Heat semigroup, harmonic functions and heat flows on manifolds by stochastic analysis

Marc Arnaudon

Université de Poitiers, France

10ème Colloque Franco-Roumain de Mathématiques Appliquées
26-31 août 2010, Poitiers, France

1. Harnack inequality with power $\alpha > 1$
   - Parallel coupling
   - Parallel coupling with drift
   - Change of probability
   - Results
2. Gradient estimates
   - Bismut formula
   - Applications, extensions
3. Li-Yau type estimates
   - Some elementary submartingales
   - Entropy estimate
   - Li-Yau local estimate

(M, g) a Riemannian manifold

Heat equation

$$\frac{\partial}{\partial t} u = \frac{1}{2} \Delta u.$$

Minimal solution is represented as

$$u(x, T) = P_T f(x) = \mathbb{E}\left[1_{\{T<\xi(x)\}} f(X_T(x))\right]$$

where

- $f = u(\cdot, 0)$;
- $X_\cdot(x)$ Brownian motion started at x;
- $\xi(x)$ its lifetime.

(M, g) a Riemannian manifold

Heat equation

$$\frac{\partial}{\partial t} u = \frac{1}{2} \Delta u.$$

Minimal solution is represented as

$$u(x, T) = P_T f(x) = \mathbb{E}\left[1_{\{T<\xi(x)\}} f(X_T(x))\right]$$

where

- $f = u(\cdot, 0)$;
- $X_\cdot(x)$ Brownian motion started at x;
- $\xi(x)$ its lifetime.

(M, g) a Riemannian manifold

Heat equation

$$\frac{\partial}{\partial t} u = \frac{1}{2} \Delta u.$$

Minimal solution is represented as

$$u(x, T) = P_T f(x) = \mathbb{E}\left[1_{\{T<\xi(x)\}} f(X_T(x))\right]$$

where

- $f = u(\cdot, 0)$;
- $X_\cdot(x)$ Brownian motion started at x;
- $\xi(x)$ its lifetime.

Harnack inequality with power $\alpha > 1$

Assume M is stochastically complete ($\xi(x) = \infty$). Then

$$u(x, T) = P_T f(x) = \mathbb{E}[f(X_T(x))]$$

Fix $x_0 \in M$

$X_t(x_0)$ Brownian motion started at x_0.

For $x \in M$, let $X_t(x)$ satisfy $X_0(x) = x$ and

$$dX_t(x) = P_{X_t(x_0), X_t(x)} dX_t(x_0),$$

with $P_{y,z}$ parallel transport along the minimal geodesic from y to z.

Then

$$d\rho(X_t(x_0), X_t(x))) = I(X_t(x_0), X_t(x))\, dt$$

with

$$I(y, z) \le \sup_{w \in \gamma(y,z)} (-\mathrm{Ric}_w^+)$$

Assume M is stochastically complete ($\xi(x) = \infty$). Then

$$u(x, T) = P_T f(x) = \mathbb{E}[f(X_T(x))]$$

Fix $x_0 \in M$

$X_t(x_0)$ Brownian motion started at x_0.

For $x \in M$, let $X_t(x)$ satisfy $X_0(x) = x$ and

$$dX_t(x) = P_{X_t(x_0),X_t(x)} dX_t(x_0),$$

with $P_{y,z}$ parallel transport along the minimal geodesic from y to z.

Then

$$d\rho(X_t(x_0), X_t(x))) = I(X_t(x_0), X_t(x))\, dt$$

with

$$I(y, z) \le \sup_{w \in \gamma(y,z)} (-\mathrm{Ric}_w^+)$$

Assume M is stochastically complete ($\xi(x) = \infty$). Then

$$u(x, T) = P_T f(x) = \mathbb{E}[f(X_T(x))]$$

Fix $x_0 \in M$

$X_t(x_0)$ Brownian motion started at x_0.

For $x \in M$, let $X_t(x)$ satisfy $X_0(x) = x$ and

$$dX_t(x) = P_{X_t(x_0),X_t(x)} dX_t(x_0),$$

with $P_{y,z}$ parallel transport along the minimal geodesic from y to z.

