A Lie algebraic generalization of the Mumford system, its symmetries and its multi-Hamiltonian structure

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Abstract

In this paper we generalize the Mumford system which describes for any fixed g all linear flows on all hyperelliptic Jacobians of dimension g. The phase space of the Mumford system consists of triples of polynomials, subject to certain degree constraints, and is naturally seen as an affine subspace of the loop algebra of $\mathfrak{sl}(2)$. In our generalizations to an arbitrary simple Lie algebra $\mathfrak{g}$ the phase space consists of $\dim \mathfrak{g}$ polynomials, again subject to certain degree constraints. This phase space and its multi-Hamiltonian structure is obtained by a Poisson reduction along a subvariety N of the loop algebra $\mathfrak{g}((\lambda^{-1}))$ of $\mathfrak{g}$. Since N is not a Poisson subvariety for the whole multi-Hamiltonian structure we prove an (algebraic) Poisson reduction theorem for reduction along arbitrary subvarieties of an affine Poisson variety; this theorem is similar in spirit to the Marsden-Ratiu reduction theorem.

We also give a different perspective on the multi-Hamiltonian structure of the Mumford system (and its generalizations) by introducing a master symmetry; this master symmetry can be described on the loop algebra $\mathfrak{g}((\lambda^{-1}))$ as the derivative in the direction of λ and is shown to survive the Poisson reduction. When acting (as a Lie derivative) on one of the Poisson structures of the system it produces a next one, similarly when acting on one of the Hamiltonians (in involution) or their (commuting) vector fields it produces a next one. In this way we arrive at several multi-Hamiltonian hierarchies, built up by a master symmetry.

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1. Introduction

In its original form the Mumford system consists, for every positive integer g, of a family of vector fields on an affine space (of dimension $3g+1$) of triples of polynomials $(U(\lambda), V(\lambda), W(\lambda))$ (see [Mum] p. 3.43). The simplest member of this family has the form

$$\begin{aligned}
\dot{U}(\lambda) &= V(\lambda),\\
\dot{V}(\lambda) &= \frac{1}{2}[-W(\lambda) + (\lambda - U_{g-1} + W_g)U(\lambda)],\\
\dot{W}(\lambda) &= -(\lambda - U_{g-1} + W_g)V(\lambda).
\end{aligned} \tag{1.1}$$

The three polynomials are subject to the restrictions that U and W are monic of degrees g and $g+1$, while V has degree less than g. U_i is the coefficient of λ^i in $U(\lambda)$ and similarly for V and W. A simple computation shows that

$$(U(\lambda)W(\lambda) + V^2(\lambda))^{\cdot} = 0$$

for the above vector field, and similarly for the other members of the family. It follows that if one associates an algebraic curve (of genus g) to every point $(U(\lambda), V(\lambda), W(\lambda))$ of phase space by the equation

$$\mu^2 = U(\lambda)W(\lambda) + V^2(\lambda), \tag{1.2}$$

then this curve is invariant under the flow of these vector fields. This property is "explained" by Mumford who shows that the generic orbit, traced out by the flow of these vector fields, is an affine part of the Jacobian of the curve (1.2) associated to any of its points $(U(\lambda), V(\lambda), W(\lambda))$ and that the flows of these vector fields are linear (the Jacobian of a curve is a complex torus, hence has a linear structure). Note that this implies automatically that these vector fields commute, a property reminiscent of integrable systems. Upon introducing a Hamiltonian structure for which Mumford's vector fields are Hamiltonian it turns out that the Mumford system is indeed an example of an integrable system (such a Hamiltonian structure was however only introduced later).

It turns out that the Mumford system and some of its generalizations appear in many different contexts, although sometimes in a disguised form and often without reference to its Hamiltonian structure. It appears in the description of rings of commuting differential operators, going back to the early papers of Burchnall and Chaundy (see [BC] or [Pre]; for a different but equivalent description see [Sch]), it is a limit of the classical Schlesinger equations which describe isomonodromy deformation (see [Gar]), many classical integrable systems are isomorphic to a subsystem of the Mumford system, sometimes up to a cover (see [Van3]) and the Mumford system appears as the simplest of a large class of integrable systems on the moduli space of Higgs bundles on a Riemann surface, the latter being in this case just the Riemann sphere (see [DM]).

The purpose of this paper is to combine the ideas in [MM], [MR], [RS3] and [Sch] to generalize the Mumford system and to describe the symmetries and the multi-Hamiltonian structure of its generalizations. Let us describe these ideas.

(1) The main idea from [RS3], which is recalled in Section 2.1, is that the loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g}((\lambda^{-1}))$ of any semi-simple Lie algebra has a (multi-) Hamiltonian structure which restricts to the finite-dimensional spaces $\tilde{\mathfrak{g}}_n^\beta$ of polynomials with leading term $\beta\lambda^n$, where $\beta \in \mathfrak{g}$. A natural class of functions in involution leads, in many cases, to an integrable system on $\tilde{\mathfrak{g}}_n^\beta / G_\beta$ where G_β is the isotropy group of β. We will show that for well-chosen β a Poisson reduction on an affine subspace N of $\tilde{\mathfrak{g}}_n^\beta$ with respect to a subgroup G_β^- of G_β will lead to the generalization of the Mumford system:

while $\tilde{\mathfrak{g}}_n^\beta/G_\beta$ is never an affine space the quotient which we describe will be a space of $(\dim\mathfrak{g})$-tuples of polynomials with degree constraints, precisely as in the case of the Mumford system.

(2) The multi-Hamiltonian structure on $\tilde{\mathfrak{g}}_n^\beta$, which is given as a family of compatible Poisson brackets, does not restrict to any subvariety of $\tilde{\mathfrak{g}}_n^\beta$ (although some brackets do). Therefore we prove a general (algebraic) Poisson reduction theorem which is similar in spirit to the Marsden-Ratiu reduction theorem (see [MR]). In our theorem we consider a subvariety N of a Poisson variety M on which an affine Poisson group G acts (leaving N invariant). Assuming that the action is Poisson we give a necessary and sufficient condition for the Poisson structure on M to descend to a Poisson structure on N/G. The reduction theorem will be proven in Section 3.

(3) The next question then, which turns out to be Lie-algebraic in nature, is how to pick the subspace N and the group G_β^- such that the quotient is an affine space which can be naturally identified with a subspace of N. If we pick in the case of $\mathfrak{sl}(r+1)$ the leading coefficient β to be a generic lower triangular matrix, then our condition for Poisson reducibility implies that we can only reduce along the hyperplane $N \subset \tilde{\mathfrak{g}}_n^\beta$ which is obtained by fixing one of the entries (the entry at position $(1, r+1)$) of the coefficient of λ^{n-1}. In this case the quotient N/G_β is an affine space if and only if this entry has been fixed to a value different from 0. Notice that in this case $\alpha+\beta$ is regular, where

$$\alpha = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}.$$

The precise Lie-algebraic condition imposed on β is that it is a principal nilpotent element; since all principal nilpotent elements of a simple Lie algebra are conjugate we may take β to be given by $\beta = \sum_{i=1}^r F_i$, where $\{H_i, E_i, F_i\}_{i=1,\dots,r}$ is a Weyl basis of $\mathfrak{g}$. Then the condition which defines $N \subset \tilde{\mathfrak{g}}_n^\beta$ is that its elements $\beta\lambda^n + \sum_{i=0}^{n-1} x_i\lambda^i$ satisfy $\Pi_k x_{n-1} = \alpha$; α is any non-zero top-element in the gradation $\oplus_{i=-k}^k \mathfrak{g}_i$ of $\mathfrak{g}$, which is associated to the Weyl basis, i.e., $0 \neq \alpha \in \mathfrak{g}_k$ and Π_k is the projection onto $\mathfrak{g}_k$. This leads to the proper Lie algebraic setup for a first generalization of the Mumford system to any simple Lie algebra. Indeed, if we define α, β and n in the above way for an arbitrary simple Lie algebra $\mathfrak{g}$ then the whole multi-Hamiltonian hierarchy of Poisson structures and the algebra of functions in involution descend to the quotient which is naturally identified with an affine subspace N_0 of $\tilde{\mathfrak{g}}_n^\beta$. Notice that the group by which we reduce is in this case the full group G_β, which is Abelian, and that the action is Hamiltonian. In the case of $\mathfrak{sl}(2)$ we recover the Mumford system, while in the case of $\mathfrak{sl}(r+1)$ we find a generalization of the Mumford system due to Donagi-Markman (see [DM]).

(4) When β is not a principal nilpotent element then the whole structure theory of simple Lie algebras comes into play. Indeed, we will rely heavily on the beautiful paper [Kos1] by Kostant. As we learned from A. Schwarz, for any d coprime to $r+1$, the space of matrices in $\mathfrak{sl}(r+1)$ of the form

$$\begin{pmatrix} 0 & 0 \\ I_{r+1-d} & 0 \end{pmatrix}\lambda^n + \begin{pmatrix} \star & M_d \\ \star & \star \end{pmatrix}\lambda^{n-1} + \sum_{i=0}^{n-2} x_i\lambda^i, \tag{1.3}$$

where M_d is any lower triangular matrix of size d with ones on the diagonal, appears in the description of the solutions to the string equation $[P,Q]=1$, or, in an analogous way, of the solutions to the commutativity equation $[P,Q]=0$; in these equations P and Q are differential operators subject to certain normalizations (see [Sch] and [KV]). Notice that the matrix $\begin{pmatrix} 0 & I_d \\ I_{r+1-d} & 0 \end{pmatrix}$, which is obtained from the leading coefficients, is regular due to the fact that d and $r+1$ are coprime. Applying our reduction theorem to the subspace N of matrices of the form (1.3) we find that the multi-Hamiltonian hierarchy reduces and that the quotient space can be naturally identified with an affine subspace of N. A key information which we also learned from Schwarz' description is that we should not act with the full isotropy group G_β but with the subgroup G_β^- of lower triangular matrices in G_β (with ones on the diagonal). In Section 4 we implement these ideas in the case of an arbitrary simple Lie algebra $\mathfrak{g}$ and find for any homogeneous β the corresponding subspaces N of $\tilde{\mathfrak{g}}_n^\beta$ to which the multi-Hamiltonian hierarchies reduce; moreover we give the choices of β which lead to a quotient which is affine, thereby giving the Lie algebraic interpretation of the coprime condition which appears in the case of $\mathfrak{sl}(r+1)$. Notice that in this more general case G_β^- is not Abelian; moreover it can be shown that the action is not Hamiltonian.

(5) Another idea, which we learned from [MM], is that multi-Hamiltonian structures are often built up from a basic one by applying a master symmetry, i.e., there is a basic Poisson structure whose successive Lie derivatives with respect to a certain vector field $\mathcal{V}$ provides a linear basis for all the Poisson structures. This vector field $\mathcal{V}$ is given on $\tilde{\mathfrak{g}}_n^\beta$ by

$$\dot{X}(\lambda) = \frac{\partial}{\partial\lambda} X(\lambda),$$

and it generates all the Hamiltonians and commuting vector fields starting from a few basic ones. We show that, as a consequence of our reduction theorem, the vector field $\mathcal{V}$ projects on the quotient to a master symmetry which builds up the multi-Hamiltonian structure on the quotient space N_0. Since the operations of reduction and taking the Lie derivative commute it follows that it suffices to compute the reduction of one Poisson structure on $\tilde{\mathfrak{g}}_n^\beta$ (in fact precisely the unique linear one) and apply successive Lie derivatives to it to find the other reduced Poisson structures. The basic properties of the vector field $\mathcal{V}$ will be given in Section 2.2, we discuss the reduction of symmetries at the end of Section 3 and we find the reduced master symmetry in the case of the loop algebra in Section 4.2.

We will end this paper with a list of examples (Section 5). We will first show how our construction specializes in the case of $\mathfrak{sl}(2)$ to the Mumford system. In this case we will explicitly compute all reduced brackets. We will give an explicit description of the quotient space (as a space of polynomials) for the classical Lie algebras and for G_2.

In conclusion we have a complete description of the multi-Hamiltonian structure of the Mumford system and its generalizations to arbitrary simple Lie algebras. It seems non-trivial but interesting to do the same for the even master system (see [Van1]), which also describes all linear flows on all hyperelliptic Jacobians by equations which are similar to (1.1). A proof of the integrability of the systems on the reduced space N_0 involves algebraic geometric arguments, revealing also their algebraic complete integrability (this is done for the case of $\mathfrak{sl}(n)$ in [DM]); we leave this and a study of the algebraic geometry of the fibers of the Hamiltonians — some of which are certainly interesting Abelian varieties — for the future.

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2. Multi-Hamiltonian hierarchies and master symmetries on loop algebras

In this section we introduce a large class of multi-Hamiltonian hierarchies on the loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g}((\lambda^{-1}))$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra, which is equipped with an ad-invariant non-degenerate inner product. The multi-Hamiltonian structure of interest here was first introduced in [RS3] by using several R-brackets and is recalled in Paragraph 2.1. We exhibit several multi-Hamiltonian hierarchies, whose Hamiltonians are seen to be in involution by the classical R-matrix argument; we provide an alternative proof which uses the classical Lenard-Magri scheme. Following an idea of [MM] we show in Paragraph 2.2 that the different Poisson brackets which make up the multi-Hamiltonian structure are connected by the Lie derivative along a master symmetry $\mathcal{V}$, thereby giving another, more geometric, construction of these brackets. The vector field $\mathcal{V}$ allows one to pass from one Hamiltonian (and its vector field with respect to any of the Poisson brackets) to another, hence playing a similar role as the recursion operator in the case of Poisson-Nijenhuis manifolds (see [KM]).

2.1. The loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g}((\lambda^{-1}))$ and its Poisson brackets

Let $\mathfrak{g}$ be a (finite-dimensional) Lie algebra and $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ a non-degenerate inner product which is ad-invariant, $\langle x,[y,z]\rangle_{\mathfrak{g}} = \langle [x,y],z\rangle_{\mathfrak{g}}$. We fix a basis $\{e_a\}_{a\in I}$ for $\mathfrak{g}$ and define linear forms $\xi_a\colon \mathfrak{g}\to \mathbf{C}$ by $\xi_a = \langle\cdot,e_a\rangle_{\mathfrak{g}}$. We look at $\mathfrak{g}$ as an affine (algebraic) variety, in particular we consider $\mathcal{O}(\mathfrak{g}) = \mathbf{C}\left[\xi_a\right]_{a\in I}$ as its algebra of (regular) functions. For any $F\in\mathcal{O}(\mathfrak{g})$, its gradient $\nabla F(x)$ at $x\in\mathfrak{g}$ is defined by

$$\langle \nabla F(x), y\rangle_{\mathfrak{g}} = \frac{d}{dt}_{|t=0} F(x+ty) \qquad \forall y\in\mathfrak{g}.$$

For any $a\in I$ and $F\in\mathcal{O}(\mathfrak{g})$ the map $x\mapsto \langle\nabla F(x),e_a\rangle_{\mathfrak{g}}$ belongs to $\mathcal{O}(\mathfrak{g})$. It follows that for any $F,G\in\mathcal{O}(\mathfrak{g})$ the Poisson bracket $\{F,G\}$, defined by

$$\{F,G\}(x) = \langle x, [\nabla F(x), \nabla G(x)]\rangle_{\mathfrak{g}}, \tag{2.1}$$

also belongs to $\mathcal{O}(\mathfrak{g})$, making $\mathcal{O}(\mathfrak{g})$ into a Poisson algebra.

