

Lower Bounds for the Density of Asian Type SDE's

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1 The Problem

We consider the SDE

$$\begin{aligned}X_t^1 &= x^1 + \int_0^t \sigma(X_s) dW_s + \int_0^t b_1(X_s) ds \\X_t^2 &= x^2 + \int_0^t b_2(X_s) ds\end{aligned}$$

and we look for lower bounds for the density

$$p_T(x, y) = E(\delta_0(X_T - y)) \geq ???$$

Hypothesis and results are in terms of the **Skelton** : given a control $\phi \in L^2(0, T)$ we define $x_t = x_t(\phi)$ as the solution of

$$\begin{aligned}x_t^1 &= x^1 + \int_0^t \sigma(x_s) \phi_s ds + \int_0^t b_1(x_s) ds \\x_t^2 &= x^2 + \int_0^t b_2(x_s) ds.\end{aligned}$$

Previous results for diffusion processes

Uniform elliptic diffusions : Arenson.

Uniform strong Hörmander condition : :

C.L.Fefferman, A.Sanchez-Calle. (1986)

S. Kusuoka, D. Stroock. (1987).

Our framework (weak Hörmander condition).

A. Pascucci and S. Polidoro. (2006).

F. Delarue and S. Menozzi. (2010).

Hypothesis $H(\phi)$. Give the skeleton $x_t = x_t(\phi)$ we assume :

$$A(\phi). \quad |\sigma(x_t)| + |b(x_t)| \leq C_\phi(t)$$

$$B. \quad \|\partial_\alpha \sigma\|_\infty + \|\partial_\alpha b\|_\infty \leq C \quad 1 \leq |\alpha| \leq 5.$$

$$C(\phi). \quad \min\{|\sigma(x_t)|, |\sigma \partial_1 b_2(x_t)|\} \geq \varepsilon_\phi(t) > 0.$$

Theorem. Let $x, y \in \mathbb{R}^2$. Suppose that there exists a control ϕ such that $x_0(\phi) = x, x_T(\phi) = y$ and $H(\phi)$ holds. Then

$$p_T(x, y) \geq \frac{C_1}{h_\phi(T)} \exp\left(-C_2 \int_0^T \frac{1 + \phi_t^2}{h_\phi(t)} dt\right)$$

with

$$h_\phi(t) = \left(\frac{\varepsilon_\phi(t)}{C_\phi(t)}\right)^p.$$

Corollary (Uniform "elliptic" case) Suppose that

$$A. \quad \|\sigma\|_\infty + \|b\|_\infty \leq C$$

$$B. \quad \|\partial_\alpha \sigma\|_\infty + \|\partial_\alpha b\|_\infty \leq C \quad 1 \leq |\alpha| \leq 5.$$

$$C. \quad \min\{|\sigma(x)|, |\sigma \partial_1 b_2(x)|\} \geq \varepsilon_0 > 0 \quad \forall x \in R^2.$$

Then

$$p_T(x, y) \geq C_1 \exp\left(-C_2\left(T + \frac{|y^1 - x^1|^2}{T} + \frac{|y^2 - x^2 - b_2(x)T|^2}{T^3}\right)\right)$$

with $C_i = C_i(C, \varepsilon_0)$, $i = 1, 2$.

Log normal type diffusions.

$$X_t^1 = x^1 + \int_0^t \alpha(X_s^1) X_s^1 dW_s + \int_0^t \beta_1(X_s^1) X_s^1 ds, \quad X_t^2 = x^2 + \int_0^t b_2(X_s^1) ds.$$

Remark : The coefficients are not uniformly bounded and the ellipticity constant is not uniform.

Problem :

1. Construction of skeletons such that

$$x_t^1 = x^1 + \int_0^t \alpha(x_s^1) x_s^1 \phi_s ds + \int_0^t \beta_1(x_s^1) x_s^1 ds, \quad x_t^2 = x^2 + \int_0^t b_2(x_s^1) ds$$
$$x_0 = x, \quad x_T = y$$

and computation of the energy $\int_0^T |\phi_t|^2 dt$.

2. Optimization

Idea of the proof :

$$X_t^1 = x^1 + \int_0^t \sigma(X_s) dW_s + \int_0^t b_1(X_s) ds \quad X_t^2 = x^2 + \int_0^t b_2(X_s) ds$$

$$x_t^1 = x^1 + \int_0^t \sigma(x_s) \phi_s + b_1(x_s) ds \quad x_t^2 = x^2 + \int_0^t b_2(x_s) ds$$

Decomposition : let $t, h > 0$ be fixed.

$$X_{t+h}^1 = X_t^1 + \sigma(X_t)(W_{t+h} - W_t) + \int_t^{t+h} (\sigma(X_s) - \sigma(X_t)) dW_s + \int_t^{t+h} b_1(X_s) ds$$

and (this is the point!)

$$X_{t+h}^2 = X_t^2 + \int_t^{t+h} b_2(X_s) ds = X_t^2 + b_2(X_t)h + \int_t^{t+h} (b_2(X_s) - b_2(X_t)) ds.$$