Then

$$d\rho(X_t(x_0), X_t(x))) = I(X_t(x_0), X_t(x))\, dt$$

with

$$I(y, z) \le \sup_{w \in \gamma(y,z)} (-\mathrm{Ric}_w^+)$$

Assume M is stochastically complete ($\xi(x) = \infty$). Then

$$u(x, T) = P_T f(x) = \mathbb{E}[f(X_T(x))]$$

Fix $x_0 \in M$

$X_t(x_0)$ Brownian motion started at x_0.

For $x \in M$, let $X_t(x)$ satisfy $X_0(x) = x$ and

$$dX_t(x) = P_{X_t(x_0),X_t(x)} dX_t(x_0),$$

with $P_{y,z}$ parallel transport along the minimal geodesic from y to z.

Then

$$d\rho(X_t(x_0), X_t(x))) = I(X_t(x_0), X_t(x))\,dt$$

with

$$I(y, z) \le \sup_{w \in \gamma(y,z)} \left(-\mathrm{Ric}_w^+\right)$$

Assume M is stochastically complete ($\xi(x) = \infty$). Then

$$u(x, T) = P_T f(x) = \mathbb{E}[f(X_T(x))]$$

Fix $x_0 \in M$

$X_t(x_0)$ Brownian motion started at x_0.

For $x \in M$, let $X_t(x)$ satisfy $X_0(x) = x$ and

$$dX_t(x) = P_{X_t(x_0), X_t(x)} dX_t(x_0),$$

with $P_{y,z}$ parallel transport along the minimal geodesic from y to z.

Then

$$d\rho(X_t(x_0), X_t(x))) = I(X_t(x_0), X_t(x))\, dt$$

with

$$I(y, z) \le \sup_{w \in \gamma(y,z)} (-\operatorname{Ric}_w^+)$$

Idea: add a drift to $X_t(x)$ and compensate it by a change of probability.

For $x \in M$, let $\tilde{X}_t(x)$ satisfy $\tilde{X}_0(x) = x$ and

$$d\tilde{X}_t(x) = P_{X_t(x_0),\tilde{X}_t(x)} dX_t(x_0) + \left(\frac{\rho(x, x_0)}{T} + I(X_t(x_0), \tilde{X}_t(x))\right) n_{\tilde{X}_t(x),X_t(x_0)}\, dt$$

with $n(y, z) \in T_yM$ speed at y of the unit speed geodesic $y \to z$.

Let $\tau = \inf\{t \geq 0,\ \tilde{X}_t(x) = X_t(x_0)\}$ the coupling time. Then $\tau \leq T$. After time τ we let $\tilde{X}_t(x) = X_t(x_0)$.

Idea: add a drift to $X_t(x)$ and compensate it by a change of probability.

For $x \in M$, let $\tilde{X}_t(x)$ satisfy $\tilde{X}_0(x) = x$ and

$$\begin{aligned} d\tilde{X}_t(x) =& P_{X_t(x_0),\tilde{X}_t(x)} dX_t(x_0) \\ &+ \left(\frac{\rho(x, x_0)}{T} + I(X_t(x_0), \tilde{X}_t(x)) \right) n_{\tilde{X}_t(x), X_t(x_0)}\, dt \end{aligned}$$

with $n(y, z) \in T_y M$ speed at y of the unit speed geodesic $y \to z$.

Let $\tau = \inf\{t \geq 0,\ \tilde{X}_t(x) = X_t(x_0)\}$ the coupling time. Then $\tau \leq T$. After time τ we let $\tilde{X}_t(x) = X_t(x_0)$.

Idea: add a drift to $X_t(x)$ and compensate it by a change of probability.

For $x \in M$, let $\tilde X_t(x)$ satisfy $\tilde X_0(x) = x$ and

$$d\tilde X_t(x) = P_{X_t(x_0),\tilde X_t(x)} dX_t(x_0) + \left(\frac{\rho(x,x_0)}{T} + I(X_t(x_0),\tilde X_t(x))\right) n_{\tilde X_t(x),X_t(x_0)}\, dt$$

with $n(y,z) \in T_yM$ speed at y of the unit speed geodesic $y \to z$.