From $\mathfrak{g}$ we construct the loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g}((\lambda^{-1})) = \mathfrak{g}[\lambda]\oplus\lambda^{-1}\mathfrak{g}[[\lambda^{-1}]]$. Elements of the loop algebra will be denoted by capital letters; for an element $X = X(\lambda) = \sum x_i\lambda^i\in\tilde{\mathfrak{g}}$ we write $X = X_+ + X_-$ according to the above (vector space) decomposition. The inner product $\langle\cdot\,,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ leads to an inner product $\langle\cdot\,,\cdot\rangle$ on $\tilde{\mathfrak{g}}$ via

$$\langle X(\lambda), Y(\lambda)\rangle = \sum_{i+j=-1}\langle x_i, y_j\rangle_{\mathfrak{g}}.$$

By a slight abuse of notation one often writes $\operatorname{Res}\langle X(\lambda),Y(\lambda)\rangle_{\mathfrak{g}}$ for the above right hand side; here $\operatorname{Res}\sum x_i\lambda^i = x_{-1}$. Clearly $\langle\cdot\,,\cdot\rangle$ is ad-invariant and non-degenerate just as $\langle\cdot\,,\cdot\rangle_{\mathfrak{g}}$ is. For $a\in I,\ i\in\mathbf{Z}$ we define elements $E_a^i = e_a\lambda^i$ of $\tilde{\mathfrak{g}}$ and linear functions $\xi_a^i = \left\langle\cdot, E_a^{-i-1}\right\rangle$. We wish to introduce an algebra $\mathcal{O}(\tilde{\mathfrak{g}})$ of functions on $\tilde{\mathfrak{g}}$ for which we can define a gradient and a Poisson bracket as in the case of $\mathfrak{g}$, but which is large enough to contain functions of the type $X(\lambda)\mapsto \operatorname{Res} H(X(\lambda))$ (for $H\in\mathcal{O}(\mathfrak{g})$), which will be important later. To do this we first define on $\tilde{\mathfrak{g}}_{\le n} = \lambda^n\mathfrak{g}[[\lambda^{-1}]]$ an algebra of functions by

$$\mathcal{O}(\tilde{\mathfrak{g}}_{\le n}) = \mathbf{C}\left[\xi_a^i\right]_{\substack{a\in I\\ i\le n}}$$

and obtain from it the following algebra of functions on $\tilde{\mathfrak{g}}$:

$$\mathcal{O}(\tilde{\mathfrak{g}}) = \left\{ F : \tilde{\mathfrak{g}} \to \mathbf{C} \mid \forall n \in \mathbf{Z} : F_{|\tilde{\mathfrak{g}}_{\leq n}} \in \mathcal{O}(\tilde{\mathfrak{g}}_{\leq n}) \right\}.$$

Thus, elements of $\mathcal{O}(\tilde{\mathfrak{g}})$ restrict to polynomials on all subspaces $\tilde{\mathfrak{g}}_{\leq n}$. As in the case of $\mathfrak{g}$ the gradient $\nabla F(X)$ of a function $F \in \mathcal{O}(\tilde{\mathfrak{g}})$ at $X \in \tilde{\mathfrak{g}}$ is defined by

$$\langle \nabla F(X), Y \rangle = \frac{d}{dt}_{|t=0} F(X + tY) \qquad \forall Y \in \tilde{\mathfrak{g}}. \tag{2.2}$$

Proposition 2.1 *For any $X \in \tilde{\mathfrak{g}}$ and $F \in \mathcal{O}(\tilde{\mathfrak{g}})$, $\nabla F(X)$ is well-defined by (2.2) and belongs to $\tilde{\mathfrak{g}}$. For any $F, G \in \mathcal{O}(\tilde{\mathfrak{g}})$ the Poisson bracket $\{F, G\}$, defined by*

$$\{F, G\}(X) = \langle X, [\nabla F(X), \nabla G(X)] \rangle ,$$

belongs to $\mathcal{O}(\tilde{\mathfrak{g}})$, making $\mathcal{O}(\tilde{\mathfrak{g}})$ into a Poisson algebra.

Proof

The fact that the gradient is well-defined follows from non-degeneracy of $\langle \cdot, \cdot \rangle$; in fact, for any $j \in \mathbf{Z}$ the coefficient $(\nabla F(X))_j \in \mathfrak{g}$ is given by

$$\langle (\nabla F(X))_j, e_a \rangle_{\mathfrak{g}} = \left\langle \nabla F(X), E_a^{-j-1} \right\rangle = \frac{d}{dt}_{|t=0} F(X + tE_a^{-j-1}).$$

If $X \in \tilde{\mathfrak{g}}_{\leq n}$ then $F(X + tE_a^j)$ is independent of t for j sufficiently small, since $F_{|\tilde{\mathfrak{g}}_{\leq n}}$ is a polynomial. Thus, $(\nabla F(X))_j$ is zero for j sufficiently large and $\nabla F(X) \in \tilde{\mathfrak{g}}$. Further, $X \mapsto \left\langle \nabla F(X), E_a^j \right\rangle$ belongs to $\mathcal{O}(\tilde{\mathfrak{g}})$ for any $j \in \mathbf{Z}$ since the restriction to any $\tilde{\mathfrak{g}}_{\leq n}$ of the map

$$X \mapsto \frac{d}{dt}_{|t=0} F(X + tE_a^j)$$

is just a polynomial (in this formula, use $F_{|\tilde{\mathfrak{g}}_{\leq m}}$ where $m = \max\{n, j\}$). As a corollary, if F, $G \in \mathcal{O}(\tilde{\mathfrak{g}})$ then the map

$$X \mapsto \langle X, [\nabla F(X), \nabla G(X)] \rangle$$

belongs to $\mathcal{O}(\tilde{\mathfrak{g}})$, giving a bracket $\{\cdot\,,\cdot\} : \mathcal{O}(\tilde{\mathfrak{g}}) \times \mathcal{O}(\tilde{\mathfrak{g}}) \to \mathcal{O}(\tilde{\mathfrak{g}})$. The fact that it satisfies the Jacobi identity follows from the fact that (2.1) satisfies the Jacobi identity. ∎

Following [RS3] we introduce a family R_l of endomorphisms of $\tilde{\mathfrak{g}}$ by

$$R : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}} : X \mapsto X_+ - X_-,$$
$$R_l : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}} : X \mapsto R(\lambda^l X).$$

Proposition 2.2 ([RS3]) *For any $l \in \mathbf{Z}$ a Poisson bracket on $\mathcal{O}(\tilde{\mathfrak{g}})$ is defined by*

$$\{F, G\}_l(X) = \frac{1}{2} \langle X, [R_l \nabla F(X), \nabla G(X)] + [\nabla F(X), R_l \nabla G(X)] \rangle .$$

Moreover the brackets $\{\cdot,\cdot\}_l$, $l \in \mathbf{Z}$ form a family of compatible Poisson brackets, i.e., any linear combination of these brackets is a Poisson bracket.

As above these brackets are taken as brackets on $\mathcal{O}(\tilde{\mathfrak{g}})$. We call them R-brackets and call $\{\cdot,\cdot\}$ the canonical Lie-Poisson bracket on $\tilde{\mathfrak{g}}$. If we denote the structure constants of $\mathfrak{g}$ with respect to the basis $\{e_a\}_{a\in I}$ by C^c_{ab}, i.e., $[e_a, e_b] = \sum_{c\in I} C^c_{ab} e_c$, then one easily finds by using $\nabla \xi^i_a = E^{-i-1}_a$ that

$$\{\xi^i_a, \xi^j_b\}_l = \epsilon^{ij}_l \sum_{c\in I} C^c_{ab} \xi^{i+j+1-l}_c, \tag{2.3}$$

where $\epsilon^{ij}_l = 1$ if $i, j < l$ and $\epsilon^{ij}_l = -1$ if $i, j \geq l$; otherwise $\epsilon^{ij}_l = 0$. The R-brackets have two remarkable properties which make them more relevant for integrable systems than the canonical Lie-Poisson bracket on $\tilde{\mathfrak{g}}$. The first property, which follows immediately from (2.3), is that if $-p \leq l \leq q+1$ then $\{\cdot,\cdot\}_l$ restricts to the following natural finite-dimensional subspace of $\tilde{\mathfrak{g}}$,

$$\tilde{\mathfrak{g}}_{-p,q} = \left\{ \sum_{i=-p}^{q} x_i \lambda^i \mid x_i \in \mathfrak{g} \right\}. \tag{2.4}$$

Since multiplication by λ^p induces an isomorphism $\left(\tilde{\mathfrak{g}}_{-p,q}, \{\cdot,\cdot\}_l\right) \longrightarrow \left(\tilde{\mathfrak{g}}_{0,p+q}, \{\cdot,\cdot\}_{l+p}\right)$ we may restrict ourselves to the spaces $\tilde{\mathfrak{g}}_{0,n}$ of matrices which are polynomial (in λ) of degree at most n. In fact we will be interested in the affine subspaces of $\tilde{\mathfrak{g}}_{0,n}$ defined by

$$\tilde{\mathfrak{g}}^\beta_n = \left\{ \sum_{i=0}^{n} x_i \lambda^i \in \tilde{\mathfrak{g}} \mid x_n = \beta \right\}, \tag{2.5}$$

where β is any fixed element in $\mathfrak{g}$. The family of R-brackets which restricts to $\tilde{\mathfrak{g}}^\beta_n$ is also computed at once from (2.3) and is given in the following proposition.

Proposition 2.3 *If β is not a central element in $\mathfrak{g}$ then the Poisson structure $\sum_{l=-\infty}^{\infty} c_l \{\cdot,\cdot\}_l$ restricts to $\tilde{\mathfrak{g}}^\beta_n$ if and only if $c_l = 0$ for $l < 0$ and for $l > n$.*

The second remarkable property of the R-brackets is that the Ad-invariant functions on $\mathfrak{g}$ lead to a large subalgebra $\mathcal{A}$ of $\mathcal{O}(\tilde{\mathfrak{g}})$ which is involutive with respect to all these brackets. Indeed, a function $H \in \mathcal{O}(\mathfrak{g})$ induces a function $H : \tilde{\mathfrak{g}} \to \mathbf{C}((\lambda^{-1}))$ and hence leads for any $i \in \mathbf{Z}$ to a function H_i on $\tilde{\mathfrak{g}}$, defined by

$$H_i(X(\lambda)) = \operatorname{Res} \frac{H(X(\lambda))}{\lambda^{i+1}}. \tag{2.6}$$

Clearly any such function H_i belongs to $\mathcal{O}(\tilde{\mathfrak{g}})$.

Proposition 2.4 ([RS3]) *Let H and K be two* Ad*-invariant functions in $\mathcal{O}(\mathfrak{g})$. Then for any $i, j \in \mathbf{Z}$ the functions H_i and K_j are in involution with respect to all R-brackets $\{\cdot,\cdot\}_l$.*

Proof

Ad-invariant functions in $\mathcal{O}(\mathfrak{g})$ are those functions which are invariant for the adjoint action of a Lie group G for which $\mathfrak{g} = \operatorname{Lie} G$. It may be impossible[1] to pick G algebraic but this is irrelevant

[1] If $\mathfrak{g}$ is semi-simple then G is algebraic, see [OV], p.29.

here because we only use the fact that such a function $H \in \mathcal{O}(\mathfrak{g})$ satisfies the infinitesimal condition $[x, \nabla H(x)] = 0$. To show the latter, use ad-invariance of $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ and Ad-invariance of H to find

$$\langle [x, \nabla H(x)], y \rangle = \langle \nabla H(x), [y, x] \rangle = \frac{d}{dt}_{|t=0} H(x + t[y, x]) = \frac{d}{dt}_{|t=0} H(\mathrm{Ad}_{z(t)}\, x) = 0,$$

when setting $z(t) = \exp ty$. In particular, if $H \in \mathcal{O}(\mathfrak{g})$ is Ad-invariant then, for any $i \in \mathbf{Z}$, the function $H_i \in \mathcal{O}(\tilde{\mathfrak{g}})$ defined by (2.6) is Ad-invariant and $[X, \nabla H_i(X)] = 0$. It follows that if H and K are Ad-invariant functions on $\mathfrak{g}$ then for any $i, j, l \in \mathbf{Z}$

$$\{H_i, K_j\}_l(X) = \frac{1}{2} \langle X, [R_l \nabla H_i(X), \nabla K_j(X)] + [\nabla H_i(X), R_l \nabla K_j(X)] \rangle = 0,$$

showing that all functions on $\mathcal{O}(\tilde{\mathfrak{g}})$ which come from Ad-invariant functions on $\mathfrak{g}$ are in involution with respect to all R-brackets. ∎

The algebra of Ad-invariant functions on $\mathfrak{g}$ is denoted by $\mathcal{O}(\mathfrak{g})^G$ and the involutive algebra generated by all H_i, $i \in \mathbf{Z}$, $H \in \mathcal{O}(\mathfrak{g})^G$ is denoted by $\mathcal{A}$. If we define for any $F \in \mathcal{O}(\tilde{\mathfrak{g}})$ a vector field on $\tilde{\mathfrak{g}}$ by $\mathcal{X}_F = \{\cdot, F\}_0$ then the i-th vector field $\mathcal{X}_{H_i}$ ($i \in \mathbf{Z}$) which comes from an Ad-invariant function $H \in \mathcal{O}(\mathfrak{g})^G$ is given by the Lax equation

$$\dot{X} = -\frac{1}{2} [X, R \nabla H_i(X)]. \tag{2.7}$$

Two alternative ways to write this are

$$\dot{X} = -[X, (\nabla H_i(X))_+] = [X, (\nabla H_i(X))_-]. \tag{2.8}$$

The vector fields $\mathcal{X}_{H_i}$ are in fact Hamiltonian with respect to all brackets $\{\cdot, \cdot\}_l$. To see this, check that for any $H \in \mathcal{O}(\mathfrak{g})$,

$$\frac{d}{dt}_{|t=0} \operatorname{Res} \frac{H(X + tY)}{\lambda^{i+1}} = \frac{d}{dt}_{|t=0} \operatorname{Res} \frac{H(X + t\lambda Y)}{\lambda^{i+2}},$$

showing that $\nabla H_i(X) = \lambda \nabla H_{i+1}(X)$. It follows that (2.7) can be written in Lax form with respect to all endomorphisms R_l and that for any $H \in \mathcal{O}(\mathfrak{g})^G$ the functions $\{H_i\}_{i \in \mathbf{Z}}$ form a multi-Hamiltonian hierarchy in the sense that

$$\{\cdot, H_i\}_0 = \{\cdot, H_{i+l}\}_l \qquad (i, l \in \mathbf{Z}). \tag{2.9}$$

The relations (2.9), which are called Lenard relations, can be used to give an alternative proof of Proposition 2.4. For functions belonging to the same hierarchy the classical argument applies (see, e.g., [CMP]), giving $\{H_i, H_j\}_l = \{H_j, H_i\}_l = 0$. For members of different hierarchies, coming from different functions H, $K \in \mathcal{O}(\mathfrak{g})^G$ some care is needed since none of the H_i or K_j is a Casimir for any of the R-brackets. However, we see from (2.8) that for any $X \in \tilde{\mathfrak{g}}$ the Hamiltonian vector field $\mathcal{X}_{H_s}$ vanishes at X for s large enough since then $(\nabla H_s(X))_+ = 0$. Thus also in this case the Lenard relations give (e.g., for the zeroth R-bracket)

$$\{H_i, K_j\}_0(X) = \{H_s, K_{j-s+i}\}_0(X) = 0.$$

which shows that functions which belong to different hierarchies are also in involution.