Then using Ito + equation

$$\begin{aligned}\int_t^{t+h} (b_2(X_s) - b_2(X_t))ds &= \int_t^{t+h} \left(\int_t^s \sigma \partial_1 b_2(X_u) dW_u \right) ds + rest \\ &= \sigma \partial_1 b_2(X_t) \int_t^{t+h} \left(\int_t^s 1 dW_u \right) ds + \\ &\quad + \int_t^{t+h} \left(\int_t^s (\sigma \partial_1 b_2(X_u) - \sigma \partial_1 b_2(X_t)) dW_u \right) ds + rest \\ &= \sigma \partial_1 b_2(X_t) \int_t^{t+h} (t+h-u) dW_u + rest\end{aligned}$$

We denote

$$\Delta = \sigma(X_t)(W_{t+h} - W_t), \quad \bar{\Delta} = \sigma \partial_1 b_2(X_t) \int_t^{t+h} (t+h-u) dW_u$$

Then

$$X_{t+h}^1 = X_t^1 + \Delta + rest, \quad X_{t+h}^2 = X_t^2 + b_2(X_t)h + \bar{\Delta} + rest$$

Covariance matrix for $(\Delta, \bar{\Delta})$

We denote

$$a = \sigma(X_t), \quad b = \sigma \partial_1 b(X_t)$$

The covariance of $(\Delta, \bar{\Delta})$ is given by

$$\text{cov}(\Delta, \bar{\Delta}) = \begin{pmatrix} a^2 \times h & ab \times \frac{h^2}{2} \\ ab \times \frac{h^2}{2} & b^2 \times \frac{h^3}{3} \end{pmatrix}$$
$$\det = a^2 b^2 \times h^4 \times \left(\frac{1}{3} - \frac{1}{4} \right) = a^2 b^2 \times \frac{h^4}{12}.$$

Idea of proof. Framework : we consider a time grid $0 = t_0 < t_1 < \dots t_N = T$ and a sequence of random variables $F_k = (F_k^1, \dots, F_k^d)$ with

$$F_{k+1}^i = F_k^i + \sum_{j=1}^d \int_{t_k}^{t_{k+1}} h_{k,j}^i(s) dW_s^j + R_k^i$$

where $h_{k,j}^i(s)$, $s \in (t_k, t_{k+1})$ is F_{t_k} measurable.

$$G_k = \sum_{j=1}^d \int_{t_k}^{t_{k+1}} h_{k,j}(s) dW_s^j \quad \text{Gaussian}, \quad R_k = \text{Remainder}.$$

Covariance matrix

$$M_k^{ij} = \sum_{l=1}^d \int_{t_k}^{t_{k+1}} h_{k,l}^i(s) h_{k,l}^j(s) ds, \quad i, j = 1, \dots, d$$

EX. Diffusion processes :

$$\begin{aligned} X_{t_{k+1}}^i &= X_{t_k}^i + \sum_{j=1}^d \int_{t_k}^{t_{k+1}} \sigma_j^i(X_s) dW_s^i + \int_{t_k}^{t_{k+1}} b^i(X_s) ds \\ &= X_{t_k}^i + \sum_{j=1}^d \int_{t_k}^{t_{k+1}} \sigma_j^i(X_{t_k}) dW_s^i + R_k^i \end{aligned}$$

with

$$R_k^i = \sum_{j=1}^d \int_{t_k}^{t_{k+1}} (\sigma_j^i(X_s) - \sigma_j^i(X_{t_k})) dW_s^i + \int_{t_k}^{t_{k+1}} b^i(X_s) ds.$$

Covariance

$$M_k = \sigma \sigma^*(X_{t_k}).$$

Elliptic evolution sequence. We consider deterministic positive definite matrixes \overline{M}_k and we introduce the norm on R^d

$$\|x\|_k^2 = \langle \overline{M}_k^{-1} x, x \rangle, \quad B_r^{(k)}(x) = \{y : \|x - y\|_k < r\}.$$

We also consider a **sequence of points** $x_k, k = 1, \dots, N$. We consider the **tube**

$$A_k = \{\|x_i - F_{i-1}\|_k \leq \frac{1}{2} \sqrt{\delta_i}, i = 1, \dots, k\}, \quad \delta_i = t_i - t_{i-1}$$

THEOREM Suppose that

$$i) \quad \frac{1}{a_k} \overline{M}_k \geq M_k \geq a_k \overline{M}_k > \varepsilon_k \times Id \quad \text{"locally"} \quad (\text{on } A_k).$$

and, for $|\alpha| \leq d + 4$

$$ii) \quad (E(\|D^\alpha R_k\|_k^p))^{1/p} \leq C_p \times a_k^p \quad \text{"locally"} \quad (\text{on } A_k).$$