Let $\tau = \inf\{t \geq 0,\ \tilde X_t(x) = X_t(x_0)\}$ the coupling time. Then $\tau \leq T$. After time τ we let $\tilde X_t(x) = X_t(x_0)$.

Let

$$b_t = \left(\frac{\rho(x, x_0)}{T} + l(X_t(x_0), \tilde{X}_t(x)) \right) n_{\tilde{X}_t(x), X_t(x_0)},$$

$$N_t(x) = -\int_0^t \langle b_s, P_{X_s(x_0), \tilde{X}_s(x)} dX_s(x_0) \rangle,$$

$$R_t(x) = \exp\left(N_t(x) - \frac{1}{2}[N(x), N(x)]_t \right).$$

Under $\mathbb{Q} = R(x) \cdot \mathbb{P}$, $\tilde{X}_t(x)$ is a Brownian motion.

As a consequence, for $\alpha > 1$ and $\beta = \frac{\alpha}{\alpha - 1}$ the conjugate exponent,

$$P_T f(x) = \mathbb{E}\left[R_T(x) f(\tilde{X}_T(x)) \right] = \mathbb{E}\left[R_T(x) f(X_T(x_0)) \right]$$

$$P_T f(x) \leq \mathbb{E}\left[f^\alpha(X_T(x_0)) \right]^{1/\alpha} \|R_T(x)\|_\beta.$$

Let

$$b_t = \left(\frac{\rho(x, x_0)}{T} + l(X_t(x_0), \tilde{X}_t(x)) \right) n_{\tilde{X}_t(x), X_t(x_0)},$$

$$N_t(x) = -\int_0^t \langle b_s, P_{X_s(x_0), \tilde{X}_s(x)} dX_s(x_0) \rangle,$$

$$R_t(x) = \exp\left(N_t(x) - \frac{1}{2}[N(x), N(x)]_t \right).$$

Under $\mathbb{Q} = R(x) \cdot \mathbb{P}$, $\tilde{X}_t(x)$ is a Brownian motion.

As a consequence, for $\alpha > 1$ and $\beta = \frac{\alpha}{\alpha - 1}$ the conjugate exponent,

$$P_T f(x) = \mathbb{E}\left[R_T(x) f(\tilde{X}_T(x)) \right] = \mathbb{E}\left[R_T(x) f(X_T(x_0)) \right]$$

$$P_T f(x) \le \mathbb{E}\left[f^\alpha(X_T(x_0)) \right]^{1/\alpha} \|R_T(x)\|_\beta.$$

Let

$$b_t = \left(\frac{\rho(x, x_0)}{T} + I(X_t(x_0), \tilde{X}_t(x))\right) n_{\tilde{X}_t(x), X_t(x_0)},$$

$$N_t(x) = -\int_0^t \langle b_s, P_{X_s(x_0), \tilde{X}_s(x)} dX_s(x_0)\rangle,$$

$$R_t(x) = \exp\left(N_t(x) - \frac{1}{2}[N(x), N(x)]_t\right).$$

Under $\mathbb{Q} = R(x) \cdot \mathbb{P}$, $\tilde{X}_t(x)$ is a Brownian motion.

As a consequence, for $\alpha > 1$ and $\beta = \frac{\alpha}{\alpha - 1}$ the conjugate exponent,

$$P_T f(x) = \mathbb{E}\left[R_T(x) f(\tilde{X}_T(x))\right] = \mathbb{E}\left[R_T(x) f(X_T(x_0))\right]$$

$$P_T f(x) \leq \mathbb{E}\left[f^\alpha(X_T(x_0))\right]^{1/\alpha} \|R_T(x)\|_\beta.$$

Let

$$b_t = \left(\frac{\rho(x, x_0)}{T} + l(X_t(x_0), \tilde{X}_t(x)) \right) n_{\tilde{X}_t(x), X_t(x_0)},$$

$$N_t(x) = -\int_0^t \langle b_s, P_{X_s(x_0), \tilde{X}_s(x)} dX_s(x_0) \rangle,$$

$$R_t(x) = \exp\left(N_t(x) - \frac{1}{2}[N(x), N(x)]_t \right).$$

Under $\mathbb{Q} = R(x) \cdot \mathbb{P}$, $\tilde{X}_t(x)$ is a Brownian motion.