2.2. Master symmetries and the deformation property

In this paragraph we show how the Poisson brackets $\{\cdot\,,\cdot\}_l$ are related by a vector field $\mathcal{V}$ which is a master symmetry[2] for the involutive algebra $\mathcal{A}$ (introduced after Proposition 2.4). We mean by this that $\mathcal{V}$ has the property $[[\mathcal{V},\mathcal{X}_F]\,,\mathcal{X}_G]=0$ for all $F,\,G\in\mathcal{A}$ (a symmetry has the stronger property $[\mathcal{V},\mathcal{X}_F]=0$ for all $F\in\mathcal{A}$). The vector field $\mathcal{V}$ has in addition the deformation property with respect to the brackets $\{\cdot\,,\cdot\}_l$; this means that the Lie derivative of any bracket $\{\cdot\,,\cdot\}_l$ in the direction of $\mathcal{V}$ is also a Poisson bracket[3]. As was shown in [MM] this implies that any bracket $\{\cdot\,,\cdot\}_l$ is compatible with its Lie derivative in the direction of $\mathcal{V}$.

The vector field $\mathcal{V}$ is defined as the infinitesimal generator of the action of $\mathbf{C}$ on $\tilde{\mathfrak{g}}$ given by "shift in λ",

$$\left(s,\sum x_i\lambda^i\right)\mapsto\sum x_i(\lambda+s)^i;$$

here we use for negative powers of λ the formal expansion

$$(\lambda+s)^{-1}=\sum_{i\geq 0}(-1)^i s^i\lambda^{-i-1},$$

which is actually convergent for small s, in particular it is the right definition if one wants to consider the fundamental vector field $\mathcal{V}$ of this action: the latter is easily computed as

$$\dot{X}(\lambda)=\frac{\partial}{\partial\lambda}X(\lambda)\qquad\text{i.e.}\qquad\mathcal{L}_{\mathcal{V}}\xi_a^j=(j+1)\xi_a^{j+1},$$

where $\mathcal{L}_{\mathcal{V}}$ denotes the Lie derivative along $\mathcal{V}$. The two mentioned properties of $\mathcal{V}$ are given by the following proposition.

Proposition 2.5 *Let $i,l\in\mathbf{Z}$ and $H\in\mathcal{O}(\mathfrak{g})^G$ be arbitrary.*

a) $\mathcal{V}$ has the deformation property with respect to all brackets $\{\cdot\,,\cdot\}_l$, more precisely the relation

$$\mathcal{L}_{\mathcal{V}}\{F,G\}_l-\{\mathcal{L}_{\mathcal{V}}F,G\}_l-\{F,\mathcal{L}_{\mathcal{V}}G\}_l=-l\{F,G\}_{l-1}\tag{2.10}$$

holds, i.e., the Lie derivative of the l-th R-bracket is (up to a factor $-l$) the $(l-1)$-th R-bracket;

b) $\mathcal{L}_{\mathcal{V}}H_i=(i+1)H_{i+1}$;

c) $[\mathcal{V},\mathcal{X}_{H_i}]=\mathcal{X}_{\mathcal{L}_{\mathcal{V}}H_i}=(i+1)\mathcal{X}_{H_{i+1}}$;

d) $\mathcal{V}$ is a master symmetry for $\mathcal{A}$.

Proof

It suffices to verify a) for $F=\xi_a^i$ and $G=\xi_b^j$ with say $i\leq j$. We can use (2.3); since for this particular F and G all terms in (2.10) are proportional to $\sum_c C_{ab}^c\xi_c^{i+j-l+2}$ it actually suffices to keep track of the coefficients and the proof of (2.10) amounts to the verification of the following identity,

$$(i+j-l+2)\epsilon_l^{ij}-(i+1)\epsilon_l^{i+1,j}-(j+1)\epsilon_l^{i,j+1}=-l\epsilon_{l-1}^{ij}.$$

---

[2] The concept of a master symmetry was first introduced by Fuchssteiner (see [Fuc]). The notion we use here is slightly more general.

[3] In many important examples the master symmetries for an algebra which is involutive with respect to some Poisson bracket have the deformation property with respect to this Poisson bracket, however these two properties are independent in general.

As for b),

$$\begin{aligned}
\mathcal{L}_{\mathcal{V}} H_i(X) &= \frac{d}{ds}_{|s=0} \operatorname{Res} \frac{H(X(\lambda+s))}{\lambda^{i+1}} \\
&= \operatorname{Res} \frac{1}{\lambda^{i+1}} \frac{d}{d\lambda} H(X(\lambda)) \\
&= \operatorname{Res} \left[\frac{d}{d\lambda} \left(\frac{H(X(\lambda))}{\lambda^{i+1}} \right) + (i+1) \frac{H(X(\lambda))}{\lambda^{i+2}} \right] \\
&= (i+1) \operatorname{Res} \frac{H(X(\lambda))}{\lambda^{i+2}},
\end{aligned}$$

which is precisely $(i+1)H_{i+1}(X)$. For c) we substitute $l=0$ and $G=H_i$ in part a) to find

$$\mathcal{L}_{\mathcal{V}}\{F, H_i\}_0 = \{\mathcal{L}_{\mathcal{V}} F, H_i\}_0 + \{F, \mathcal{L}_{\mathcal{V}} H_i\}_0,$$

which can also be written as $\mathcal{L}_{\mathcal{V}}(\mathcal{X}_{H_i}(F)) = \mathcal{X}_{H_i}(\mathcal{L}_{\mathcal{V}} F) + \mathcal{X}_{\mathcal{L}_{\mathcal{V}} H_i}(F)$; using b) we conclude c). In order to show d) first notice that $[\mathcal{X}_F, \mathcal{X}_G] = -\mathcal{X}_{\{F,G\}_0} = 0$ for any $F, G \in \mathcal{A}$. Then c) implies that $[[\mathcal{V}, \mathcal{X}_{H_i}], \mathcal{X}_G] = 0$ for any $H \in \mathcal{O}(\mathfrak{g})^G$ and for any $G \in \mathcal{A}$. By the Jacobi identity we also have that $[[\mathcal{V}, \mathcal{X}_G], \mathcal{X}_{H_i}] = 0$. The more general statement that $[[\mathcal{V}, \mathcal{X}_G], \mathcal{X}_F] = 0$ for any $F, G \in \mathcal{A}$ follows from b) upon using the fact that $\mathcal{A}$ is generated by the functions H_i where i runs over $\mathbf{Z}$ and H runs over $\mathcal{O}(\mathfrak{g})^G$. ∎

Picking any two Poisson structures such as $\{\cdot\,,\cdot\}_0$ and $\{\cdot\,,\cdot\}_l$ the relations (2.9) and Proposition 2.5 can be depicted in the following diagram (we omit the coefficients; $\mathcal{L}^l_{\mathcal{V}} := \mathcal{L}_{\mathcal{V}} \circ \mathcal{L}^{l-1}_{\mathcal{V}}$),

$$\begin{tikzcd}
\cdots & H_i \arrow[r, "\mathcal{L}^l_{\mathcal{V}}"] \arrow[d, "0"'] & H_{i+l} \arrow[r, "\mathcal{L}^l_{\mathcal{V}}"] \arrow[d, "0"'] \arrow[dl, "l"] & H_{i+2l} \arrow[d, "0"'] \arrow[dl, "l"] & \cdots \\
& \mathcal{X}_{H_i} \arrow[r, "\mathcal{L}^l_{\mathcal{V}}"] & \mathcal{X}_{H_{i+l}} \arrow[r, "\mathcal{L}^l_{\mathcal{V}}"] & \mathcal{X}_{H_{i+2l}} &
\end{tikzcd}$$

Remark 2.6 An R-bracket on a Lie algebra $\mathfrak{g}$ leads also to a quadratic and a cubic bracket, assuming that the Lie algebra derives from an associative algebra, with a pairing $\langle\cdot\,,\cdot\rangle_{\mathfrak{g}}$ which derives from a traceform (see [LP] and [OR]). Explicitly the quadratic bracket $\{\cdot\,,\cdot\}_Q$ and the cubic bracket $\{\cdot\,,\cdot\}_C$ are given for F, $G \in \mathcal{O}(\mathfrak{g})$ by

$$\begin{aligned}
\{F,G\}_Q(x) &= \frac{1}{2}\langle [x, \nabla F(x)], R(x\nabla G(x) + \nabla G(x)x)\rangle_{\mathfrak{g}} - \frac{1}{2}\langle [x, \nabla G(x)], R(x\nabla F(x) + \nabla F(x)x)\rangle_{\mathfrak{g}} \\
\{F,G\}_C(x) &= \langle [x, \nabla F(x)], R(x\nabla G(x)x)\rangle_{\mathfrak{g}} - \langle [x, \nabla G(x)], R(x\nabla F(x)x)\rangle_{\mathfrak{g}}.
\end{aligned}$$

When applied to the R-bracket on the loop algebra $\tilde{\mathfrak{g}}$ of $\mathfrak{g} = \mathfrak{gl}(N)$ we get a quadratic and a cubic Poisson bracket on $\mathcal{O}(\tilde{\mathfrak{g}})$. It was shown in [LP] that the linear, the quadratic and the cubic bracket are related by the vector field $\mathcal{U}_{X(\lambda)} = X^2(\lambda)$. It is easy to prove that $\mathcal{U}$ is a master symmetry for the algebra $\mathcal{A}$, which is in the case $\mathfrak{g} = \mathfrak{gl}(N)$ generated by the functions

$$I_{ij}(X) = \operatorname{Res} \frac{\operatorname{Tr} X^i(\lambda)}{\lambda^{j+1}}, \qquad i > 0,\ j \in \mathbf{Z};$$

[LP] gives Lenard relations for the functions I_{ij} with respect to these brackets. Using the fact that $\mathcal{U}$ and $\mathcal{V}$ commute it is easy to show that $\mathcal{V}$ also has the deformation property with respect to both the quadratic and the cubic brackets, e.g., the Lie derivative in the direction of $\mathcal{V}$ of the quadratic bracket which corresponds to R_l is ($-l$ times) the quadratic bracket which corresponds to R_{l-1}; this leads in particular to another set of Lenard relations for the functions I_{ij}. It follows that on the loop algebra $\tilde{\mathfrak{g}}$ the cubic and the quadratic bracket have all properties which the R-brackets have: $\mathcal{A}$ is involutive with respect to these brackets, the corresponding Hamiltonian vector fields are multi-Hamiltonian with respect to these brackets and the brackets are connected by the Lie derivative with respect to the vector fields $\mathcal{U}$ and $\mathcal{V}$ which are master symmetries for $\mathcal{A}$. The higher order brackets differ however from the linear structures in one crucial aspect: as it is easy to see they do not restrict to any of the finite-dimensional spaces $\tilde{\mathfrak{g}}_{-p,q}$, defined in (2.4). Similarly the vector field $\mathcal{U}$ clearly does not restrict to any of the subspaces $\tilde{\mathfrak{g}}_{-p,q}$ (except in the trivial case $p = q = 0$).

3. Poisson reduction and reduction of symmetries

In order to construct our examples we need a reduction theorem which leads to a Poisson structure in the following situation: for N a subvariety of an affine Poisson variety $(M, \{\cdot\,,\cdot\})$ with an algebraic group G acting on it (leaving N stable) we want an inherited Poisson structure on the quotient space N/G. By an affine Poisson variety we mean an affine variety whose algebra of regular functions is equipped with the structure of a Poisson algebra. We will assume that our group G also carries a Poisson structure (which may be trivial).

If N is a Poisson subvariety of M then a Poisson structure on N/G, or, more precisely, on the ring $\mathcal{O}(N)^G$ of G-invariant regular functions on N, will exist if the map $\chi : G \times N \to N$ is a Poisson map with respect to some Poisson structure on G; such an action is called a Poisson action[4] and the bracket is called a reduced bracket. If N is *not* a Poisson subvariety of M then N/G may still inherit a bracket from M: we will give below necessary and sufficient conditions for this to happen.

The following notation will be useful: the algebra of regular functions on M which restrict to G-invariant functions on N is denoted by $\mathcal{O}(M,N)^G$; we have a natural restriction[5] map $\rho : \mathcal{O}(M,N)^G \to \mathcal{O}(N)^G$. The ideal of N is denoted by $I(N)$ and we have an inclusion map $\imath : N \to M$. Also, if $\phi : M_1 \to M_2$ is a regular map between affine varieties then we denote by ϕ^* the induced map $\mathcal{O}(M_2) \to \mathcal{O}(M_1)$ defined by $\phi^*(f) = f \circ \phi$.

Definition 3.1 Let $(M, \{\cdot\,,\cdot\})$ be an affine Poisson variety, $\chi : G \times M \to M$ a Poisson action and N a subvariety of M which is G-stable. Then the triple (M, G, N) is called *Poisson-reducible* if $\mathcal{O}(M,N)^G$ is a Poisson subalgebra of $\mathcal{O}(M)$ and if there exists a Poisson bracket on $\mathcal{O}(N)^G$ such that

$$\{\rho(F_1), \rho(F_2)\}_{\mathcal{O}(N)^G} = \rho\{F_1, F_2\} \tag{3.1}$$

holds for all $F_1,\ F_2 \in \mathcal{O}(M,N)^G$.

Formula (3.1) says that in order to compute the Poisson bracket of two G-invariant functions on N one computes the Poisson bracket of any extensions to M and then restricts the result to N. Note also that (3.1) uniquely defines a bracket on $\mathcal{O}(N)^G$ (if it exists) since ρ is surjective. In the following theorem, which is similar in spirit to the Marsden-Ratiu reduction theorem (see [MR]), we give necessary and sufficient conditions for (M, G, N) to be Poisson-reducible.

Theorem 3.2 *Let $(M, \{\cdot\,,\cdot\})$ be an affine Poisson variety, $\chi : G \times M \to M$ a Poisson action and N a subvariety of M which is G-stable. Then (M, G, N) is Poisson-reducible if and only if*

$$\rho\left\{\mathcal{O}(M,N)^G, I(N)\right\} = 0; \tag{3.2}$$

it is implicit in this condition that its left hand side makes sense.

Proof

Suppose first that condition (3.2) is satisfied. We proceed to show that

$$\left\{\mathcal{O}(M,N)^G, \mathcal{O}(M,N)^G\right\} \subset \mathcal{O}(M,N)^G.$$

[4] Some authors, e.g., [LM] use this term in the more restricted sense in which G is given the trivial Poisson structure; then χ being a Poisson action means that for any $g \in G$ the induced map $\chi_g : N \to N$ is a Poisson map.

[5] This restriction map is onto, although the restriction map $\mathcal{O}(M)^G \to \mathcal{O}(N)^G$ is not onto in general.

