma-t)\mathcal{L}} P_S(F(\sigma,x))d\sigma = \sum_{1\leq j\leq 6} \nu_j(t) n_j(x),$$

where

$$\nu_1' = 2\nu_4 - 2\nu_6 + \mu_1 \quad , \quad \nu_{2,k}' = 2\nu_{3,k} + \mu_{2,k} \quad , \quad \nu_{3,k}' = \mu_3,$$

$$\nu_4' = 2\nu_5 + \mu_4 \quad , \quad \nu_5' = -2\nu_6 + \mu_5 \quad , \quad \nu_6' = \mu_6,$$

$$\mu_\alpha(t) = \langle F(t,x), m_\alpha(x)\rangle = \int \Re F(t) \Re m_\alpha + \int \Im F(t) \Im m_\alpha,$$

$$m_1 = i\tilde{Q} \ , \ m_{2,k} = x_k Q \ , \ m_{3,k} = -\partial_k Q \ , \ m_4 = -\frac{1}{2}|x|^2 Q - cQ$$

$$m_5 = i(Q + x.\nabla Q) \ , \ m_6 = -Q \ , \ n_j \in \Sigma \ , \ span\{n_j\} = S.$$

Simple modulation argument

Starting again from the pseudo-conformal transformed equation

$$i\partial_t v + \Delta v - \frac{1}{t^2} V\left(\frac{x}{t}\right) v + |v|^2 v = 0.$$

We first modulate $v = \tilde{v} e^{i\theta(t)}$ and then we define h by $\tilde{v} = e^{it}(Q + h)$, so

$$h(t,x) = -\int_t^{+\infty} e^{(\sigma - t)\mathcal{L}} \left(\frac{i}{\sigma^2} V\left(\frac{x}{\sigma}\right) + i\theta'(\sigma) \right) Q(x)\, d\sigma + \dots .$$

We can choose θ' such that $\nu_1' = 0$, so that we gain another power in time, and we can have a weaker flatness assumption on V at zero.

Modulation argument

For any modulation,

$$v(t,x) = \frac{e^{i\theta_1(t)+i\theta_2(t).x+i\theta_3(t)|x|^2}}{\lambda(t)} \tilde{v}\left(\gamma(t), \frac{x}{\lambda(t)} - \beta(t)\right) \quad , \quad \dot{\gamma}(t) = \frac{1}{\lambda^2(t)},$$

by defining $\tilde{v} = e^{i\tau}(Q+h)$ in variables $y = \frac{x}{\lambda(t)}$, $\tau = \gamma(t)$,

$$\partial_\tau h + iLh = F_p(h) + Z_p(h) + Z_p(Q),$$

where

$$Z_p(f) = -i(p_1 f + p_2.y\, f + p_3|y|^2 f) + p_4.\nabla f + p_p(f + y.\nabla f).$$

The p_j's are real functions of (θ, λ, β). In particular $Z_p(Q) \in S$.

Given p_j's with appropriate decay we recover by a fixed point (θ, λ, β).

Construction of a solution

We want to find a fonction h such that there exists p for which

$$\begin{cases} \partial_\tau h_S + iLh_S = P_S F_p(h) + P_S Z_p(h) + Z_p(Q), \\ \partial_\tau h_M + iLh_M - P_M Z_p(h_M) = P_M F_p(h) + P_M Z_p(h_S). \end{cases}$$

- Given a fonction h, we find p such that

$$\Phi_1(h)(\tau) = \int_\tau^\infty e^{(\sigma-\tau)\mathcal{L}} \left(P_S F_p(h) + P_S Z_p(h) + Z_p(Q)\right) d\sigma$$

  is spanned by only one secular mode (we assume V radial).
- We prove the existence of $\Phi_2(h) = \phi$ solution (in M for all τ) of

$$\partial_\tau \phi + (iL - P_M Z_p)(\phi) = P_M F_p(h) + P_M Z_p(h_S),$$

  by showing first energy estimates.
- We perform a fixed point argument for $(\Phi_1 + \Phi_2)(h)$.