As a consequence, for $\alpha > 1$ and $\beta = \frac{\alpha}{\alpha - 1}$ the conjugate exponent,

$$P_T f(x) = \mathbb{E}\left[R_T(x) f(\tilde{X}_T(x)) \right] = \mathbb{E}\left[R_T(x) f(X_T(x_0)) \right]$$

$$P_T f(x) \le \mathbb{E}\left[f^\alpha(X_T(x_0)) \right]^{1/\alpha} \| R_T(x) \|_\beta.$$

Let

$$b_t = \left(\frac{\rho(x, x_0)}{T} + l(X_t(x_0), \tilde{X}_t(x)) \right) n_{\tilde{X}_t(x), X_t(x_0)},$$

$$N_t(x) = -\int_0^t \langle b_s, P_{X_s(x_0), \tilde{X}_s(x)} dX_s(x_0) \rangle,$$

$$R_t(x) = \exp\left(N_t(x) - \frac{1}{2}[N(x), N(x)]_t \right).$$

Under $\mathbb{Q} = R(x) \cdot \mathbb{P}$, $\tilde{X}_t(x)$ is a Brownian motion.

As a consequence, for $\alpha > 1$ and $\beta = \frac{\alpha}{\alpha - 1}$ the conjugate exponent,

$$P_T f(x) = \mathbb{E}\left[R_T(x) f(\tilde{X}_T(x)) \right] = \mathbb{E}\left[R_T(x) f(X_T(x_0)) \right]$$

$$P_T f(x) \le \mathbb{E}\left[f^\alpha(X_T(x_0)) \right]^{1/\alpha} \|R_T(x)\|_\beta.$$

$$P_T f(x) \le \mathbb{E}\left[f^\alpha(X_T(x_0))\right]^{1/\alpha} \|R_T\|_\beta$$ Harnack inequality with power $\alpha > 0$

Theorem

Assume $\mathrm{Ric}_x \ge -c(1+\rho_o(x)^2)$*. For any* $\varepsilon \in]0,1]$ *there exists* $c(\varepsilon) > 0$ *such that*

$$|P_t f|^\alpha(x) \le P_t|f|^\alpha(x_0) \exp\left[\frac{\alpha(\varepsilon\alpha+1)\rho(x,y)^2}{2(2-\varepsilon)(\alpha-1)t} + \frac{c(\varepsilon)\alpha^2(\alpha+1)^2}{(\alpha-1)^3}(1+\rho(x,y)^2)\rho(x,y)^2 + \frac{\alpha-1}{2}(1+\rho_o(x)^2)\right]$$

for all bounded measurable function f *on* M*.*

[Arnaudon-Thalmaier-F.Y Wang, Bull. Sci. Math. 2006]

$$P_T f(x) \le \mathbb{E}\left[f^\alpha(X_T(x_0))\right]^{1/\alpha} \|R_T\|_\beta \quad \text{Harnack inequality with power } \alpha > 0$$

Theorem

Assume $\mathrm{Ric}_x \ge -c(1+\rho_o(x)^2)$. *For any* $\varepsilon \in]0,1]$ *there exists* $c(\varepsilon) > 0$ *such that*

$$|P_t f|^\alpha(x) \le P_t |f|^\alpha(x_0) \exp\left[\frac{\alpha(\varepsilon\alpha+1)\rho(x,y)^2}{2(2-\varepsilon)(\alpha-1)t} + \frac{c(\varepsilon)\alpha^2(\alpha+1)^2}{(\alpha-1)^3}(1+\rho(x,y)^2)\rho(x,y)^2 + \frac{\alpha-1}{2}(1+\rho_o(x)^2)\right]$$

for all bounded measurable function f on M.

