NILPOTENT ORBITS IN SIMPLE LIE ALGEBRAS AND THEIR TRANSVERSE POISSON STRUCTURES

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Abstract. We consider nilpotent coadjoint orbits in complex simple Lie algebras and we examine their transverse Poisson structures. We specialize to the two extreme and most interesting cases, i.e. the subregular and minimal orbits.

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SOME GENERAL RESULTS

In this first section we prove some general results on the transverse Poisson structures to coadjoint orbits. In the next section we specialize to the two extreme and most interesting cases, i.e. the subregular and minimal orbits. The results on the minimal orbit are new and detailed proofs will appear in a future paper. We refer to [4] for proofs of the remaining Theorems.

Transverse Poisson Structures to adjoint orbits

The splitting theorem of A.Weinstein [7] says that locally a Poisson manifold is a direct product of a symplectic manifold with another singular Poisson manifold whose tensor vanishes at the point. More precisely: Let x_0 be a point in a Poisson manifold M of dimension n. Let S_{x_0} be the symplectic leaf through x_0, $\dim S_{x_0} = 2s$. Let N be an arbitrary $n-2s$ dimensional manifold transverse to S_{x_0} at x_0. Then N inherits a Poisson structure from M vanishing at x_0. This Poisson structure on N is in a neighborhood of x_0, unique up to Poisson diffeomorphism and is called the transverse Poisson structure at x_0.

We consider the case where $M = \mathbf{g}^*$, where $\mathbf{g}$ is a complex Lie algebra, equipped with its standard Lie-Poisson structure; see [1] for the definitions. As is well known, the symplectic leaf through $\mu \in \mathbf{g}^*$ is the co-adjoint orbit $\mathbf{G} \cdot \mu$ of the adjoint Lie group $\mathbf{G}$ of $\mathbf{g}$. A natural transverse slice to $\mathbf{G} \cdot \mu$ is obtained in the following way: we choose any complement $\mathbf{n}$ to the centralizer $\mathbf{g}(\mu)$ of μ in $\mathbf{g}$ and we take N to be the affine subspace $\mu + \mathbf{n}^{\perp}$ of $\mathbf{g}^*$. It follows easily that N is a transverse slice to $\mathbf{G} \cdot \mu$ at μ. Furthermore, defining on $\mathbf{n}^{\perp}$ any system of linear coordinates $(q_1, \ldots, q_k)$, and using the Dirac constraint formula, one can write down explicit formulas for the Poisson matrix $\Lambda_{ij} := \{q_i, q_j\}$ of the transverse Poisson structure. As a corollary, in the Lie-Poisson case, the transverse Poisson structure is always rational. Let us give a specific example of such computation using the Dirac reduction formula in the case of the simple Lie algebra $\mathbf{sp}_4$.

Example 1 *We realize the type C_2 Lie algebra as the set of matrices of the form*

$$\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^t \end{pmatrix}$$

where $Z_i \in M_2(\mathbf{C})$ and Z_2, Z_3 are symmetric.

We look at the subregular nilpotent orbit corresponding to the partition $(2,2)$. It is an orbit through the nilpotent element

$$X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

A simple calculation shows that a typical element in the transverse slice is given by

$$Q = \begin{pmatrix} 0 & -q_1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2q_4 & q_3 & 0 & 0 \\ q_3 & 2q_2 & q_1 & 0 \end{pmatrix} .$$

The Dirac constraint formula gives the structure matrix for the transverse Poisson structure Λ:

$$\Lambda = \begin{pmatrix} 0 & q_3 & 2q_4 - 2q_2 - q_1^2 & -q_3 \\ -q_3 & 0 & -2q_1q_4 & 0 \\ -2q_4 + 2q_2 + q_1^2 & 2q_1q_4 & 0 & -2q_1q_4 \\ q_3 & 0 & 2q_1q_4 & 0 \end{pmatrix} . \tag{1}$$

The coefficients of the characteristic polynomial of q are the Casimirs of the transverse Poisson structure. In our case the characteristic polynomial is

$$x^4 - 2(q_2 + q_4)x^2 + 2q_1^2 q_4 + 4q_2q_4 - q_3^2 .$$

What is striking in this example is that the transverse Poisson structure is actually polynomial. Therefore it is an interesting question to determine conditions on the Lie algebra $\mathbf{g}$, the particular co-adjoint orbit, and the particular complement $\mathbf{n}$ so that the transverse Poisson structure is polynomial.

