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Height-2 Toda systems

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We announce some results which involve some new, evidently integrable systems of Toda type. More specifically we construct a large family of Hamiltonian systems which interpolate between the classical Kostant–Toda lattice and the full Kostant–Toda lattice and we discuss their integrability. There is one such system for every nilpotent ideal $\mathcal{I}$ in a Borel subalgebra $\mathfrak{b}_+$ of an arbitrary simple Lie algebra $\mathfrak{g}$. The classical Kostant–Toda lattice corresponds to the case of $\mathcal{I} = [\mathfrak{n}_+, \mathfrak{n}_+]$, where $\mathfrak{n}_+$ is the unipotent ideal of $\mathfrak{b}_+$, while the full Kostant–Toda lattice corresponds to $\mathcal{I} = \{0\}$. We mainly focus on the case of $\mathfrak{g}$ being of type A, B or C with $\mathcal{I} = [[\mathfrak{n}_+, \mathfrak{n}_+], \mathfrak{n}_+]$ which we call the height-2 Toda lattice. Complete proofs of the announced results will appear in a future publication.

1 Introduction

The classical Toda lattice is the mechanical system with Hamiltonian function

$$H(q_1, \ldots, q_N, p_1, \ldots, p_N) = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}}.$$

It describes a system of N particles on a line connected by exponential springs. The differential equations which govern this lattice can be transformed via a change of variables due to Flaschka [9] to a Lax equation $\dot{L} = [L_+, L]$, where L is the Jacobi matrix

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & & \cdots & a_{N-1} & b_N \end{pmatrix}, \tag{1}$$

and L_+ is the skew-symmetric part of L in the Lie algebra decomposition lower triangular plus skew-symmetric. Lax equations define isospectral deformations; though the entries of L vary over time, the eigenvalues of L remain constant. It follows that the functions $H_i = \frac{1}{i}\operatorname{Tr} L^i$ are constants of motion. Moreover they are in involution with respect to a Poisson structure associated to the above Lie algebra decomposition.

There is a generalization due to Deift, Li, Nanda and Tomei [5] who showed that the system remains integrable when L is replaced by a full symmetric $N \times N$ matrix. The resulting system is called the full symmetric Toda lattice. The functions $H_i := \frac{1}{i}\operatorname{Tr} L^i$ are still in involution, but they are not enough to ensure integrability. It was shown in [5] that there are additional integrals, called chop integrals, which are rational functions of the entries of L. They are constructed as follows. For $k = 0, \ldots, [\frac{(N-1)}{2}]$, denote by $(L - \lambda \operatorname{Id}_N)_k$ the result of removing the first k rows and the last k columns from $L - \lambda \operatorname{Id}_N$ and let

$$\det(L - \lambda \operatorname{Id}_N)_k = E_{0k}\lambda^{N-2k} + \cdots + E_{N-2k,k}. \tag{2}$$

Set

$$\frac{\det\,(L - \lambda \operatorname{Id}_N)_k}{E_{0k}} = \lambda^{N-2k} + I_{1k}\lambda^{N-2k-1} + \cdots + I_{N-2k,k}. \tag{3}$$

The functions I_{rk}, where $r = 1, \ldots, N-2k$ and $k = 0, \ldots, [\frac{N-1}{2}]$, are independent constants of motion, they are in involution and sufficient to account for the integrability of the full Toda lattice.

1.1 Bogoyavlensky–Toda

The classical Toda lattice was generalized in another direction. One can define a Toda-type system for each simple Lie algebra. The finite, nonperiodic Toda lattice corresponds to a root system of type A_ℓ. This generalization is due to Bogoyavlensky [3]. These systems were studied extensively in [10] in which the solution of the system was connected intimately with the representation theory of simple Lie groups. See also Olshanetsky–Perelomov [11] and Adler–van Moerbeke [1]. We call these systems the Bogoyavlensky–Toda lattices. They can be described as follows.