Then

$$p_{F_N}(x_N) \geq \frac{C}{\sqrt{\det \overline{M}_N}} e^{-N\theta}.$$

Diffusion processes We consider the diffusion process

$$X_t = x + \sum_{j=1}^d \int_0^t \sigma_j(X_s) dW_s^j + \int_0^t b(X_s) ds,$$

and the skeleton

$$x_t = x + \sum_{j=1}^d \int_0^t \sigma_j(x_s) \phi_j(s) ds + \int_0^t b(x_s) ds.$$

Suppose that

$$\sigma_j, b \in C_b^{d+4}, \quad \sigma \sigma^*(x_t) \geq \varepsilon_0 Id.$$

Then, if $x = x_0$ and $y = x_T$

$$p_T(x, y) \geq C_1 \exp(-C_2(T + \int_0^T |\phi_t|^2 dt))$$

Remark : In this case $M_k = \sigma \sigma^*(X_{t_k})$ and $\bar{M}_k = \sigma \sigma^*(x_{t_k})$.

In our case the ellipticity assumption fails. **Construction of the evolution sequence :**

$$X_t^1 = x^1 + \int_0^t \sigma(X_s) dW_s + \int_0^t b_1(X_s) ds \quad X_t^2 = x^2 + \int_0^t b_2(X_s) ds$$

$$x_t^1 = x^1 + \int_0^t \sigma(x_s) \phi_s + b_1(x_s) ds \quad x_t^2 = x^2 + \int_0^t b_2(x_s) ds$$

We fix the time grid $0 = t_0 < t_1 < \dots < t_N$ and we make the **Decomposition :**

$$X_{t_{k+1}}^1 = X_{t_k}^1 + \sigma(X_{t_k})(W_{t_{k+1}} - W_{t_k}) + \int_{t_k}^{t_{k+1}} (\sigma(X_s) - \sigma(X_{t_k})) dW_s + \int_{t_k}^{t_{k+1}} b_1(X_s) ds$$

and (this is the point !)

$$X_{t_{k+1}}^2 = X_{t_k}^2 + \int_{t_k}^{t_{k+1}} b_2(X_s) ds = X_{t_k}^2 + b_2(X_{t_k})(t_{k+1} - t_k) + \int_{t_k}^{t_{k+1}} (b_2(X_s) - b_2(X_{t_k})) ds.$$

Then using Ito + equation

$$\begin{aligned}
\int_{t_k}^{t_{k+1}} (b_2(X_s) - b_2(X_{t_k})) ds &= \int_{t_k}^{t_{k+1}} \left(\int_{t_k}^s \sigma \partial_1 b_2(X_u) dW_u \right) ds + rest \\
&= \sigma \partial_1 b_2(X_{t_k}) \int_{t_k}^{t_{k+1}} \left(\int_{t_k}^s 1 dW_u \right) ds + \\
&+ \int_{t_k}^{t_{k+1}} \left(\int_{t_k}^s (\sigma \partial_1 b_2(X_u) - \sigma \partial_1 b_2(X_{t_k})) dW_u \right) ds + rest \\
&= \sigma \partial_1 b_2(X_{t_k}) \int_{t_k}^{t_{k+1}} (t_{k+1} - u) dW_u + rest
\end{aligned}$$

We denote

$$\Delta_k = W_{t_{k+1}} - W_{t_k}, \quad \bar{\Delta}_k = \int_{t_k}^{t_{k+1}} (t_{k+1} - u) dW_u$$

and we use the evolution sequence

$$F_k = X_{t_k} - x_{t_k}, \quad y_k = 0$$

with the decomposition

$$F_{k+1} = F_k + G_k + R_k \quad G_k = (\sigma(X_{t_k})\Delta_k, \partial_\sigma b(X_{t_k})\bar{\Delta}_k)$$

Covariance matrix we denote

$$\delta_k = t_{k+1} - t_k, \quad \sigma_k = \sigma(X_{t_k}), \quad b_k = \partial_\sigma b(X_{t_k}), \quad c_k = \sigma_k + \frac{\delta_k}{2} b_k$$

and we construct the matrix

$$N_k = \sqrt{\delta_k} \begin{pmatrix} c_{k,1} & b_{k,1} \times \frac{\delta_k}{\sqrt{12}} \\ c_{k,2} & b_{k,2} \times \frac{\delta_k}{\sqrt{12}} \end{pmatrix}$$

Then

$$M_k = \text{Cov}(G_k) = N_k \times N_k^*$$

and

$$\bar{M}_k = \bar{N}_k \times \bar{N}_k \quad \text{with} \quad \bar{N}_k = \sqrt{\delta_k} \begin{pmatrix} \bar{c}_{k,1} & \bar{b}_{k,1} \times \frac{\delta_k}{\sqrt{12}} \\ \bar{c}_{k,2} & \bar{b}_{k,2} \times \frac{\delta_k}{\sqrt{12}} \end{pmatrix}$$

$$\bar{\sigma}_k = \sigma(x_{t_k}), \quad \bar{b}_k = \partial_\sigma b(x_{t_k}), \quad \bar{c}_k = \bar{\sigma}_k + \frac{\delta_k}{2} \bar{b}_k$$