If we denote by $\pi_{2,N}$ the projection $G \times N \to N$ onto the second factor then G-invariance of a function $f \in \mathcal{O}(N)$ is conveniently expressed by the formula $\chi^* f = \pi_{2,N}^* f$. Thus we need to show that

$$\chi^* \imath^* \{F_1, F_2\} = \pi_{2,N}^* \imath^* \{F_1, F_2\} \tag{3.3}$$

for any F_1, $F_2 \in \mathcal{O}(M,N)^G$. Since χ and $\pi_{2,N}$ are the restrictions to $G \times N$ of the corresponding maps χ and $\pi_{2,M}$ on $G \times M$ and since these maps are Poisson maps, (3.3) is equivalent to

$$(1_G \times \imath)^* \left(\{\chi^* F_1, \chi^* F_2\}_{G \times M} - \{\pi_{2,M}^* F_1, \pi_{2,M}^* F_2\}_{G \times M} \right) = 0, \tag{3.4}$$

where 1_G is the identity map on G. For $g \in G$ and $n \in N$ we define maps $\chi_g : M \to M$ and $\chi_n : G \to M$ by inserting g resp. n in χ. Then $\chi_n^* F$ is constant for any $F \in \mathcal{O}(M,N)^G$ so that

$$\begin{aligned}\{\chi^* F_1, \chi^* F_2 - \pi_{2,M}^* F_2\}_{G \times M}(g,n) &= \{\chi_g^* F_1, \chi_g^* F_2 - F_2\}(n) + \{\chi_n^* F_1, \chi_n^* F_2 - F_2(n)\}_G(g) \\ &= \{\chi_g^* F_1, \chi_g^* F_2 - F_2\}(n),\end{aligned}$$

which vanishes by the assumption (3.2). Therefore

$$(1_G \times \imath)^* \{\chi^* F_1, \chi^* F_2 - \pi_{2,M}^* F_2\}_{G \times M} = 0, \tag{3.5}$$

and similarly

$$(1_G \times \imath)^* \{\chi^* F_1 - \pi_{2,M}^* F_1, \pi_{2,M}^* F_2\}_{G \times M} = 0. \tag{3.6}$$

Summing (3.5) and (3.6) we find (3.4) which shows that $\{F_1, F_2\} \in \mathcal{O}(M,N)^G$.

It follows that we can actually use (3.1) to define $\{\cdot,\cdot\}_{\mathcal{O}(N)^G}$: on the one hand ρ is surjective, on the other hand the bracket on $\mathcal{O}(N)^G$ given by (3.1) is well-defined since if $\rho(\tilde{F}_2) = \rho(F_2)$ then $\imath^* \{F_1, \tilde{F}_2 - F_2\} = 0$, another application of (3.2). From the definition it is also immediate that $\{\cdot,\cdot\}_{\mathcal{O}(N)^G}$ satisfies the Jacobi identity so we get a Poisson bracket on $\mathcal{O}(N)^G$ which satisfies (3.1).

This shows the if part; the only if part is trivial since $\rho(I(N)) = 0$. ∎

Remark 3.3 Suppose that all algebras under consideration are finitely generated. Then $\mathcal{O}(N)^G$ is the algebra of functions on an affine variety N/G which can be considered as the quotient of N by G. Similarly $\mathcal{O}(M,N)^G$ corresponds then to an affine variety $(M,N)/G$, obtained by taking the quotient of M with respect to G but along N only, i.e., only N is shrunk inside M into its orbit space N/G while the other points of M remain intact. In geometric terms formula (3.2) states that the Hamiltonian vector fields which are associated to functions on M which are G-invariant on N, are tangent to N (at points of N). It follows from the proof of Theorem 3.2 that if condition (3.2) holds then $(M,N)/G$ inherits a Poisson bracket from M and in turn N/G inherits a Poisson bracket from $(M,N)/G$, the latter because all Hamiltonian vector fields on $(M,N)/G$ are tangent to N/G.

As an application of this theorem let us show that if a vector field $\mathcal{V}$ which descends to the quotient has the deformation property with respect to some Poisson-reducible bracket then this deformation property is conserved after the reduction. We need the following lemma.

Lemma 3.4 *Let M be an affine variety, $\mathcal{V}$ a vector field on M and G a linear algebraic group acting on M; let N be an affine subvariety, stable for G, and suppose that $\mathcal{V}$ is tangent to N, $\mathcal{W} = \mathcal{V}_{|N}$. Then*

$$\mathcal{L}_{\mathcal{W}} \mathcal{O}(N)^G \subset \mathcal{O}(N)^G \tag{3.7}$$

is equivalent to

$$\mathcal{L}_{\mathcal{V}}\mathcal{O}(M,N)^G \subset \mathcal{O}(M,N)^G \tag{3.8}$$

and implies the commutativity of the following diagram.

$$\begin{array}{ccc} \mathcal{O}(M,N)^G & \xrightarrow{\rho} & \mathcal{O}(N)^G \\ {\scriptstyle \mathcal{L}_{\mathcal{V}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{L}_{\mathcal{W}}} \\ \mathcal{O}(M,N)^G & \xrightarrow[\rho]{} & \mathcal{O}(N)^G \end{array} \tag{3.9}$$

Proof

Let $\pi_N^* : \mathcal{O}(N)^G \to \mathcal{O}(N)$ denote the inclusion map (which may be thought of as coming from the quotient map $\pi_N : N \to N/G$) and note the obvious relation $\imath^* = \pi_N^*\rho$, which holds on $\mathcal{O}(M,N)^G$. Then formula (3.8) follows from (3.7),

$$\imath^*\mathcal{L}_{\mathcal{V}}\mathcal{O}(M,N)^G = \mathcal{L}_{\mathcal{W}}\imath^*\mathcal{O}(M,N)^G = \mathcal{L}_{\mathcal{W}}\pi_N^*\mathcal{O}(N)^G = \pi_N^*\mathcal{L}_{\mathcal{W}}\mathcal{O}(N)^G \subset \pi_N^*\mathcal{O}(N)^G;$$

for the proof of the other direction surjectivity of ρ is essential:

$$\pi_N^*\mathcal{L}_{\mathcal{W}}\mathcal{O}(N)^G = \mathcal{L}_{\mathcal{W}}\pi_N^*\rho\, \mathcal{O}(M,N)^G = \imath^*\mathcal{L}_{\mathcal{V}}\mathcal{O}(M,N)^G \subset \imath^*\mathcal{O}(M,N)^G = \pi_N^*\mathcal{O}(N)^G.$$

Moreover, for $F \in \mathcal{O}(M,N)^G$ we have

$$\pi_N^*\mathcal{L}_{\mathcal{W}}\rho(F) = \mathcal{L}_{\mathcal{W}}\pi_N^*\rho(F) = \mathcal{L}_{\mathcal{W}}\imath^*F = \imath^*\mathcal{L}_{\mathcal{V}}F = \pi_N^*\rho\mathcal{L}_{\mathcal{V}}F,$$

which shows that the diagram is commutative. ∎

Theorem 3.5 *Let (M,G,N) be Poisson-reducible with respect to a Poisson bracket $\{\cdot,\cdot\}$ on M and suppose that $\mathcal{V}$ is a vector field on M which is tangent to N, $\mathcal{W} = \mathcal{V}_{|N}$, and which has the deformation property with respect to $\{\cdot,\cdot\}$. If $\mathcal{L}_{\mathcal{W}}\mathcal{O}(N)^G \subset \mathcal{O}(N)^G$ then*

- *a) (M,G,N) is Poisson-reducible with respect to $\{\cdot,\cdot\}'$, the Lie derivative of $\{\cdot,\cdot\}$ in the direction of $\mathcal{V}$;*
- *b) $\mathcal{W}$ has the deformation property with respect to $\{\cdot,\cdot\}_{\mathcal{O}(N)^G}$;*
- *c) the Lie derivative of $\{\cdot,\cdot\}_{\mathcal{O}(N)^G}$ in the direction of $\mathcal{W}$ is the reduced bracket of $\{\cdot,\cdot\}'$.*

Thus the deformation property survives the reduction and the operations of reduction and deformation commute.

Proof

To show that (M,G,N) is Poisson-reducible with respect to $\{\cdot,\cdot\}'$, we use the necessary and sufficient condition (3.2) of Theorem 3.2. Since

$$\{F,G\}' = \mathcal{L}_{\mathcal{V}}\{F,G\} - \{\mathcal{L}_{\mathcal{V}}F,G\} - \{F,\mathcal{L}_{\mathcal{V}}G\},$$

we have that

$$\rho\{\mathcal{O}(M,N)^G, I(N)\}' = \rho\mathcal{L}_{\mathcal{V}}\{\mathcal{O}(M,N)^G, I(N)\} - \rho\{\mathcal{L}_{\mathcal{V}}\mathcal{O}(M,N)^G, I(N)\} - \rho\{\mathcal{O}(M,N)^G, \mathcal{L}_{\mathcal{V}}I(N)\}$$

and each term of the right hand side vanishes because (M,G,N) is Poisson-reducible with respect to $\{\cdot,\cdot\}$: for the first term use commutativity of (3.9), for the second one use (3.8) and the last is zero because $\mathcal{V}$ is tangent to N, $\mathcal{L}_{\mathcal{V}}I(N) = 0$.

Next we show that the Lie derivative of $\{\cdot\,,\cdot\}_{\mathcal{O}(N)^G}$ in the direction of $\mathcal{W}$ is the reduced bracket $\{\cdot\,,\cdot\}'_{\mathcal{O}(N)^G}$ of $\{\cdot\,,\cdot\}'$. This means that if $f_1,\ f_2 \in \mathcal{O}(N)^G$ then

$$\{f_1, f_2\}'_{\mathcal{O}(N)^G} = \mathcal{L}_{\mathcal{W}}\{f_1, f_2\}_{\mathcal{O}(N)^G} - \{\mathcal{L}_{\mathcal{W}} f_1, f_2\}_{\mathcal{O}(N)^G} - \{f_1, \mathcal{L}_{\mathcal{W}} f_2\}_{\mathcal{O}(N)^G}. \tag{3.10}$$

Let $f_1 = \rho(F_1),\ f_2 = \rho(F_2)$ and use (3.1) and commutativity of (3.9):

$$\begin{aligned}\{f_1, f_2\}'_{\mathcal{O}(N)^G} &= \rho\{F_1, F_2\}' \\ &= \rho\mathcal{L}_{\mathcal{V}}\{F_1, F_2\} - \rho\{\mathcal{L}_{\mathcal{V}} F_1, F_2\} - \rho\{F_1, \mathcal{L}_{\mathcal{V}} F_2\} \\ &= \mathcal{L}_{\mathcal{W}}\{f_1, f_2\}_{\mathcal{O}(N)^G} - \{\mathcal{L}_{\mathcal{W}} f_1, f_2\}_{\mathcal{O}(N)^G} - \{f_1, \mathcal{L}_{\mathcal{W}} f_2\}_{\mathcal{O}(N)^G}.\end{aligned}$$

Since we have proved that $\{\cdot\,,\cdot\}'_{\mathcal{O}(N)^G}$ is a Poisson bracket on $\mathcal{O}(N)^G$ we have shown in particular that $\mathcal{L}_{\mathcal{W}}$ has the deformation property with respect to $\{\cdot\,,\cdot\}_{\mathcal{O}(N)^G}$ and we are done. ∎

Remark 3.6 Under the conditions of Remark 3.3 the conditions (3.7) and (3.8) mean that the vector fields $\mathcal{W}$ and $\mathcal{V}$ are tangent to the quotient spaces N/G and $(M,N)/G$.

Remark 3.7 The conditions of Theorem 3.5 are also sufficient to conclude that a master symmetry for a subalgebra $\mathcal{A} \subset \mathcal{O}(M,N)^G$ descends to a master symmetry on the quotient. To prove this let $F \in \mathcal{O}(M,N)^G$ and note that $\mathcal{X}_F = \{\cdot, F\}$ is tangent to N. If we denote by $\mathcal{Y}_F$ the restriction of $\mathcal{X}_F$ to N then $\mathcal{Y}_F$ is given as a derivation of $\mathcal{O}(N)^G$ by $\mathcal{Y}_F = \{\cdot, \rho(F)\}_{\mathcal{O}(N)^G}$ and we have that $\mathcal{Y}_F\rho = \rho\mathcal{X}_F$. Using (3.9), written as $\mathcal{W}\rho = \rho\mathcal{V}$, we get

$$[\mathcal{Y}_F, [\mathcal{Y}_G, \mathcal{W}]]\,\rho = \rho\,[\mathcal{X}_F, [\mathcal{X}_G, \mathcal{V}]] = 0,$$

since $\mathcal{V}$ is a master symmetry for $\mathcal{A}$. Since ρ is surjective $\mathcal{W}$ is a master symmetry for $\rho(\mathcal{A})$.

4. Reduction for simple Lie algebras

In this section we apply our two reduction theorems to the finite-dimensional subspaces $\tilde{\mathfrak{g}}_n^\beta$ of $\tilde{\mathfrak{g}}$, defined in (2.5), in case $\tilde{\mathfrak{g}}$ is the loop algebra of a complex simple Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(N)$ of rank r (see [Hum] and [Ser]). We denote by G any algebraic group whose Lie algebra equals $\mathfrak{g}$. We fix a Weyl basis $\{H_i, E_i, F_i\}_{i=1}^r$ of $\mathfrak{g}$, i.e., a collection of $3r$ generators for $\mathfrak{g}$ such that H_i spans a Cartan subalgebra $\mathfrak{h}$, and the following commutation relations[6] hold:

$$[E_i, F_j] = \delta_{ij} F_i, \quad [H_i, E_j] = n_{ji} E_j, \quad [H_i, F_j] = -n_{ji} F_j.$$

Here (n_{ij}) is the Cartan matrix of $\mathfrak{g}$ and the indices i, j take values between 1 and r. The Weyl basis leads to a gradation $\mathfrak{g} = \oplus_{i=-k}^k \mathfrak{g}_i$ of $\mathfrak{g}$: $\mathfrak{g}_0 = \mathfrak{h}$ and for i positive (negative) $\mathfrak{g}_i$ is spanned by the i-fold commutators of the elements $E_1, \ldots, E_r$ $(F_1, \ldots, F_r)$. An element of $\mathfrak{g}_i$ is called a homogeneous element of degree i and $h = k+1$ is called the Coxeter number of $\mathfrak{g}$. The projection of $\mathfrak{g}$ on $\mathfrak{g}_i$ is denoted by Π_i. We will also use the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^\pm = \oplus_{i=\pm 1}^{\pm k} \mathfrak{g}_i$. We will consider a set $\{I_1, \ldots, I_r\}$ of Chevalley invariants of $\mathfrak{g}$. They are homogeneous polynomials which generate the algebra $\mathcal{O}(\mathfrak{g})^G$ of invariants for the adjoint action of G on $\mathfrak{g}$ (see [Var] p. 333). We denote the degree of I_j by d_j and call the numbers $q_j = d_j - 1$ the exponents of $\mathfrak{g}$. We will always assume the invariants I_j to be ordered by degree. Then the exponents bear the following relations (see [Kos1]):

$$1 = q_1 < q_2 \leq q_3 \leq \cdots \leq q_{r-1} < q_r = k. \tag{4.1}$$

The Chevalley invariants lead to the following G-invariant functions on $\tilde{\mathfrak{g}}$:

$$I_{ij}(X) = \operatorname{Res} \frac{I_i(X)}{\lambda^{j+1}},$$

which by definition generate the involutive algebra $\mathcal{A}$ introduced in Section 2.