[Arnaudon-Thalmaier-F.Y Wang, Bull. Sci. Math. 2006]

From Harnack inequalities with power one can deduce heat kernel upper bounds with methods developped by Grigor'yan (J. Diff. Geom. 1997), F.-Z. Gong and F.Y Wang (Q. J. Math. 2001).

Corollary

For any $\delta > 2$, there exists $c(\delta) > 0$ such that

$$p_t(x,y) \leq \frac{\exp\left[-\frac{\rho(x,y)^2}{2\delta t} + c(\delta)(1+t+t^2+\rho_o(x)^2+\rho_o(y)^2)\right]}{\sqrt{\mu(B(x,\sqrt{2t}))\mu(B(y,\sqrt{2t}))}}$$

[A-T-W, Bull. Sci. Math. 2006]

Harnack inequalities with power are also useful to obtain log Sobolev inequalities, estimating eigenvalues of Laplacian, transportation cost inequalities, short time behaviour of transition probabilities.

From Harnack inequalities with power one can deduce heat kernel upper bounds with methods developped by Grigor'yan (J. Diff. Geom. 1997), F.-Z. Gong and F.Y Wang (Q. J. Math. 2001).

Corollary

For any $\delta > 2$, there exists $c(\delta) > 0$ such that

$$p_t(x,y) \le \frac{\exp\left[-\frac{\rho(x,y)^2}{2\delta t} + c(\delta)(1+t+t^2+\rho_o(x)^2+\rho_o(y)^2)\right]}{\sqrt{\mu(B(x,\sqrt{2t}))\mu(B(y,\sqrt{2t}))}}$$

[A-T-W, Bull. Sci. Math. 2006]

Harnack inequalities with power are also useful to obtain log Sobolev inequalities, estimating eigenvalues of Laplacian, transportation cost inequalities, short time behaviour of transition probabilities.

Gradient estimates

We obtained with the coupling and change of probability method

$$u(x,T) = \mathbb{E}\left[R_\tau(x)u(X_\tau(x_0), T-\tau)\right].$$

Differentiating with respect to x at $x = x_0$ yields formulas of the type

Theorem

$$\langle d_{x_0}u(\cdot,T), v\rangle = -\mathbb{E}\left[u(X_\tau(x_0), T-\tau)\int_0^\tau \langle W_s\dot{h}_s, dX_s(x_0)\rangle\right]$$

where $W_s : T_{x_0}M \to T_{X_s(x_0)}M$ is the so-called deformed parallel translation $DW_s = -\frac{1}{2}\,\mathrm{Ric}^\sharp\, ds$ and $h_s \in T_{x_0}M$ satisfies $h_0 = v$, $h_\tau = 0$ with some integrability condition.

Note the right hand side does not involve derivatives of u, and τ can be chosen smaller than the exit time of any domain.

We obtained a formula of the type

$$\langle d_{x_0} u(\cdot, T), v\rangle = -\mathbb{E}\left[u(X_\tau(x_0), T-\tau)\Phi_\tau(v)\right]$$

where $\Phi_\tau = \int_0^\tau \langle W_s \dot{h}_s, dX_s(x_0)\rangle$ depends only on the Brownian paths in a domain around x_0, and $\Phi_\tau \in \cap_{p>1} L^p$.

Applications, extensions:

- This formula extends to hypoelliptic diffusion $X_t(x)$;
- This formula extends to $u : [0, T] \times M \to N$, N a Riemannian manifold, such that $\frac{\partial}{\partial t} u = \frac{1}{2}\tau(u)$, $\tau(u)$ the tension field of u:

We obtained a formula of the type

$$\langle d_{x_0} u(\cdot, T), v \rangle = -\mathbb{E}\left[u(X_\tau(x_0), T - \tau)\Phi_\tau(v)\right]$$

where $\Phi_\tau = \int_0^\tau \langle W_s \dot{h}_s, dX_s(x_0) \rangle$ depends only on the Brownian paths in a domain around x_0, and $\Phi_\tau \in \cap_{p>1} L^p$.