In 1989 P. A. Damianou [3] made such a conjecture for gl_n. In 2002 R. Cushman and M. Roberts [2] proved that there exists for any nilpotent adjoint orbit of a semi-simple Lie algebra a special choice of a complement $\mathbf{n}$ such that the corresponding transverse Poisson structure is polynomial. In 2005 H. Sabourin in [5] gave a more general class of complements where the transverse structure is polynomial, using in an essential way the machinery of semi-simple Lie algebras. In this paper the transverse slice is always chosen to lie in the class of complements prescribed by Sabourin.

It turns out that the transverse Poisson structure to any adjoint orbit $\mathbf{G}\cdot x$ of a semi-simple (or reductive) algebra $\mathbf{g}$ is essentially determined by the transverse Poisson structure to the underlying nilpotent orbit $\mathbf{G}(s)\cdot e$ defined by its Jordan-Chevalley decomposition $x = s + e$ where s is semisimple, e is nilpotent and $[s,e] = 0$. In fact, as is proved in [4], there is an isomorphism between the transverse Poisson structure at x and the transverse Poisson structure at e. It follows that in well-chosen coordinates, the transverse Poisson structure to any adjoint orbit in a semi-simple Lie algebra is polynomial

THE SUBREGULAR AND MINIMAL CASES

We will give an explicit description of the transverse Poisson structure in the case of the subregular orbit $\mathscr{O}_{sr} \subset \mathbf{g}$, where $\mathbf{g}$ is a semi-simple Lie algebra. Recall that an element Z in $\mathbf{g}$ is subregular if $\dim \mathbf{g}(Z) = Rk(\mathbf{g}) + 2$. In this case, the generic rank of the transverse Poisson structure on N is 2 and we know $\dim N - 2$ independent Casimirs, namely the basic Ad-invariant functions on $\mathbf{g}$, restricted to N. It follows that the transverse Poisson structure is the determinantal structure, determined by these Casimirs, up to multiplication by a function. What is much less trivial to show is that this function is actually just a non-zero *constant.*

Simple singularities

Consider the adjoint quotient map given by

$$\begin{array}{ccccc} G & : & \mathbf{g} & \to & \mathbf{C}^{\ell} \\ & & x & \mapsto & (G_1(x), G_2(x), \ldots, G_{\ell}(x)), \end{array}$$

where G_i are the Chevalley invariants of $\mathbf{g}$ and ℓ is the rank of $\mathbf{g}$. These functions are Casimirs of the Lie-Poisson structure on $\mathbf{g}$. If we denote by χ_i the restriction of G_i to the transverse slice N, then it follows that these functions are independent Casimirs of the transverse Poisson structure. The zero-fiber $G^{-1}(0)$ of G is exactly the nilpotent variety $\mathscr{N}$ of $\mathbf{g}$. We are interested in $N \cap \mathscr{N} = N \cap G^{-1}(0) = \chi^{-1}(0)$, which is an affine surface with an isolated, simple singularity.

Up to conjugacy, there are five types of finite subgroups of $\mathbf{SL}_2 = \mathbf{SL}_2(\mathbf{C})$, the cyclic, dihedral and three exceptional types, denoted by $\mathscr{C}_p, \mathscr{D}_p, \mathscr{T}, \mathscr{O}$ and $\mathscr{I}$. Given such a subgroup $\mathbf{F}$, one looks at the corresponding ring of invariant polynomials $\mathbf{C}[u,v]^{\mathbf{F}}$. In each of the five cases, $\mathbf{C}[u,v]^{\mathbf{F}}$ is generated by three fundamental polynomials X, Y, Z, subject to only one relation $R(X,Y,Z) = 0$, hence the quotient space $\mathbf{C}^2/\mathbf{F}$ can be identified, as an affine surface, with the singular surface in $\mathbf{C}^3$, defined by $R=0$. The origin is its only singular point; it is called a *(homogeneous) simple singularity*.

For the other simple Lie algebras (of type B_ℓ, C_ℓ, F_4 or G_2), there exists a similar correspondence. By definition, an *(inhomogeneous) simple singularity* of type Δ is a couple (V,Γ) consisting of a homogeneous simple singularity $V=\mathbf{C}^2/\mathbf{F}$ and a group $\Gamma=\mathbf{F}'/\mathbf{F}$ of automorphisms of V.