Let $\mathfrak{g}$ be any simple Lie algebra equipped with its Killing form $\langle \cdot \,|\, \cdot \rangle$. One chooses a Cartan subalgebra, $\mathfrak{h}$ of $\mathfrak{g}$, and a basis Π of simple roots for the root system Δ of $\mathfrak{h}$ in $\mathfrak{g}$. The corresponding set of positive roots is denoted by Δ^+. To each positive root α one can associate a triple $(X_\alpha, X_{-\alpha}, H_\alpha)$ of vectors in $\mathfrak{g}$ which generate a Lie subalgebra isomorphic to $\mathrm{sl}_2(\mathbf{C})$. The set $(X_\alpha, X_{-\alpha})_{\alpha\in\Delta^+} \cup (H_\alpha)_{\alpha\in\Pi}$ is basis of $\mathfrak{g}$ and is called a root basis. To these data one associates the Lax equation $\dot{L} = [L_+, L]$, where L and L_+ are defined as follows:

$$L = \sum_{i=1}^{\ell} b_i H_{\alpha_i} + \sum_{i=1}^{\ell} a_i (X_{\alpha_i} + X_{-\alpha_i}), \quad L_+ = \sum_{i=1}^{\ell} a_i (X_{\alpha_i} - X_{-\alpha_i}).$$

The affine space M of all elements L of $\mathfrak{g}$ of the above form is the phase space of the Bogoyavlensky–Toda lattice associated to $\mathfrak{g}$. The functions which yield the integrability of the system are the Ad-invariant functions on $\mathfrak{g}$which are restricted to M.

1.2 Kostant form

Let D be the diagonal $N \times N$ matrix with entries $d_i := \prod_{j=1}^{i-1} a_j$. In [10] Kostant conjugates the matrix L, given by (1), by the matrix D to obtain a matrix of the form

$$X = \begin{pmatrix} b_1 & 1 & 0 & \cdots & \cdots & 0 \\ c_1 & b_2 & 1 & \ddots & & \vdots \\ 0 & c_2 & b_3 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & 0 \\ \vdots & & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & c_{N-1} & b_N \end{pmatrix}. \tag{4}$$

The Lax equation takes the form $\dot{X} = [X_+, X]$, where X_+ is the strictly lower triangular part of X, according to the Lie algebra decomposition strictly lower plus upper triangular. This form is convenient in applying Lie theoretic techniques to describe the system. Note that the diagonal elements correspond to the Cartan subalgebra while the subdiagonal elements correspond to the set Π of simple roots. The full Kostant–Toda lattice is obtained by replacing Π with Δ^+ in the sense that one fills the lower triangular part of X in (4) with additional variables. It leads on the affine space of all such matrices to the Lax equation

$$\dot{X} = [X_+, X], \tag{5}$$

where X_+ is again the projection to the strictly lower part of X.

1.3 Adapted sets in a root system

Generalizing the above procedure we can introduce the Lax pair (L_Φ, B_Φ), where Φ is any subset of Δ^+ containing Π. Thus we have

$$L_\Phi = \sum_{\alpha \in \Pi} b_\alpha H_\alpha + \sum_{\alpha \in \Phi} a_\alpha (X_\alpha + X_{-\alpha}), \quad B_\Phi = \sum_{\alpha \in \Phi} a_\alpha (X_\alpha - X_{-\alpha}).$$

In order to have consistency in the Lax equation, since the Lax matrix is symmetric, the bracket $[B_\Phi, L_\Phi]$ should give an element of the form $\sum_{\alpha \in \Phi} c_\alpha H_\alpha + \sum_{\alpha \in \Phi} d_\alpha (X_\alpha + X_{-\alpha})$. In this case we say that Φ is adapted. A straightforward computation leads to the following result:

Proposition 1. *The set Φ is adapted if and only if it satisfies the following property:*

$$\forall \alpha, \beta \in \Phi, \quad \alpha - \beta \text{ or } \beta - \alpha \in \Phi \cup \{0\}.$$

Recall that $\alpha - \beta = 0$ means that $\alpha - \beta$ is not a root.