4.1. Poisson reduction

Let α and β be homogeneous elements of $\mathfrak{g}$ such that $\deg \alpha - \deg \beta = h$. We put $\deg \beta = -d$ and we define, as in Paragraph 2.1,

$$\tilde{\mathfrak{g}}_n^\beta = \left\{ \sum_{i=0}^n x_i \lambda^i \in \tilde{\mathfrak{g}} \mid x_n = \beta \right\},$$

together with the following affine subspace

$$N = \left\{ \sum_{i=0}^n x_i \lambda^i \in \tilde{\mathfrak{g}}_n^\beta \mid \Pi_j(x_{n-1} - \alpha) = 0 \text{ if } j \geq \deg \alpha \right\}.$$

Lemma 4.1 *Let $\mathfrak{g}_\beta$ be the isotropy algebra of β and let $\mathfrak{g}_\beta^- = \mathfrak{g}_\beta \cap \mathfrak{n}^-$. Then the action of $G_\beta^- = \exp \mathfrak{g}_\beta^-$ on $\tilde{\mathfrak{g}}_n^\beta$ leaves N invariant.*

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[6] Our definition of a Weyl basis differs from the one in [Ser] by a transposition in the Cartan matrix, i.e., [Ser] takes $[H_i, E_j] = n_{ij} E_j$; our choice simplifies the explicit formulas for the Weyl bases given in the examples.

Proof

It suffices to show that if $\Pi_j(x_{n-1}-\alpha)$ vanishes for all $j \geq \deg\alpha$ then the same holds true for $\Pi_j(\mathrm{Ad}_{\exp\nu}\, x_{n-1}-\alpha)$ when $\nu \in \mathfrak{n}^-$. The result follows at once from $\mathrm{Ad}_{\exp\nu} = \exp \mathrm{ad}_\nu$. ∎

Recall from Proposition 2.3 that the brackets $\{\cdot\,,\cdot\}_l$ restrict to $\tilde{\mathfrak{g}}_n^\beta$ for $0 \leq l \leq n$. Notice however that the bracket $\{\cdot,\cdot\}_n$ does *not* restrict to N: if e_a is a basis element such that $\deg e_a = \deg\alpha$ then $\xi_a^{n-1} - \xi_a(\alpha)$ belongs to the ideal $I(N)$ of N but $\{\xi_a^{n-1}, \xi_b^0\}_n = C_{ab}^c \xi_c^0$, which is non-zero for at least one value of b since $\mathfrak{g}$ is simple. Therefore we are precisely in the case of the reduction theorem (Theorem 3.2).

Theorem 4.2 *The triple* $\left(\tilde{\mathfrak{g}}_n^\beta, G_\beta^-, N\right)$ *is Poisson-reducible with respect to each Poisson structure* $\sum_{l=0}^n c_l\{\cdot\,,\cdot\}_l$.

Proof

We first show that the action of G_β^- on $\tilde{\mathfrak{g}}_n^\beta$ is Poisson. To do this we take the trivial Poisson structure on G, we fix any $l \in \mathbf{Z}$ and show that $(\mathrm{Ad}_g)^*\{f_1, f_2\}_l = \{(\mathrm{Ad}_g)^* f_1, (\mathrm{Ad}_g)^* f_2\}_l$ for any $g \in G$ and any f_1, $f_2 \in \mathcal{O}(\tilde{\mathfrak{g}})$. It is sufficient to show this for f_1 and f_2 linear; then $(\mathrm{Ad}_g)^* f_1$ and $(\mathrm{Ad}_g)^* f_2$ are linear too and their gradients do not depend on $X \in \tilde{\mathfrak{g}}$ (in particular we can omit the argument). Since

$$\frac{d}{dt}_{|t=0} f_1(\mathrm{Ad}_g(X+tY)) = f_1(\mathrm{Ad}_g Y) = \langle \nabla f_1, \mathrm{Ad}_g Y\rangle$$

we find that $\langle \nabla(\mathrm{Ad}_g)^* f_1, Y\rangle = \left\langle \mathrm{Ad}_{g^{-1}} \nabla f_1, Y\right\rangle$ giving $\nabla(\mathrm{Ad}_g)^* f_1 = \mathrm{Ad}_{g^{-1}} \nabla f_1$. Then

$$\begin{aligned}
\{(\mathrm{Ad}_g)^* f_1, (\mathrm{Ad}_g)^* f_2\}_l(X) &= \left\langle X, [\mathrm{Ad}_{g^{-1}} \nabla f_1, \mathrm{Ad}_{g^{-1}} \nabla f_2]_{R_l}\right\rangle \\
&= \langle \mathrm{Ad}_g X, [\nabla f_1, \nabla f_2]_{R_l}\rangle \\
&= \{f_1, f_2\}_l(\mathrm{Ad}_g X) \\
&= (\mathrm{Ad}_g)^*\{f_1, f_2\}_l(X).
\end{aligned}$$

We will see in Proposition 4.4 that in certain interesting cases the action is even Hamiltonian. We now verify condition (3.2). The ideal $I(N)$ of N is generated by those elements of the form $\xi_a^{n-1} - \xi_a(\alpha)$ for which $\deg e_a \leq d-h$. For $l = 0, \ldots, n-1$ these elements are Casimirs of $\{\cdot\,,\cdot\}_l$. Indeed, if $X \in \tilde{\mathfrak{g}}_n^\beta$ and $b \in I$ and $0 \leq k \leq n-1$ then

$$\{\xi_a^{n-1}, \xi_b^k\}_l(X) = \epsilon_l^{n-1,k} C_{ab}^c \xi_c^{n+k-l}(X) = 0$$

for $k \neq l$, while if $k = l$ then

$$\{\xi_a^{n-1}, \xi_b^l\}_l(X) = C_{ab}^c \xi_c^n(X) = C_{ab}^c \langle e_c, \beta\rangle_{\mathfrak{g}} = \langle [e_a, e_b], \beta\rangle_{\mathfrak{g}} = \langle [\beta, e_a], e_b\rangle_{\mathfrak{g}} = 0.$$

We used in the last equality that $[\beta, e_a] = 0$ if $\deg e_a \leq d-h$, which follows from $\deg[\beta, e_a] \leq -h$. This shows that (3.2) is satisfied when $l \neq n$. As for the n-th bracket, let $F \in \mathcal{O}(\tilde{\mathfrak{g}}_n^\beta, N)^{G_\beta^-}$ and let a be such that $\deg e_a \leq d-h$; notice that if F restricts to a G_β^- invariant function on N then F satisfies the infinitesimal condition

$$\langle [\nabla F(X), X], \nu\rangle = 0, \qquad \forall X \in N, \, \forall \nu \in \mathfrak{g}_\beta^-.$$

We need to show that $\{\xi_a^{n-1}, F\}_n(X) = 0$ for any $X \in N$. But

$$\begin{aligned}
\{\xi_a^{n-1}, F\}_n(X) &= \frac{1}{2}\left\langle X, [e_a, \nabla F(X)] + [\lambda^{-n} e_a, R(\lambda^n \nabla F(X))]\right\rangle \\
&= \frac{1}{2}\left\langle X, [e_a, \nabla F(X)] + [\lambda^{-n} e_a, \lambda^n \nabla F(X) - 2(\lambda^n \nabla F(X))_-]\right\rangle \\
&= \langle X, [e_a, \nabla F(X)]\rangle - \left\langle X, [\lambda^{-n} e_a, (\lambda^n \nabla F(X))_-]\right\rangle.
\end{aligned}$$

For the first term we have $\langle X, [e_a, \nabla F(X)]\rangle = \langle e_a, [\nabla F(X), X]\rangle = 0$ since $e_a \in \mathfrak{g}_\beta^-$, again because $[\beta, e_a] = 0$. Similarly we find that the second term vanishes:

$$\left\langle X, [\lambda^{-n} e_a, (\lambda^n \nabla F(X))_-]\right\rangle = \langle \beta, [e_a, (\lambda^n \nabla F(X))_{-1}]\rangle_{\mathfrak{g}} = 0.$$

∎

If β is a *generic* nilpotent element then the action of G_β^- is Hamiltonian, a fact that can be used to give an alternative proof of Theorem 4.2 for such β. The proof of this depends on several facts about simple Lie algebras which we will recall now (see [Kos1] for details). A nilpotent element is called principal when its isotropy algebra has dimension r; in this case the isotropy algebra is Abelian. A generic nilpotent element is principal and all principal nilpotent elements are conjugate to $\beta = \sum_{i=1}^r F_i$ whose isotropy algebra $\mathfrak{g}_\beta$ is contained in $\mathfrak{n}^-$. Notice that as a consequence $G_\beta^- = G_\beta$ and thus that it suffices to prove that the action of G_β on $\tilde{\mathfrak{g}}_n^\beta$ is Hamiltonian for $\beta = \sum_{i=1}^r F_i$. We will use the following lemma about the gradients of the Chevalley invariants.

Lemma 4.3 *If $\beta = \sum_{i=1}^r F_i$ then the gradient of the i-th Chevalley invariant I_i at β is homogeneous of degree $-(d_i - 1)$ and the gradients $\nabla I_1(\beta), \ldots, \nabla I_r(\beta)$ are linearly independent.*

Proof

For the first claim it suffices to show that $\langle \nabla I_i(\beta), y\rangle_{\mathfrak{g}} = 0$ for all $y \in \mathfrak{g}_j$ with $j \neq (d_i - 1)$, since $\langle \mathfrak{g}_l, \mathfrak{g}_m\rangle_{\mathfrak{g}} = 0$ if $l + m \neq 0$. To show this we introduce for all $x \in \mathfrak{g}$ the operator $\partial_x : \mathcal{O}(\mathfrak{g}) \to \mathcal{O}(\mathfrak{g})$ defined by

$$(\partial_x f)(z) = \langle \nabla f(z), x\rangle = \frac{d}{dt}_{|t=0} f(z + tx),$$

and we observe that $g(x) = \frac{1}{m!}(\partial_x^m g)(0)$ for any homogeneous polynomial g of degree m. Then, for $x = \beta$ and $f = I_i$, we get

$$\langle \nabla I_i(\beta), y\rangle = \partial_y I_i(\beta) = \frac{1}{(d_i - 1)!}(\partial_\beta^{d_i - 1} \partial_y I_i)(0).$$

But the proof of Lemma 14 in [Kos2] shows that $(\partial_\beta^{d_i-1}\partial_y I_i)(0) = 0$ if $y \in \mathfrak{g}_j$ with $j \neq d_i - 1$. Finally, the elements $\nabla I_i(\beta)$ are linearly independent since $\dim \mathfrak{g}_\beta = r$ (see [Kos2], Theorem 9). ∎

Proposition 4.4 *If $\beta = \sum_{i=1}^r F_r$ then the adjoint action of G_β on $\tilde{\mathfrak{g}}_n^\beta$ is Hamiltonian with respect to every Poisson structure $\{\cdot, \cdot\}_l$, $l = 0, \ldots, n$. We can choose a basis $\{b_i\}_{i=1}^r$ of $\mathfrak{g}_\beta$ in such a way that the corresponding infinitesimal generators $\mathcal{X}_i$ of the action are the Hamiltonian vector fields*

$$\mathcal{X}_i = \{\cdot, I_{i,n(d_i-1)-1}\}_0 = \{\cdot, I_{i,n(d_i-1)+l-1}\}_l, \qquad l = 0, \ldots, n, \quad i = 1, \ldots, r.$$

Proof

We first show that for any $X \in \tilde{\mathfrak{g}}_n^\beta$ and $i = 1, \dots, r$

$$b_i := \left(\nabla I_{i,n(d_i-1)-1}(X)\right)_+ \tag{4.2}$$

is independent of X as well as of λ and belongs to $\mathfrak{g}_\beta$. To see this, take $x \in \mathfrak{g}$ to find that

$$\begin{aligned}
\left\langle \nabla I_{i,n(d_i-1)-1}(X), x\lambda^{-l} \right\rangle &= \frac{d}{dt}_{|t=0} \operatorname{Res} \frac{I_i(X + tx\lambda^{-l})}{\lambda^{n(d_i-1)}} \\
&= \operatorname{Res} \frac{1}{\lambda^{n(d_i-1)}} \frac{d}{dt}_{|t=0} I_i(\beta\lambda^n + \dots + x_0 + tx\lambda^{-l}), \\
&= \operatorname{Res} \lambda^n \frac{d}{dt}_{|t=0} I_i(\beta + x_{n-1}\lambda^{-1} + \dots + x_0\lambda^{-n} + tx\lambda^{-l-n}) \\
&= \operatorname{Res} \lambda^{-l} \langle \nabla I_i(\beta + x_{n-1}\lambda^{-1} + \dots + x_0\lambda^{-n}), x\rangle_{\mathfrak{g}}.
\end{aligned}$$

Developing ∇I_i in a Taylor series at β we find (taking $l \geq 2$) that b_i is independent of λ, and (taking $l = 1$) that $b_i = \nabla I_i(\beta)$, independent of X. From this description we may conclude on the one hand that the elements b_i are independent, as a corollary of Lemma 4.3; on the other hand we may conclude that each b_i belongs to the isotropy algebra $\mathfrak{g}_\beta$ of β, since

$$[\beta, b_i] = [\beta, \nabla I_i(\beta)] = 0$$

by Ad-invariance of I_i. Since $\dim \mathfrak{g}_\beta = r$, it follows that the b_i $(i = 1, \dots, r)$ span $\mathfrak{g}_\beta$.

The corresponding generators are clearly the vector fields $\mathcal{X}_i$ defined by $\dot{X} = [b_i, X]$, where $X \in \tilde{\mathfrak{g}}_n^\beta$. But using (2.8) and the definition of b_i it is easily seen that $\mathcal{X}_i$ is the Hamiltonian vector field associated with $I_{i,n(d_i-1)-1}$ by means of $\{\cdot,\cdot\}_0$. Moreover, since their Hamiltonians are of the form I_{ij} the action of G_β is actually Hamiltonian with respect to any of the Poisson structures $\{\cdot,\cdot\}_l$ $(l = 0, \dots, n)$. ∎

4.2. Reduction of the master symmetry

We now turn our attention to the vector field $\mathcal{V}$ on $\tilde{\mathfrak{g}}$ which was shown to be a master symmetry for $\mathcal{A}$ and to have the deformation property with respect to the brackets $\{\cdot,\cdot\}_l$. We will now show that $\mathcal{V}$ descends to a master symmetry which has the deformation property. We will use the same notation $\{\cdot,\cdot\}_l$ for the reduced brackets (on $\mathcal{O}(N)^{G_\beta^-}$) as for the original ones (on $\mathcal{O}(\tilde{\mathfrak{g}}_n^\beta)$).