We can now state the following extension of a theorem of Brieskorn, which is due to Slodowy [6]

Theorem 1 *Let* $\mathbf{g}$ *be a simple complex Lie algebra, with Dynkin diagram of type* Δ*. Let* $\mathscr{O}_{sr}=\mathbf{G}\cdot e$ *be the subregular orbit and* $N=e+\mathbf{n}^\perp$ *a transverse slice to* $\mathscr{O}_{sr}$*. The surface* $N\cap\mathscr{N}=\chi^{-1}(0)$ *has a (homogeneous or inhomogeneous) simple singularity of type* Δ*.*

Example 2 *In example (1) the Casimirs are* $f_1=q_2+q_4$ *and* $f_2=2q_1^2q_4+4q_2q_4-q_3^2$*. The common level set of the Casimirs is obtained by eliminating all but three variables and we obtain*

$$2q_1^2q_2+4q_2^2+q_3^2=0\ ,$$

a surface with a type D_3 *singularity which agrees with the result of Slodowy.*

The determinantal Poisson structure

In terms of linear coordinates $q_1,q_2,\ldots,q_{\ell+2}$ on $\mathbf{C}^{\ell+2}$, the formula

$$\{f,g\}_{det}:=\frac{df\wedge dg\wedge d\chi_1\wedge\ldots\wedge d\chi_\ell}{dq_1\wedge dq_2\wedge\ldots\wedge dq_{\ell+2}} \tag{2}$$

defines a Poisson bracket on $\mathbf{C}^{\ell+2}$ with Casimirs $\chi_1,\ldots,\chi_\ell$.

Looking at our example, the two Casimirs can be used to compute the transverse Poisson structure via the determinant formula (2). It turns out that it gives the same result (up to a constant multiple) as (1).

This phenomenon is general. In the subregular orbit we have two polynomial Poisson structures on the transverse slice N which have $\chi_1,\ldots,\chi_\ell$ as Casimirs on $N\cong\mathbf{C}^{\ell+2}$, namely the transverse Poisson structure and the determinantal structure, constructed by using these Casimirs. It is proved in [4] that these two Poisson brackets are essentially the same:

Theorem 2 *Let* $\mathscr{O}_{sr}$ *be the subregular nilpotent adjoint orbit of a complex semi-simple Lie algebra* $\mathbf{g}$ *and let* (h,e,f) *be the canonical triple, associated to* $\mathscr{O}_{sr}$*. Let* $N=e+\mathbf{n}^\perp$ *be a transverse slice to* $\mathscr{O}_{sr}$*, where* $\mathbf{n}$ *is an* ad_h*-invariant complementary subspace to* $\mathbf{g}(e)$*. Let* $\{\cdot,\cdot\}_N$ *and* $\{\cdot,\cdot\}_{det}$ *denote respectively the transverse Poisson structure and the determinantal structure on* N*. Then* $\{\cdot,\cdot\}_N=c\{\cdot,\cdot\}_{det}$ *for some* $c\in\mathbf{C}^*$*.*

The minimal orbit

In this subsection we consider the transverse Poisson structure to the minimal orbit $\mathscr{O}_{min}$ in an arbitrary semi-simple Lie algebra **g**. This orbit is the nilpotent orbit of minimal dimension (besides the trivial orbit $\{0\}$). It is unique and is generated by a root vector E_{min}, associated to a highest root, with respect to a fixed Cartan subalgebra **h** and a choice of simple roots.

Theorem 3 *The transverse Poisson structure of the minimal orbit $\mathscr{O}_{min}$ is the sum of two Poisson structures $\Lambda_{min} = \mathscr{A} + \mathscr{Q}$, where*

1. *$\mathscr{A}$ is a linear Poisson structure, isomorphic to the Lie-Poisson structure on the dual of the Lie algebra $\mathbf{g}(E_{min})$;*
2. *$\mathscr{Q}$ is a quadratic Poisson bracket, whose generic rank is $\dim \mathscr{O}_{min} - 2$.*
3. $\operatorname{rank} \Lambda_{min} = \operatorname{rank} \mathscr{A}$.
4. *The pair $(\mathscr{A}, \mathscr{Q})$ defines a bi-hamiltonian vector field.*

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