Thus for each Φ which is adapted we obtain a corresponding Hamiltonian system and the problem is to study this system and determine whether it is integrable. We conjecture that in fact it is integrable. We prove this claim for a particular class of such systems. Note that the special case $\Phi = \Pi$ corresponds to the classical Toda lattice while the case $\Phi = \Delta^+$ corresponds to the full symmetric Toda of [5].

Example 1. We consider a Lie algebra of type B_2. The set of positive roots $\Delta^+ = \{\alpha, \beta, \alpha+\beta, \beta+2\alpha\}$ which corresponds to the full symmetric Toda lattice with Lax matrix

$$L = \begin{pmatrix} b_1 & a_1 & a_3 & a_4 & 0 \\ a_1 & b_2 & a_2 & 0 & -a_4 \\ a_3 & a_2 & 0 & -a_2 & -a_3 \\ a_4 & 0 & -a_2 & -b_2 & -a_1 \\ 0 & -a_4 & -a_3 & -a_1 & -b_1 \end{pmatrix}.$$

This system is completely integrable with integrals $h_2 = \frac{1}{2}\mathrm{Tr}L^2$ which is the Hamiltonian, $h_4 = \frac{1}{2}\mathrm{Tr}L^4$ and a rational integral which is obtained by the method of chopping as in [5].

Taking $\Phi = \{\alpha, \beta, \alpha+\beta\}$ we obtain another integrable system with Lax matrix

$$L = \begin{pmatrix} b_1 & a_1 & a_3 & 0 & 0 \\ a_1 & b_2 & a_2 & 0 & 0 \\ a_3 & a_2 & 0 & -a_2 & -a_3 \\ 0 & 0 & -a_2 & -b_2 & -a_1 \\ 0 & 0 & -a_3 & -a_1 & -b_1 \end{pmatrix}.$$

The matrix L_+ is defined as above, i.e. the skew-symmetric part of L. Again there is rational integral given by $I_{11} = (a_1a_2 - a_3b_2)/a_3$. Defining the Poisson bracket by $\{a_1, a_2\} = a_3$, $\{a_i, b_i\} = -a_i$, $i = 1, 2$, and $\{a_1, b_2\} = a_1$ we verify easily that h_2 plays the role of the Hamiltonian and I_{11} is a Casimir. The set $\{h_2, h_4, I_{11}\}$ is an independent set of functions in involution.

2 Intermediate Toda lattices

We have defined some Hamiltonian systems associated to a subset Φ consisting of positive roots (which we call adapted). The associated matrix is symmetric. As in the case of classical and full Toda there is also an analogous system defined by a Lax matrix which is lower triangular (the Kostant–Toda lattices). In this paper we restrict our attention to this version of the systems. In this section we show that these Hamiltonian systems are associated to a nilpotent ideal of a Borel subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Since for particular (extreme) choices of the ideal one finds the classical Kostant–Toda lattice or the full Kostant–Toda lattice associated to $\mathfrak{g}$, we call these Hamiltonian systems *intermediate Toda lattices.*

2.1 The phase space $M_{\mathcal{I}}$

Throughout this section $\mathfrak{g}$ is an arbitrary complex semisimple Lie algebra, the rank of which we denote by ℓ. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a basis $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ of the root system Δ of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The choice of Π amounts to the choice of a Borel subalgebra $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$ of $\mathfrak{g}$. It also leads to a Borel subalgebra $\mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{n}_-$ corresponding to the negative roots. We fix an element ε in $\mathfrak{n}_+$, satisfying $\langle \varepsilon \,|\, [\mathfrak{n}_-, \mathfrak{n}_-]\rangle = 0$, where $\langle \cdot \,|\, \cdot \rangle$ stands for the Killing form of $\mathfrak{g}$. One usually picks for ε a principal nilpotent element of $\mathfrak{n}_+$. For example, for $\mathfrak{g} = \mathrm{sl}_N(\mathbf{C})$, viewed as the Lie algebra of traceless $N \times N$ matrices, one can take for $\mathfrak{h}$ and for $\mathfrak{b}_+$ the subalgebras of diagonal, respectively upper triangular, matrices and for ε one can choose