Applications, extensions:

- This formula extends to hypoelliptic diffusion $X_t(x)$;
- This formula extends to $u : [0, T] \times M \to N$, N a Riemannian manifold, such that $\frac{\partial}{\partial t} u = \frac{1}{2}\tau(u)$, $\tau(u)$ the tension field of u:

We obtained a formula of the type

$$\langle d_{x_0} u(\cdot, T), v \rangle = -\mathbb{E}\left[u(X_\tau(x_0), T-\tau)\Phi_\tau(v) \right]$$

where $\Phi_\tau = \int_0^\tau \langle W_s \dot{h}_s, dX_s(x_0) \rangle$ depends only on the Brownian paths in a domain around x_0, and $\Phi_\tau \in \cap_{p>1} L^p$.

Applications, extensions:

- This formula extends to hypoelliptic diffusion $X_t(x)$;
- This formula extends to $u : [0, T] \times M \to N$, N a Riemannian manifold, such that $\frac{\partial}{\partial t} u = \frac{1}{2}\tau(u)$, $\tau(u)$ the tension field of u:

We obtained a formula of the type

$$\langle d_{x_0} u(\cdot, T), v \rangle = -\mathbb{E}\left[u(X_\tau(x_0), T-\tau)\Phi_\tau(v)\right]$$

where $\Phi_\tau = \int_0^\tau \langle W_s \dot{h}_s, dX_s(x_0)\rangle$ depends only on the Brownian paths in a domain around x_0, and $\Phi_\tau \in \cap_{p>1} L^p$.

Applications, extensions:

- This formula extends to hypoelliptic diffusion $X_t(x)$;
- This formula extends to $u : [0, T] \times M \to N$, N a Riemannian manifold, such that $\frac{\partial}{\partial t} u = \frac{1}{2}\tau(u)$, $\tau(u)$ the tension field of u:

Theorem

$$(T_{x_0} u_t)v$$
$$= -\mathbb{E}\left[\int_0^\tau \left\langle (W^N)_s^{-1}, d(u(X_\cdot(x_0), T - \cdot)_s\right\rangle \int_0^\tau \langle W_s^M h_s, dX_s(x_0)\rangle\right].$$

[Arnaudon-Thalmaier, J. Math. Pure Appl 1998]

Corollary

Assume $u : M \to N$ *satisfies* $\tau(u) = 0$ *and has generalized* K*-bounded dilatation:* $\lambda_1 \leq K^2(\lambda_2 + \ldots + \lambda_m)$, λ_i *eigenvalues of* $Tu^* \circ Tu$*. Assume furthermore that*

$$\mathrm{Ric}^M \geq -\alpha, \quad R^N \leq -\beta, \quad \alpha \geq 0, \ \ \beta > 0.$$

1) If M *is compact with nonempty boundary* ∂M *then*

$$\|T_a u\|^2 \leq \frac{K}{\beta} C(\mathrm{dist}(a, \partial M)),$$

$$\text{with} \quad C(r) = \frac{\pi^2}{4}(m+3)r^{-2} + \frac{\pi}{2}\sqrt{\alpha(m-1)}r^{-1} + \alpha.$$

2) If M *is complete then*

$$\|Tu\|^2 \leq \frac{\alpha K^2}{\beta} \quad \text{Shen (J. Reine Angew. Math. 1984)}$$

In particular, if $\mathrm{Ric}^M \geq 0$ *and* $R^N \leq -\beta$*, then harmonic maps* $u : M \to N$ *of generalized* K*-bounded dilatation are constant.*

Corollary

Assume $u : M \to N$ satisfies $\tau(u) = 0$ and has generalized K-bounded dilatation: $\lambda_1 \le K^2(\lambda_2 + \ldots + \lambda_m)$, λ_i eigenvalues of $Tu^* \circ Tu$. Assume furthermore that

$$\mathrm{Ric}^M \ge -\alpha, \quad R^N \le -\beta, \quad \alpha \ge 0, \ \beta > 0.$$