Proposition 4.5 *The master symmetry $\mathcal{V}$ is tangent to $\tilde{\mathfrak{g}}_n^\beta$ and $\mathcal{L}_{\mathcal{W}}\mathcal{O}(N)^{G_\beta^-} \subset \mathcal{O}(N)^{G_\beta^-}$, where $\mathcal{W}$ denotes the restriction of $\mathcal{V}$ to N. Therefore, the brackets $\{\cdot,\cdot\}_l$ and $\{\cdot,\cdot\}_{l-1}$ on $\mathcal{O}(\tilde{\mathfrak{g}}_n^\beta)$ which are connected by the Lie derivative with respect to $\mathcal{V}$ reduce to two brackets $\{\cdot,\cdot\}_l$ and $\{\cdot,\cdot\}_{l-1}$ on $\mathcal{O}(N)^{G_\beta^-}$ which are connected by the Lie derivative with respect to $\mathcal{W}$. Moreover $\mathcal{W}$ is a master symmetry for $\rho(\mathcal{A})$.*

Proof

The flow of $\mathcal{V}$ is given by

$$\phi_s : X(\lambda) \mapsto X(\lambda + s);$$

if $X(\lambda) \in N$, $X(\lambda) = \beta\lambda^n + x_{n-1}\lambda^{n-1} + \cdots$, then

$$\begin{aligned} X(\lambda+s) &= \beta(\lambda+s)^n + x_{n-1}(\lambda+s)^{n-1} + \cdots \\ &= \beta\lambda^n + (ns\beta + x_{n-1})\lambda^{n-1} + \cdots \end{aligned} \tag{4.3}$$

belongs to N since $\beta \in \mathfrak{n}_-$, showing that $\mathcal{V}$ is tangent to N. Also it is clear that $\mathcal{L}_{\mathcal{W}}\mathcal{O}(N)^{G_\beta^-} \subset \mathcal{O}(N)^{G_\beta^-}$ because G_β^- acts by simultaneous conjugation on the coefficients of λ in $X(\lambda)$, hence commutes with $\mathcal{L}_{\mathcal{W}}$. Thus Theorem 3.5 applies to yield the first statement. The fact that $\mathcal{W}$ is a master symmetry for $\rho(\mathcal{A})$ follows from Remark 3.7. ∎

4.3. The reduced space N/G_β^- as a linear subspace $N_0 \subset N$

We now show that when $\alpha + \beta$ is regular then the algebra $\mathcal{O}(N)^{G_\beta^-}$ is finitely generated (although G_β^- is not reductive) and that the quotient space N/G_β^- can be identified in a natural way with an affine subspace N_0 of N. By naturality we mean here that under the identification which we will construct the involutive algebra $\mathcal{A} \subset \mathcal{O}(N/G_\beta^-) \cong \mathcal{O}(N)^{G_\beta^-}$ and the vector field $\mathcal{W}$ correspond to their restriction to N_0, hence can easily be computed. Note however that the Poisson structures $\{\cdot\,,\cdot\}_l$ on N_0 are not obtained by restriction.

We will assume, as in Paragraph 4.1, that α and β are homogeneous with $\deg\alpha - \deg\beta = h$. We put $d = -\deg\beta$ and we assume that $\gamma = \alpha+\beta$ is regular, meaning that the isotropy subalgebra of γ is a Cartan subalgebra. We will give at the end of this section for every simple Lie algebra $\mathfrak{g}$ an important class of pairs (α,β) such that $\alpha+\beta$ is regular. The only property that we will use about the regularity of $\alpha+\beta$ is contained in the following lemma.

Lemma 4.6 *Let α and β be as above. Then $\mathfrak{g}_\alpha \cap \mathfrak{g}_\beta^- = \{0\}$.*

Proof

If $x \in \mathfrak{g}_\alpha \cap \mathfrak{g}_\beta^-$, then x belongs to the isotropy algebra of $\alpha+\beta$, which is a Cartan subalgebra, hence x is semisimple. On the other hand $x \in \mathfrak{n}^-$ hence it is nilpotent. Therefore $x = 0$. ∎

The space N_0 is constructed as follows. Let $\mathfrak{q}_i$ be a subspace of $\mathfrak{g}_i$, for $i = 1-d, \ldots, h-d-1$, such that

$$\mathfrak{g}_i = \mathfrak{q}_i \oplus \left[\mathfrak{g}_\beta \cap \mathfrak{g}_{i+d-h}, \alpha\right]. \tag{4.4}$$

If we denote

$$\mathfrak{q} = \left(\oplus_{i=-k}^{-d}\mathfrak{g}_i\right) \oplus \left(\oplus_{i=1-d}^{h-d-1}\mathfrak{q}_i\right),$$

then N_0 is defined by

$$N_0 = \{X \in N \mid x_{n-1} = \alpha + \tilde{x}_{n-1},\, \tilde{x}_{n-1} \in \mathfrak{q}\}.$$

Theorem 4.7 *If $\alpha+\beta$ is regular then the inclusion $\jmath : N_0 \to N$ induces an algebra isomorphism $\mathcal{O}(N)^{G_\beta^-} \cong \mathcal{O}(N_0)$ so that N/G_β^- is an affine space which can be identified with N_0. The functions in involution I_{ij} and the master symmetry $\mathcal{W}$ on N_0 are the restrictions of the corresponding functions I_{ij} and the master symmetry $\mathcal{W}$ on N.*

Proof

We first define a regular map $N \to G_\beta^- : X \mapsto g_X$ which has the property that $\mathrm{Ad}_{g_X} X \in N_0$ for any $X \in N$ and equals X for any $X \in N_0$. To determine g_X use the fact that $\mathfrak{g}_\beta^-$ is contained in $\mathfrak{n}^-$ to write it as

$$g_X = \exp \nu, \qquad \text{with} \qquad \nu = \sum_{j=1}^{k} \nu_{-j} \in \mathfrak{g}_\beta, \quad \nu_{-j} \in \mathfrak{g}_\beta \cap \mathfrak{g}_{-j}.$$

Then

$$\begin{aligned}(\mathrm{Ad}_{g_X})_{n-1} &= \mathrm{Ad}_{g_X} x_{n-1} \\ &= x_{n-1} + [\nu, x_{n-1}] + \cdots \\ &= \left(\alpha + \sum_{i=-k}^{k-d} \Pi_i x_{n-1}\right) + \left[\sum_{j=1}^{k} \nu_{-j}, \alpha + \sum_{i=-k}^{k-d} \Pi_i x_{n-1}\right] + \cdots,\end{aligned}$$

which has to be equal to $\alpha + \sum_{i=1-d}^{h-1-d} q_i + p$, with $q_i \in \mathfrak{q}_i$ and $p \in \oplus_{i=-k}^{-d} \mathfrak{g}_i$. The projection on $\mathfrak{g}_{h-d}$ yields $\alpha = \alpha$ while projection on $\mathfrak{g}_{h-d-1}$ leads to

$$\Pi_{h-d-1} x_{n-1} = q_{h-d-1} - [\nu_{-1}, \alpha],$$

from which q_{h-d-1} and ν_{-1} are uniquely determined because of the direct sum decomposition

$$\mathfrak{g}_{h-d-1} = \mathfrak{q}_{h-d-1} \oplus [\mathfrak{g}_\beta \cap \mathfrak{g}_{-1}, \alpha]$$

and $\mathfrak{g}_\alpha \cap \mathfrak{g}_\beta^- = \{0\}$ (Lemma 4.6). More generally, the projection on $\mathfrak{g}_j$ ($j = h-d-1, \ldots, 1-d$) yields

$$\Pi_j x_{n-1} + (\text{known stuff}) = q_j - [\nu_{j+d-h}, \alpha],$$

which gives a unique $q_j \in \mathfrak{q}_j$ and a unique $\nu_{j+d-h} \in \mathfrak{g}_\beta \cap \mathfrak{g}_{j+d-h}$. This gives us the desired map $N \to G_\beta^-$; since all q_j and ν_i are unique, all elements of N_0 map to the identity element in G_β^-. The map $N \to G_\beta^-$ is regular because the ν_i depend linearly on the entries of x_{n-1} and $\exp : \mathfrak{g}_\beta^- \to G_\beta^-$ is a regular map. Notice that only $\Pi_j x_{n-1}$, $j = 1-d, \ldots, h-d-1$ enter the construction of g_X, so that $g_X = g_{X'}$ if $\Pi_j x_{n-1} = \Pi_j x'_{n-1}$ for $j = 1-d, \ldots, h-d-1$.

We thus also have a regular map $\psi : N \to N_0$ given by $X \mapsto \mathrm{Ad}_{g_X} X$. Let us show that the image of the induced injective map $\psi^* : \mathcal{O}(N_0) \to \mathcal{O}(N)$ is precisely $\mathcal{O}(N)^{G_\beta^-}$. If $F \in \mathcal{O}(N_0)$ then $\psi^* F$ is G_β^--invariant because ψ is G_β^--invariant, hence ψ^* is injective; also, if F is a G_β^--invariant function then its restriction to N_0 maps to F under ψ^*, hence the image of ψ^* is $\mathcal{O}(N)^{G_\beta^-}$. In conclusion $\mathcal{O}(N_0)$ and $\mathcal{O}(N)^{G_\beta^-}$ are isomorphic and we can identify N_0 as the quotient N/G_β^-.

The functions I_{ij} on N are G_β^--invariant hence pass to the quotient N_0. Since the quotient map was induced by the inclusion map $\jmath : N_0 \to N$ the corresponding functions on N_0 are just obtained by restriction. Formula (4.3) implies that $\mathcal{W}$ is tangent to N_0 and also that $g_{X(\lambda)} = g_{X(\lambda+s)}$. If we denote the restriction of $\mathcal{W}$ to N_0 by $\mathcal{W}'$ then it follows that $\mathcal{L}_{\mathcal{W}} \psi^* = \psi^* \mathcal{L}_{\mathcal{W}'}$, in other words the projection of $\mathcal{W}$ on $N_0 = N/G_\beta^-$ is just $\mathcal{W}'$, the restriction of $\mathcal{W}$ to N_0. In conclusion the functions in involution and their master symmetry have a simple description on the reduced space N_0. ∎

We end this section by giving a general rule to select pairs (α, β), with $\deg \alpha - \deg \beta = h$, such that $\alpha + \beta$ is regular. We first recall some facts from [Kos1], Theorem 6.7. Let $\beta_1 = \sum_{i=1}^{r} F_i$ and let

$\alpha_1 \in \mathfrak{g}_k$, $\alpha_1 \neq 0$. Then $\gamma_1 = \alpha_1 + \beta_1$ is regular, so that $\mathfrak{h}' = \mathfrak{g}_{\gamma_1}$ is a Cartan subalgebra. Moreover, there exists a basis $\{\gamma_1, \ldots, \gamma_r\}$ of $\mathfrak{h}'$ with the following properties (here $\Pi^{\pm}$ is the projection onto $\mathfrak{n}^{\pm}$ and $q_1 \leq q_2 \ldots \leq q_r$ are the exponents of $\mathfrak{g}$):

1) $\beta_s = \Pi^{-}(\gamma_s)$ is homogeneous of degree $-q_s$;
2) $\alpha_s = \Pi^{+}(\gamma_s)$ is homogeneous of degree $h - q_s$.

The next proposition allows us to determine which pairs (α_s, β_s) are such that $\gamma_s = \alpha_s + \beta_s$ is regular. We are grateful to B. Kostant for providing us with a proof.

Proposition 4.8 *The element γ_s is regular if and only if q_s is coprime to the Coxeter number h.*

Proof

Let H_0 be the unique element in $\mathfrak{h}$ such that $[H_0, E_i] = E_i$ for all $i = 1, \ldots, r$ (the existence and uniqueness of H_0 follow from the fact that the Cartan matrix of $\mathfrak{g}$ is invertible). Then it is easily seen that

$$[H_0, x] = jx \qquad \forall\, x \in \mathfrak{g}_j. \tag{4.5}$$

If we define $P_0 \in G$ by $P_0 = \exp(\frac{2\pi\sqrt{-1}}{h} H_0)$, then (4.5) implies that $\mathrm{Ad}_{P_0}\, \gamma_i = \omega^{-q_i}\gamma_i$, where $\omega = e^{2\pi\sqrt{-1}/h}$. In particular we have that $\mathrm{Ad}_{P_0}(\mathfrak{h}') \subset \mathfrak{h}'$, so that P_0 belongs to the normalizer $N(H')$ of $H' = \exp \mathfrak{h}'$ in G. We denote the element of $N(H')/H'$ which corresponds to P_0 by c. The group $W = N(H')/H'$ is called the Weyl group of $\mathfrak{g}$. Clearly each element $w \in W$ acts on $\mathfrak{h}'$ by the adjoint action; we will use $w(x)$ to stand for $\mathrm{Ad}_g\, x$, where $g \in N(H')$ is any representative of w. Since $q_1 = 1$, it follows from the fact that $c(\gamma_s) = \omega^{-q_s}\gamma_s$ for $s = 1, \ldots, r$ that the order of c is h.

Now let us suppose that $m > 1$ is a common divisor of q_s and h. Then we can write $h = h'm$, $q_s = q_s' m$ for some $h', q_s' \in \mathbf{N}$. We show that γ_s cannot be regular by proving that $c^{h'}$ is a nontrivial element of W that leaves γ_s fixed (see, e.g., [Kna], p.426–427). Indeed,

$$c^{h'}(\gamma_s) = \omega^{-q_s h'}\gamma_s = \omega^{-h q_s'}\gamma_s = \gamma_s,$$

and $c^{h'}$ is not identity because the order of c is $h > h'$.

Conversely, assume that q_s is coprime to h. Then $\omega_s = \omega^{-q_s}$ is still a primitive h-root of unity. If $\{I_j\}_{j=1,\ldots,r}$ are the Chevalley invariants, $\deg I_j = q_j + 1$, then we have that $I_j(\gamma_s) = 0$ for all $j < r$. Indeed,

$$I_j(\gamma_s) = I_j(c(\gamma_s)) = I_j(\omega_s\gamma_s) = \omega_s^{q^j+1} I_j(\gamma_s),$$

while (4.1) implies that $q_j + 1 < q_r + 1 = h$, so that $\omega_s^{q_j+1} \neq 1$ for $j < r$. On the other hand, $I_r(\gamma_s)$ cannot vanish, because any element of $\mathfrak{g}$ at which all invariants vanish is nilpotent (see Theorem 9.1 of [Kos1]). Therefore there exists a non-zero $b \in \mathbf{C}$ such that $I_j(b\gamma_1) = I_j(\gamma_s)$ for $j = 1, \ldots, r$. Now, Lemma 9.2 of [Kos1] states that two elements of a Cartan subalgebra at which all invariants take the same values are W-conjugate. Since γ_1 is regular, γ_s is regular too. ∎

5. Examples

In this section we elaborate on the examples of the classical Lie algebras and G_2. Each will be realized as a subalgebra of $\mathfrak{sl}(N)$ and we will use $\langle x,y\rangle_{\mathfrak{g}} = \mathrm{Tr}(xy)$ as ad-invariant inner product; the representation will be such that $\mathfrak{g}$ has a Weyl basis of a simple form. We denote by e_{ij} the $N\times N$ matrix whose only non-zero entry is a one at position (i,j).

5.1. The Mumford system

We first show that when our construction is applied to the case of $\mathfrak{g}=\mathfrak{sl}(2)$ we get the Mumford system. We explicitly describe the reduced brackets, exhibit the multi-Hamiltonian hierarchies and check the deformation property of the master symmetry.