$$\varepsilon := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix}.$$

Let $\mathcal{I}$ be a nilpotent ideal of $\mathfrak{b}_+$. The quotient map $\mathfrak{b}_+ \to \mathfrak{b}_+/\mathcal{I}$ is denoted by $P_{\mathcal{I}}$. Using the isomorphism $\mathfrak{b}_+^* \simeq \mathfrak{b}_-$ induced by the Killing form, we can think of the orthogonal $\mathcal{I}^\perp$ of $\mathcal{I}$ in $\mathfrak{b}_+^*$ as a vector subspace of $\mathfrak{b}_-$. We consider the affine space $M_{\mathcal{I}} := \varepsilon + \mathcal{I}^\perp$. Explicitly

$$M_{\mathcal{I}} = \{X + \varepsilon \mid X \in \mathfrak{b}_- \text{ and } \langle X \,|\, \mathcal{I}\rangle = 0\}.$$

When $\mathcal{I} = \{0\}$, $M_{\mathcal{I}} = \mathfrak{b}_- + \varepsilon$, which is the phase space of the full Kostant–Toda lattice. On the other extreme, taking $\mathcal{I} = [\mathfrak{n}_+ \,, \mathfrak{n}_+]$ the manifold $M_{\mathcal{I}}$ is the phase space of the classical Kostant–Toda lattice. We therefore call $M_{\mathcal{I}}$ the intermediate Kostant–Toda phase space. Notice that, if $\mathcal{I} \subset \mathcal{J}$, then $M_{\mathcal{J}} \subset M_{\mathcal{I}}$.

2.2 Hamiltonian structure

We show that $M_{\mathcal{I}}$ has a natural Poisson structure. To do this we prove that $M_{\mathcal{I}}$ is a Poisson submanifold of $\mathfrak{g}$ equipped with a Poisson structure $\{\cdot\,,\cdot\}$ the construction of which[1] we firstly recall. We use the theory of R-matrices (see [2, Chapter 4.4]

[1]See the appendix of [6] for an alternative construction using symplectic reduction to the cotangent bundle $T^*\mathbf{G}$, where $\mathbf{G}$ is any Lie group integrating $\mathfrak{g}$.

for the general theory of R-matrices). Write $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ where $\mathfrak{g}_+ := \mathfrak{b}_+$ and $\mathfrak{g}_- := \mathfrak{n}_-$. For $X \in \mathfrak{g}$ its projection in $\mathfrak{g}_\pm$ is denoted by $X_\pm$. The endomorphism $R : \mathfrak{g} \to \mathfrak{g}$, defined for all $X \in \mathfrak{g}$ by $R(X) := X_+ - X_-$, is an R-matrix which means that the bracket on $\mathfrak{g}$, defined by

$$[X, Y]_R := \frac{1}{2}([R(X), Y] + [X, R(Y)]) = [X_+, Y_+] - [X_-, Y_-]$$

for all $X, Y \in \mathfrak{g}$, is a (new) Lie bracket on $\mathfrak{g}$. The Lie–Poisson bracket on $\mathfrak{g}$, which corresponds to $[\cdot, \cdot]_R$ and which we denote simply by $\{\cdot\,, \cdot\}$ (since it is the only Poisson bracket on $\mathfrak{g}$ which we use), is given by

$$\{F, G\}(X) = \langle X \mid [(\nabla_X F)_+, (\nabla_X G)_+]\rangle - \langle X \mid [(\nabla_X F)_-, (\nabla_X G)_-]\rangle \tag{6}$$

for every pair of functions, F and G, on $\mathfrak{g}$ and for all $X \in \mathfrak{g}$. In this formula the gradient $\nabla_X F$ of F at X is the element of $\mathfrak{g}$ defined by

$$\langle \nabla_X F \mid Y\rangle = \langle \mathrm{d}_X F, Y\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(X + tY). \tag{7}$$