1) If M is compact with nonempty boundary ∂M then

$$\|T_a u\|^2 \le \frac{K}{\beta} C(\mathrm{dist}(a, \partial M),$$

$$\text{with} \quad C(r) = \frac{\pi^2}{4}(m+3)r^{-2} + \frac{\pi}{2}\sqrt{\alpha(m-1)}r^{-1} + \alpha.$$

2) If M is complete then

$$\|Tu\|^2 \le \frac{\alpha K^2}{\beta} \quad \text{Shen (J. Reine Angew. Math. 1984)}$$

In particular, if $\mathrm{Ric}^M \ge 0$ and $R^N \le -\beta$, then harmonic maps $u : M \to N$ of generalized K-bounded dilatation are constant.

Corollary

Assume $u : M \to N$ satisfies $\tau(u) = 0$ and has generalized K-bounded dilatation: $\lambda_1 \le K^2(\lambda_2 + \ldots + \lambda_m)$, λ_i eigenvalues of $Tu^* \circ Tu$. Assume furthermore that

$$\mathrm{Ric}^M \ge -\alpha, \quad R^N \le -\beta, \quad \alpha \ge 0, \ \beta > 0.$$

1) If M is compact with nonempty boundary ∂M then

$$\|T_a u\|^2 \le \frac{K}{\beta} C(\mathrm{dist}(a, \partial M)),$$

$$\text{with} \quad C(r) = \frac{\pi^2}{4}(m+3)r^{-2} + \frac{\pi}{2}\sqrt{\alpha(m-1)}r^{-1} + \alpha.$$

2) If M is complete then

$$\|Tu\|^2 \le \frac{\alpha K^2}{\beta} \quad \textit{Shen (J. Reine Angew. Math. 1984)}$$

In particular, if $\mathrm{Ric}^M \ge 0$ and $R^N \le -\beta$, then harmonic maps $u : M \to N$ of generalized K-bounded dilatation are constant.

Li-Yau type estimates

Recall $\frac{\partial}{\partial t}u = \frac{1}{2}\Delta u$, $\quad u(x,T) = P_T f(x) = \mathbb{E}\left[f(X_T(x))\right]$, where $f = u(\cdot, 0)$, $X_\cdot(x)$ Brownian motion started at x.

Assume there exists $k \geq 0$ such that $\mathrm{Ric} \geq -k$ on M. Fix $T > 0$ and let for $t \in [0, T]$

$$u_t = u(T-t, X_t(x)), \quad \nabla u_t = \nabla u(T-t, \cdot)(X_t(x)).$$

An elementary calculation proves that

$$N_t := \frac{T-t}{2(1+k(T-t))}\frac{|\nabla u_t|^2}{u_t} + u_t \log u_t$$

is a local submartingale.

Recall $\dfrac{\partial}{\partial t}u = \dfrac{1}{2}\Delta u$, $\quad u(x,T) = P_T f(x) = \mathbb{E}\left[f(X_T(x))\right]$, where $f = u(\cdot, 0)$, $X_\cdot(x)$ Brownian motion started at x.

Assume there exists $k \geq 0$ such that $\mathrm{Ric} \geq -k$ on M. Fix $T > 0$ and let for $t \in [0,T]$

$$u_t = u(T-t, X_t(x)), \quad \nabla u_t = \nabla u(T-t, \cdot)(X_t(x)).$$

An elementary calculation proves that

$$N_t := \frac{T-t}{2(1+k(T-t))}\frac{|\nabla u_t|^2}{u_t} + u_t \log u_t$$

is a local submartingale.

Recall $\dfrac{\partial}{\partial t}u = \dfrac{1}{2}\Delta u, \quad u(x,T) = P_T f(x) = \mathbb{E}\left[f(X_T(x))\right]$, where $f = u(\cdot, 0)$, $X_\cdot(x)$ Brownian motion started at x.