A Weyl basis for $\mathfrak{g}=\mathfrak{sl}(2)$ is given by $E=e_{12}$, $F=e_{21}$, $H=e_{11}-e_{22}$, leading to $\beta=e_{21}$ and $\alpha=e_{12}$. Then $\tilde{\mathfrak{g}}_n^\beta$ consists of those matrices

$$\begin{pmatrix} v(\lambda) & u(\lambda)\\ w(\lambda) & -v(\lambda)\end{pmatrix}$$

for which $w(\lambda)$ is monic of degree n and both $u(\lambda)$ and $v(\lambda)$ have degree less than n. We will write $u(\lambda)=\sum_{i=0}^{n-1}u_i\lambda^i$ and similarly for $v(\lambda)$ and $w(\lambda)$. The hyperplane N of $\tilde{\mathfrak{g}}_n^\beta$ is defined by the extra condition that $u(\lambda)$ is monic of degree $n-1$, i.e., it is defined by $u_{n-1}=1$. The group G_β consists of all matrices of the form $\begin{pmatrix} 1 & 0\\ a & 1\end{pmatrix}$ with Lie algebra $\mathfrak{g}_\beta=\mathbf{C}\beta=\mathfrak{g}_{-1}$. For $i=0$ the decomposition (4.4) gives

$$\mathfrak{g}_0=\mathfrak{q}_0\oplus[\mathfrak{g}_{-1},\alpha],$$

leading to $\mathfrak{q}_0=0$. Therefore the quotient space N/G_β is identified with the affine space N_0 of all matrices

$$\begin{pmatrix} V(\lambda) & U(\lambda)\\ W(\lambda) & -V(\lambda)\end{pmatrix}\quad\text{such that}\quad\begin{cases} U(\lambda)\text{ monic, } \deg U(\lambda)=n-1,\\ \deg V(\lambda)<n-1,\\ W(\lambda)\text{ monic, } \deg W(\lambda)=n.\end{cases}$$

Again we will write $U(\lambda)=\sum_{i=0}^{n-1}U_i\lambda^i$, where $U_{n-1}=1$, and similarly for $V(\lambda)$ and $W(\lambda)$. Since the algebra of invariant polynomials on $\mathfrak{g}$ is generated by $x\mapsto \mathrm{Tr}\,x^2$ we find that the algebra $\mathcal{A}$ on $\tilde{\mathfrak{g}}_n^\beta$ is generated by the coefficients (in λ) of the polynomial

$$u(\lambda)w(\lambda)+v^2(\lambda),$$

and on N_0 by the coefficients of

$$U(\lambda)W(\lambda)+V^2(\lambda).$$

In order to describe the reduced Poisson structures on N_0 we define $u_i, U_i,\dots$ to be zero for all values for which those variables have not been previously defined (e.g., $u_{-1}=u_n=0$). Then formula (2.3) for the brackets $\{\cdot,\cdot\}_l$ $(0\le l\le n)$ gives

$$\begin{aligned}
\{u_i,v_j\}_l&=\epsilon_l^{ij}u_{i+j+1-l},\\
\{v_i,w_j\}_l&=\epsilon_l^{ij}w_{i+j+1-l},\\
\{w_i,u_j\}_l&=2\epsilon_l^{ij}v_{i+j+1-l},
\end{aligned}$$

and all other brackets (between linear functions) are zero. The map $N \to G_\beta : X \mapsto g_X$ sends

$$\begin{pmatrix} v(\lambda) & u(\lambda) \\ w(\lambda) & -v(\lambda) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ v_{n-1} & 1 \end{pmatrix}$$

so that the quotient map $N \to N_0 : X \mapsto \mathrm{Ad}_{g_X} X$ is given explicitly by

$$\begin{aligned} U_i &= u_i, \\ V_i &= v_i - u_i v_{n-1}, \qquad\qquad i = 0, 1, \dots, n-1. \\ W_i &= w_i + 2v_i v_{n-1} - u_i v_{n-1}^2, \end{aligned} \tag{5.1}$$

The reduced brackets are computed by extending the functions at the right hand side of (5.1) (which are G_β-invariant functions on N) to functions on $\tilde{\mathfrak{g}}_n^\beta$ and by taking their bracket; we will[7] do this simply by taking the same expressions, but forgetting that $u_{n-1} = 1$. For example if $l \neq n$ then $\{V_i, W_j\}_l$ is found from

$$\{v_i - u_i v_{n-1}, w_j + 2v_j v_{n-1} - u_j v_{n-1}^2\}_l = \epsilon_l^{ij} w_{i+j+1-l} - \epsilon_l^{ij} u_{i+j+1-l} v_{n-1}^2 + 2\epsilon_l^{ij} v_{i+j+1-l} v_{n-1} + u_i \delta_j^l,$$

giving $\{V_i, W_j\}_l = \epsilon_l^{ij} W_{i+j+1-l} + U_i \delta_j^l$. In this way the reduced brackets $\{\cdot\,,\cdot\}_l$ are found to be given, for $l = 0, 1, \dots, n-1$, by

$$\begin{aligned} \{U_i, V_j\}_l &= \epsilon_l^{ij} U_{i+j+1-l}, & \{U_i, U_j\}_l &= 0, \\ \{V_i, W_j\}_l &= \epsilon_l^{ij} W_{i+j+1-l} + U_i \delta_j^l, & \{V_i, V_j\}_l &= 0, \\ \{W_i, U_j\}_l &= 2\epsilon_l^{ij} V_{i+j+1-l}, & \{W_i, W_j\}_l &= 2\delta_i^l V_j - 2\delta_j^l V_i, \end{aligned} \tag{5.2}$$

while the bracket $\{\cdot\,,\cdot\}_n$ is quadratic and is given by

$$\begin{aligned} \{U_i, V_j\}_n &= \epsilon_n^{ij} U_{i+j+1-n} - U_i U_j, & \{U_i, U_j\}_n &= 0, \\ \{V_i, W_j\}_n &= \epsilon_n^{ij} W_{i+j+1-n} - U_i W_j, & \{V_i, V_j\}_n &= 0, \\ \{W_i, U_j\}_n &= 2\epsilon_n^{ij} V_{i+j+1-n} - 2U_j V_i, & \{W_i, W_j\}_n &= 2V_i W_j - 2V_j W_i. \end{aligned} \tag{5.3}$$

Using these explicit formulas it is easy to verify that $\mathcal{V}$ has the deformation property with respect to all these brackets. For example, for the n-th bracket (which is quadratic) we find

$$\begin{aligned} \mathcal{L}_\mathcal{V}&\{U_i, V_j\}_n - \{\mathcal{L}_\mathcal{V} U_i, V_j\}_n - \{U_i, \mathcal{L}_\mathcal{V} V_j\}_n \\ &= [(i+j+2-n)\epsilon_n^{ij} - (i+1)\epsilon_n^{i+1,j} - (j+1)\epsilon_n^{i,j+1}] U_{i+j+2-n} \\ &= -n\epsilon_{n-1}^{ij} U_{i+j+2-n} \\ &= -n\{U_i, V_j\}_{n-1}. \end{aligned}$$

[7] Note that we can e.g. extend $v_i - u_i v_{n-1}$ also to the more symmetric expression $u_{n-1} v_i - u_i v_{n-1}$, but according to Theorem 3.2 the final result is independent of the chosen extensions.

Similarly (5.2) and (5.3) can be used to compute the Hamiltonian vector fields $\mathcal{X}_{I_i} = \{\cdot\,, I_i\}_0$ on N_0, where I_i is the i-th coefficient of $U(\lambda)W(\lambda)+V^2(\lambda)$. For example

$$\begin{aligned}
\mathcal{X}_i(U(\lambda)) &= \{U(\lambda), I_i\}_0\\
&= \sum_{j+k=i}\sum_{l=0}^{n-2}\{U_l, U_jW_k + V_jV_k\}_0\lambda^l\\
&= \sum_{j+k=i}\sum_{l=0}^{n-2}\left(2U_jV_{k+l+1} - V_jU_{k+l+1} - U_{l+j+1}V_k\right)\lambda^l\\
&= 2U(\lambda)\left[\frac{V(\lambda)}{\lambda^{i+1}}\right]_+ - 2V(\lambda)\left[\frac{U(\lambda)}{\lambda^{i+1}}\right]_+.
\end{aligned}$$

If we denote

$$A = \begin{pmatrix} V(\lambda) & U(\lambda)\\ W(\lambda) & -V(\lambda)\end{pmatrix} \quad\text{and}\quad B_i = \begin{pmatrix} 0 & 0\\ -U_i & 0\end{pmatrix}$$

then we recover the Lax equations

$$\dot{A} = -[A, (\lambda^{-i-1}A)_+ + B_i] \tag{5.4}$$

of which Mumford's vector field (1.1) is a special case (up to a factor -2; here $n = g+1$). Another way to obtain the vector field $\mathcal{X}_{I_i}$ on N_0 is to project the corresponding vector field on N along the tangent space to the orbits of G_β. Since this is spanned by $[A,\beta]$, one has to write

$$[A, (\lambda^{-i-1}A)_-] = \dot{A} + c(A)[A,\beta], \qquad\qquad A\in N_0,$$

where c is a function on N_0. The entry (1,1) of the coefficient of λ^{n-1} of this equation gives $c(A) = U_i$, and then (5.4) follows.

5.2. A_r

We now discuss the case of $\mathfrak{sl}(r+1)$ and obtain for every positive integer which is smaller than $r+1$ and coprime to $r+1$ a generalization of the Mumford system to matrices of size $r+1$. We will label the entries of elements of $\mathfrak{sl}(r+1)$ with indices $0,\dots,r$.

A Weyl basis $\{H_i, E_i, F_i\}_{i=1}^r$ is defined by $H_i = e_{i-1,i-1} - e_{i,i}$, $E_i = e_{i,i-1}$, and $F_i = E_i^t$. Clearly then $\mathfrak{g}_i$ is spanned by the elements $e_{j,i+j}$ so that $\dim\mathfrak{g}_i = r-i+1$ for $i>0$ and $h = r+1$. The elements $\alpha_1\in\mathfrak{g}_{r+1}$ and $\beta_1 = \sum_{i=1}^r F_i$ look as follows:

$$\alpha_1 = \begin{pmatrix} 0 & \cdots & 0 & 1\\ 0 & \cdots & 0 & 0\\ \vdots & & \vdots & \vdots\\ 0 & \cdots & 0 & 0\end{pmatrix} \qquad \beta_1 = \begin{pmatrix} 0 & & \cdots & & 0\\ 1 & 0 & & & \\ 0 & 1 & 0 & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & 0\end{pmatrix}.$$

The isotropy algebra of $\alpha_1+\beta_1$ is the algebra of matrices (a_{ij}) for which $a_{ij} = a_{i+1,j+1}$, where the indices i, j take values in $\mathbf{Z}_{r+1}$. It follows that β_d and α_d are given by

$$\alpha_d = \begin{pmatrix} 0 & I_d\\ 0 & 0\end{pmatrix} \qquad \beta_d = \begin{pmatrix} 0 & 0\\ I_{r+1-d} & 0\end{pmatrix}.$$

We fix a d coprime to $r+1$ and let $\alpha = \alpha_d$ and $\beta = \beta_d$. Then the space $\tilde{\mathfrak{g}}_n^\beta$ consists of polynomials of degree n with coefficients in $\mathfrak{g}$ whose top coefficient equals β. The elements of the subvariety N are those for which the second coefficient equals α plus arbitrary terms of degree less than $r+1-d$. The Lie algebra $\mathfrak{g}_\beta^-$ consists of the strictly lower triangular matrices of the form a_{ij} for which $a_{i+d,j+d} = a_{ij}$; here $0 \le j < i \le r-d$.

Proposition 5.1 *If $\mathfrak{g} = \mathfrak{sl}(r+1)$ and d is coprime to $h = r+1$ then the quotient space N_0 is given by* [8]

$$N_0 = \left\{ \begin{pmatrix} 0 & 0 \\ I_{h-d} & 0 \end{pmatrix} \lambda^n + \begin{pmatrix} 0 & I_d \\ \star & \star \end{pmatrix} \lambda^{n-1} + \sum_{i=0}^{n-2} x_i \lambda^i \mid x_i \in \mathfrak{sl}(r+1) \right\}.$$

Proof

Taking into account Theorem 4.7, we need to show that the spaces $\mathfrak{q}_i$, for $1-d \le i \le h-d-1$, can be chosen in such a way that the elements of $\mathfrak{q}$ have the form $\begin{pmatrix} 0_{d,r+1} \\ \star \end{pmatrix}$. In other words, if $\mathfrak{q}_i$ is the span of $\{e_{j,j+i}\}_{j=d,\dots,h-1-i}$, we must check that $\mathfrak{q}_i \oplus [\mathfrak{g}_\beta \cap \mathfrak{g}_{i+d-h}, \alpha] = \mathfrak{g}_i$. Since $\mathfrak{g}_\beta^- \cap \mathfrak{g}_a = \{0\}$ (by Lemma 4.6), we have that $\dim[\mathfrak{g}_\beta \cap \mathfrak{g}_{i+d-h}, \alpha] = \dim(\mathfrak{g}_\beta \cap \mathfrak{g}_{i+d-h})$; then from the explicit description of $\mathfrak{g}_\beta^-$ it is easily seen that

$$\dim \mathfrak{q}_i + \dim[\mathfrak{g}_\beta \cap \mathfrak{g}_{i+d-h}, \alpha] = \dim \mathfrak{g}_i,$$

so that we are left with showing that $\mathfrak{q}_i \cap [\mathfrak{g}_\beta \cap \mathfrak{g}_{i+d-h}, \alpha] = \{0\}$. To this aim, let us suppose that $M \in \mathfrak{g}_\beta \cap \mathfrak{g}_{i+d-h}$ and $[M, \alpha] \in \mathfrak{q}_i$; then $[M,\alpha]_{s,s+i} = 0$ for all $s = \max\{-i, 0\}, \dots, d-1$, that is,

$$M_{s,s+i+d-h} = M_{h+s-d,s+i} \qquad \text{for any } s = 0, \dots, d-1, \tag{5.5}$$

where we have put $M_{jk} = 0$ for indices j and k outside the range $0, \dots, r = h-1$. Let us define for $t = 0, \dots, h-1$ the elements $m_t = M_{t,t+i+d-h}$; then we have that $m_t = 0$ for $0 \le t \le h-d-i-1$. Moreover, equation (5.5) takes the form

$$m_t = m_{t+h-d} \qquad \text{for } t = 0, \dots, d-1. \tag{5.6}$$

If $i \le 0$, it is not difficult to show that this implies $m_t = 0$ for all t, that is, $M = 0$. For $i \ge 1$, we have to use also the fact that $M \in \mathfrak{g}_\beta$, i.e., that

$$m_t = m_{t+d} \qquad \text{for } t = h-d-i, \dots, h-d-1. \tag{5.7}$$

Now, equation (5.6) says that we can think of the indices in m_t as belonging to $\mathbf{Z}_{h-d}$. We already know that $m_t = 0$ for $0 \le t \le h-d-i-1$. In order to show that $m_s = 0$ for $h-d-i \le s \le h-d-1$, we fix such an m_s and we observe that $m_s = m_{s+d}$ on account of (5.7). If $s + d = s_1 + t_1(h-d)$ with $0 \le s_1 \le h-d-i-1$ then we are done. Otherwise, we can add d again to s_1, and we are sure that finally we will obtain an s_i such that $0 \le s_i \le h-d-i-1$ since the equivalence class of d is a generator of $\mathbf{Z}_{h-d}$ (because d and $h-d$ are coprime). ∎

A set of Chevalley invariant of $\mathfrak{g}$ is given by the polynomials $I_i : x \mapsto \operatorname{Tr} x^{i+1}$, $i = 1, \dots, r$, or, equivalently, by the coefficients of the characteristic polynomial $\det(x - \mu \operatorname{Id})$. Therefore the coefficients of $\det(X(\lambda) - \mu \operatorname{Id})$ give generators for $\mathcal{A}$.