Assume there exists $k \geq 0$ such that $\mathrm{Ric} \geq -k$ on M. Fix $T > 0$ and let for $t \in [0,T]$

$$u_t = u(T-t, X_t(x)), \quad \nabla u_t = \nabla u(T-t, \cdot)(X_t(x)).$$

An elementary calculation proves that

$$N_t := \frac{T-t}{2(1+k(T-t))}\frac{|\nabla u_t|^2}{u_t} + u_t \log u_t$$

is a local submartingale.

Theorem

We have

$$\left|\frac{\nabla P_T f}{P_T f}\right|^2 (x) \le 2\left(\frac{1}{T} + k\right) P_T\left(\frac{f}{P_T f(x)} \log \frac{f}{P_T f(x)}\right)(x)$$

Proof.

$N_0 \le \mathbb{E}[N_T]$. □

Theorem

We have

$$\left|\frac{\nabla P_T f}{P_T f}\right|^2 (x) \leq 2\left(\frac{1}{T}+k\right) P_T\left(\frac{f}{P_T f(x)} \log \frac{f}{P_T f(x)}\right)(x)$$

Proof.

$N_0 \leq \mathbb{E}[N_T]$. □

Corollary

For all $\delta > 0$,

$$\begin{aligned}|\nabla P_T f|(x) \leq & \frac{1}{2\delta}\left(\frac{1}{T}+k\right) P_T f(x) \\ & + \delta\left[P_T(f\log f)(x) - P_T f(x)\log P_T f(x)\right].\end{aligned}$$

Integrating this inequality yields Harnack inequalities with power.

Corollary

For all $\delta > 0$,

$$|\nabla P_T f|(x) \leq \frac{1}{2\delta}\left(\frac{1}{T}+k\right) P_T f(x) + \delta\left[P_T(f\log f)(x) - P_T f(x)\log P_T f(x)\right].$$

Integrating this inequality yields Harnack inequalities with power.

Corollary

If M is complete, $\mathrm{Ric} \geq -k$, $A = \sup_{M\times[0,T]} u$ *then*

$$\frac{|\nabla u|^2}{u^2}(x,T) \leq 2\left(\frac{1}{T}+k\right)\log\frac{A}{u(x,T)}.$$

[Arnaudon-Thalmaier Adv. Stud. Pure Math. 2010]
(for compact M, Hamilton An. Geom. 1993).

Other elementary submartingales yield with the same method Li-Yau global and local estimates:

Theorem

Let u be a solution of the heat equation on $D \times [0, T]$, with $D \subset M$ open and relatively compact. Assume that u is positive and continuous on $\bar{D} \times [0, T]$. Furthermore let $-k$ ($k \geq 0$) be a lower bound for the Ricci curvature on D. Fix $x \in D$ and let $a \in]1, 2[$. For any $\beta > 0$ we have

$$\left|\frac{\nabla u_0}{u_0}\right|^2 - a\frac{\Delta u_0}{u_0} \leq \frac{4n}{(4-a^2)T} + \frac{\pi^2 n\left[\left(1+\beta+\frac{1}{a-1}\right)n+3\right]}{(4-a^2)\rho(x,\partial D)^2} + \left(\frac{1}{(4-a^2)\beta} + \frac{1}{a-1}\right)nk.$$

[A-T Adv. Stud. Pure Math. 2010]

Other elementary submartingales yield with the same method Li-Yau global and local estimates:

Theorem

Let u be a solution of the heat equation on $D \times [0, T]$, with $D \subset M$ open and relatively compact. Assume that u is positive and continuous on $\bar{D} \times [0, T]$. Furthermore let $-k$ ($k \geq 0$) be a lower bound for the Ricci curvature on D. Fix $x \in D$ and let $a \in]1, 2[$. For any $\beta > 0$ we have

$$\left|\frac{\nabla u_0}{u_0}\right|^2 - a\frac{\Delta u_0}{u_0} \leq \frac{4n}{(4-a^2)T} + \frac{\pi^2 n\left[\left(1+\beta+\frac{1}{a-1}\right)n+3\right]}{(4-a^2)\rho(x,\partial D)^2} + \left(\frac{1}{(4-a^2)\beta}+\frac{1}{a-1}\right)nk.$$

[A-T Adv. Stud. Pure Math. 2010]