[8] In this formula and in several formulas that follow we use stars as an abbreviation for arbitrary matrices of the appropriate size; of course it is understood that these "arbitrary" matrices must be chosen such that the resulting matrix is in $\mathfrak{g}$.

5.3. B_r and C_r

As usually these two families, which correspond to the symplectic and half of the orthogonal algebras can be treated simultaneously. The representations which we will choose are the ones for which the gradation is the restriction of the one for $\mathfrak{sl}(N)$; here $N = 2r+1$ corresponds to the orthogonal algebra B_r and $N = 2r$ to the symplectic algebra C_r. Let T denote the following $N \times N$ matrix,

$$T = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ \iddots & & & & \end{pmatrix} \qquad \text{i.e.} \quad T_{ij} = (-1)^{N-j}\delta_{i+j,N+1},$$

then $\mathfrak{g}$ is defined by $X^t T + TX = 0$, i.e.

$$X_{N+1-l,k} = (-1)^{N+k+l} X_{N+1-k,l}.$$

The meaning of this is that the main diagonal and all its parallels at even distance are skew-symmetric with respect to the secondary diagonal (hence $\operatorname{Tr} X = 0$) while the other ones are symmetric. Then $\dim \mathfrak{g} = r(2r+1)$ where $r = \left[\frac{N}{2}\right]$ is the rank of $\mathfrak{g}$. If $N = 2r+1$ then we define

$$\begin{aligned} E_i &= e_{i,i+1} + e_{2r+1-i,2r+2-i}, \qquad & i &= 1,\dots,r-1, \\ E_r &= 2(e_{r,r+1} + e_{r+1,r+2}), & & \\ F_i &= e_{i+1,i} + e_{2r+2-i,2r+1-i}, & i &= 1,\dots,r, \end{aligned}$$

while if $N = 2r$ we define

$$\begin{aligned} E_i &= e_{i,i+1} + e_{2r-i,2r+1-i}, \qquad & i &= 1,\dots,r-1, \\ E_r &= e_{r,r+1}, & & \end{aligned}$$

and $F_i = E_i^t$ for $i = 1,\dots,r$. In either case, if we introduce $H_i = [E_i, F_i]$ for $i = 1,\dots,r$, then $\{H_i, E_i, F_i\}$ is a Weyl basis for $\mathfrak{g}$. In particular the associated gradation $\mathfrak{g} = \oplus_{-k}^{k}\mathfrak{g}_i$ is the restriction of the one for $\mathfrak{sl}(N)$ and we have $\dim \mathfrak{g}_i = r - [i/2]$ for $i \geq 1$. In both cases the Coxeter number h equals $2r$. The principal nilpotent element β_1 is the same one as in the $\mathfrak{sl}(N)$ case and α_1 is for N even respectively for N odd given by

$$\alpha_1 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \qquad \text{resp.} \qquad \alpha_1 = \begin{pmatrix} 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

If N is even then $\alpha_1 + \beta_1$ is the same as in the $\mathfrak{sl}(2r)$ case and therefore its isotropy algebra can be obtained by means of a simple restriction. In particular α_d and β_d have the same form as in the $\mathfrak{sl}(2r)$ case (but only odd values of d are allowed) and when d is coprime to $2r$ (i.e., to r) the quotient space can be identified with a suitable affine subspace N_0. If $d = 1$ a possible choice for the coefficient of λ^{n-1} of the elements in N_0 is

$$\begin{pmatrix} 0 & \star & 0 & \star & \cdots & \star & 0 & 1 \\ \star & \star & \star & \star & \cdots & \star & \star & 0 \\ \star & & \cdots & & \cdots & & & \star \\ \vdots & & & & & & & \vdots \\ \star & & \cdots & & \cdots & & \star & 0 \end{pmatrix}.$$

If $d = 2r - 1$ two possible choices are

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & & & & \\ 0 & 0 & 1 & 0 & \cdots & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \\ & & \ddots & 0 & 1 & 0 & & \\ & & & \star & 0 & 1 & & \\ & & \iddots & & & \ddots & \ddots & \\ 0 & \star & 0 & \cdots & & & & 1 \\ \star & 0 & \cdots & & & & & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & 0 & \cdots & & & & \\ \star & 0 & 1 & 0 & \cdots & & & \\ 0 & 0 & \ddots & \ddots & \ddots & & & \\ \vdots & & \ddots & 0 & 1 & 0 & & \\ \vdots & & & & 0 & 1 & & \\ \star & & & & & \ddots & \ddots & \\ 0 & 0 & 0 & \cdots & & & & 1 \\ \star & 0 & \star & \cdots & & & & 0 \end{pmatrix}.$$

When N is odd the isotropy algebra of $\alpha_1 + \beta_1$ consists of those elements of the form

$$\begin{pmatrix} 0 & t_r & 0 & \cdots & & & \cdots & t_2 & 0 & t_1 & 0 \\ t_1 & 0 & 2t_r & 0 & \cdots & & & \cdots & 2t_2 & 0 & t_1 \\ 0 & t_1 & 0 & 2t_r & \cdots & 0 & & & \cdots & 2t_2 & 0 \\ t_2 & \ddots & \ddots & \ddots & \ddots & & & & & \cdots & t_2 \\ 0 & t_2 & & & & & & & & & \vdots \\ \vdots & \ddots & \ddots & & & & & & & & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ \vdots & & & & & t_2 & 0 & t_1 & 0 & 2t_r & 0 \\ t_r & & & & & & t_2 & 0 & t_1 & 0 & t_r \\ 0 & t_r & \cdots & & & & & t_2 & 0 & t_1 & 0 \end{pmatrix},$$

giving immediately the expressions for α_d and β_d. For $d = 1$ the coefficient of λ^{n-1} in N_0 can be chosen as

$$\begin{pmatrix} 0 & \star & 0 & \star & \cdots & 0 & 1 & 0 \\ \star & \star & \star & \star & \cdots & \star & 0 & 1 \\ \star & & \cdots & & \cdots & & & 0 \\ \vdots & & & & & & & \vdots \\ \star & & \cdots & & \cdots & & \star & 0 \end{pmatrix},$$

while for $d = 2r - 1$ two natural choices are

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & & & & & \\ 0 & 0 & 2 & 0 & \cdots & & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & & \\ & & \ddots & 0 & 2 & 0 & & & \\ & & & \star & 0 & 2 & & & \\ & & \iddots & & & \ddots & \ddots & & \\ 0 & \star & 0 & \iddots & & & 0 & 2 & 0 \\ \star & 0 & \star & & & & 0 & 1 & \\ 0 & \star & 0 & \cdots & & & & & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & 0 & \cdots & & & & & \\ \star & 0 & 2 & 0 & \cdots & & & & \\ 0 & 0 & \ddots & \ddots & \ddots & & & & \\ \vdots & & \ddots & 0 & 2 & 0 & & & \\ \vdots & & & & 0 & 2 & & & \\ \star & & & & & \ddots & \ddots & & \\ 0 & 0 & 0 & \cdots & & & 0 & 2 & 0 \\ \star & 0 & 0 & \cdots & & & & 0 & 1 \\ 0 & \star & 0 & \star & & & & & 0 \end{pmatrix}.$$

A set of Chevalley invariants, whose degrees are $2, 4, \ldots, 2r$ is given by the non-zero coefficients of the characteristic polynomial, viewed as functions on $\mathfrak{g}$.

5.4. D_r

We realize D_r as a subalgebra of $\mathfrak{sl}(N)$, $N = 2r$ as follows. Let T denote the following $N \times N$ matrix,

$$T = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix} \qquad \text{i.e.} \quad T_{ij} = (-1)^{\frac{|j-i|+1}{2}} \delta_{i+j,N+1},$$

then $\mathfrak{g}$ is defined as before by $X^t T + TX = 0$, i.e.

$$X_{N+1-l,k} = \pm(-1)^{k-l} X_{N-k+1,l}, \qquad \text{if} \quad \frac{2k-N-1}{2l-N-1} \lessgtr 0.$$

Thus, up to the $\pm$ sign this is the same as in the case of C_r and a generic element of D_r is written down by writing down a generic element of C_r, putting zeros at the secondary diagonal and changing all signs under this diagonal, except in the south-east $r \times r$ block. In particular $\operatorname{Tr} X = 0$ and $\dim \mathfrak{g} = r(2r-1)$. A Weyl basis for $\mathfrak{g}$ is in this case given by $\{E_i, F_i, H_i\}$ where

$$\begin{aligned} E_i &= e_{i,i+1} + e_{2r-i,2r+1-i}, \qquad & i = 1, \ldots, r-1, \\ E_r &= e_{r-1,r+1} + e_{r,r+2}, & \end{aligned}$$

$F_i = E_i^t$ and $H_i = [E_i, F_i]$ for $i = 1, \ldots, r$. The associated gradation $\mathfrak{g} = \oplus_{-k}^{k} \mathfrak{g}_i$ is now slightly more complicated; the portion above the secondary diagonal of a typical element of $\mathfrak{g}_i$, $i > 0$ has the following snake-shaped form.

$$\begin{matrix} \star & & & & & \\ & \ddots & & & & \\ & & \star & \star & & \\ & & & \star & & \\ & & & & \ddots & \\ & & & & & \star \end{matrix} \tag{5.8}$$

Precisely, a basis of $\mathfrak{g}_i$, $i > 0$ is given by

$$\begin{aligned} & e_{j,i+j} + (-1)^{i-1} e_{2r-i-j+1,2r-j+1}, \qquad & j = 1, \ldots, r-i, \\ & e_{j-1,i+j} + (-1)^{i-1} e_{2r-i-j+1,2r-j+2}, & j = \max\{2, r-i+1\}, \ldots, r - \left[\frac{i}{2}\right], \end{aligned}$$

giving $\dim \mathfrak{g}_i = r - [i/2]$ for $1 \le i < r$ and $\dim \mathfrak{g}_i = r - [i/2] - 1$ for $i \ge r$. A set of Chevalley invariants is given by the non-zero coefficients of the characteristic polynomial, with the highest order one (the determinant) being replaced with its square root. The elements α_1 and β_1 take the form

$$\alpha_1 = \begin{pmatrix} 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \beta_1 = \begin{pmatrix} 0 & & & & & & & \\ 1 & 0 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 1 & 0 & & & & \\ & & 1 & 0 & 0 & & & \\ & & & 1 & 1 & 0 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & 0 \end{pmatrix}$$

and one can show that $\alpha_{2r-2} = E_1 + 2\sum_{i=2}^{r-2} E_i + E_{r-1} + E_r$ and

$$\beta_{2r-2} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \end{pmatrix}.$$

For $d = 1$ the space $\mathfrak{q}$ may be taken consisting of all elements in $\mathfrak{g}$ of the form

$$\begin{pmatrix} 0 & \star & 0 & \star & \cdots & 0 & \star & \star & 0 & \star & 0 & \star & \cdots & & 0 & 0 & 0 \\ \star & \star & \star & \star & \cdots & \star & 0 & \star & \star & \star & \star & \star & \cdots & \star & \star & 0 & 0 \\ \star & & & & \cdots & & & \star & & & & & \cdots & & & & \star \end{pmatrix},$$

the 0 in the second row appearing at position $2\left[\frac{r+1}{2}\right]$. For $d = 2r - 2$ a possible choice for $\mathfrak{q}$ is

$$\begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \star & 0 & \cdots & \cdots & \\ 0 & 0 & & & \\ \vdots & \vdots & & & \vdots \\ \star & 0 & & & \\ \star & 0 & \cdots & \cdots & \\ 0 & 0 & \cdots & \cdots & \\ \vdots & \vdots & & & \\ \star & 0 & & & \vdots \\ 0 & 0 & & & \\ \star & 0 & \cdots & \cdots & \\ 0 & \star & \cdots & \cdots & 0 \end{pmatrix},$$

where the pair of stars in the first column appear at positions $2[r/2]$ and $2[r/2] + 1$.

5.5. G_2

Finally here is $\mathfrak{g} = \mathfrak{g}_2$ in the standard representation, as taken from [FH]. A Weyl basis is given by

$$E_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

F_1 is obtained by transposing E_1 and interchanging the middle 1 and 2, $F_2 = E_2^t$ and $H_i = [E_i, F_i]$, $(i = 1, 2)$. The spaces $\mathfrak{g}_i$ making up the gradation are spanned by the following vectors:

$$\begin{aligned}
\mathfrak{g}_1 &: E_1,\ E_2 \\
\mathfrak{g}_2 &: E_3 = [E_1, E_2] \\
\mathfrak{g}_3 &: E_4 = [E_1, [E_1, E_2]] \\
\mathfrak{g}_4 &: E_5 = [E_1, [E_1, [E_1, E_2]]] \\
\mathfrak{g}_5 &: E_6 = [E_2, [E_1, [E_1, [E_1, E_2]]]]
\end{aligned}$$

the spaces $\mathfrak{g}_{-i}$, $i > 0$, being constructed by using in the above formulas F's instead of E's. The ring of invariants is generated by $\operatorname{Tr} X^2$ and $\operatorname{Tr} X^6$, so that the exponents are 1 and 5. The elements α_1 and β_1 are given by $\alpha_1 = E_6$ and $\beta_1 = F_1 + F_2$, and the isotropy algebra of $\alpha_1 + \beta_1$ is spanned by $F_1 + F_2 + E_6$ and $F_6 + 36E_1 + 72E_2$. Therefore we also have $\alpha_5 = 6E_1 + 12E_2$ and $\beta_5 = F_6/6$. Since

$$\begin{aligned}
&[\mathfrak{g}_{\beta_1} \cap \mathfrak{g}_{-1}, \alpha_1] = \mathfrak{g}_4, \\
&[\mathfrak{g}_{\beta_1} \cap \mathfrak{g}_{-5}, \alpha_1] = \mathbf{C}[H_1 + 2H_2] = \mathbf{C}\, \mathrm{diag}\,[-1, -1, 0, 0, 0, 1, 1],
\end{aligned}$$

the quotient space can for $d = 1$ be taken as $N_0 = \beta_1 \lambda^n + \sum_{i=n-1}^{0} x_i \lambda^i$ where $x_{n-1} - \alpha_1$ lies in the 11-dimensional span of the vectors

$$H_2,\ E_1,\ E_2,\ E_3,\ E_4,\ F_1,\ F_2,\ F_3,\ F_4,\ F_5,\ F_6.$$

For $d = 5$ we have that $\mathfrak{g}_{\beta_5}^- = \mathfrak{n}^-$ and the quotient space can be taken as $N_0 = \beta_5 \lambda^n + \sum_{i=n-1}^{0} x_i \lambda^i$ where $x_{n-1} - \alpha_5$ lies in the span of the vectors F_2 and F_6.

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