Proposition 2. *Let $\mathcal{I}$ be a nilpotent ideal of $\mathfrak{b}_+$.*

(1) The affine space $M_\mathcal{I}$ is a Poisson submanifold of $(\mathfrak{g}, \{\cdot\,, \cdot\})$;

(2) Equipped with the induced Poisson structure $M_\mathcal{I}$ is isomorphic to $(\mathfrak{b}_+/\mathcal{I})^*$, which is equipped with the canonical Lie–Poisson bracket;

(3) A function F on $M_\mathcal{I}$ is a Casimir function if and only if $(\nabla_X \tilde{F})_+ \in \mathcal{I}$ for all $X \in M_\mathcal{I}$, where $\tilde{F}$ is an arbitrary extension of F to $\mathfrak{g}$.

For a function H on $M_\mathcal{I}$ we denote its Hamiltonian vector field by $\mathcal{X}_H$; our sign convention is that $\mathcal{X}_H := \{\cdot\,, H\}$ so that $\mathcal{X}_H[F] = \{F, H\}$ for all $F \in \mathcal{F}(M)$. The Hamiltonian of the intermediate Kostant–Toda lattice is the polynomial function on $M_\mathcal{I}$ given by

$$H := \frac{1}{2}\langle X \,|X\rangle \tag{8}$$

so that the vector field of the intermediate Kostant–Toda lattice is given by the Lax equation (on $M_\mathcal{I}$)

$$\dot{X} = [X_+, X]\,. \tag{9}$$

2.3 Height k Kostant–Toda lattices

In the sequel of this paper we mainly study the case for which $\mathcal{I}$ is an ideal of height 2, a notion which we introduce in this paragraph. We firstly give some information on the nilpotent ideals of $\mathfrak{b}_+$ (see [4]). If $\mathcal{I}$ is a nilpotent ideal of $\mathfrak{b}_+$, then $\mathcal{I}$ is contained in $\mathfrak{n}_+$. For example $\mathfrak{n}_+$ itself is a nilpotent ideal of $\mathfrak{b}_+$.

For $\alpha \in \Delta^+$ let $X_\mathfrak{a}$ denote an arbitrary root vector corresponding to α, i.e., $[H, X_\alpha] = \langle \alpha, H \rangle X_\alpha$ for all $H \in \mathfrak{h}$. Consider a subset, Φ, of Δ^+ which has the property that, if $\alpha \in \Phi$, then every root of the form $\alpha + \beta$ with $\beta \in \Delta^+$ belongs to Φ; we call such a set Φ an admissible set of roots. For such α and β the Jacobi identity implies that $[X_\alpha, X_\beta]$ is a multiple of $X_{\alpha+\beta}$. It follows that the (vector space) span of $\{X_\alpha \mid \alpha \in \Phi\}$ is a nilpotent ideal of $\mathfrak{b}_+$. Most importantly every nilpotent ideal of $\mathfrak{b}_+$ is of this form for a certain admissible set of roots Φ. Thus the nilpotent ideals of a given Borel subalgebra $\mathfrak{b}_+$ of $\mathfrak{g}$ are parametrized by the family of all subsets Φ of Π^+, which have the property that, if $\alpha \in \Phi$, then every root of the form $\alpha + \beta$ with $\beta \in \Delta^+$ belongs to Φ.

Every positive root $\alpha \in \Delta^+$ can be written as a linear combination of the simple roots, $\alpha = \sum_{i=1}^{\ell} n_i \alpha_i$, where all n_i are nonnegative integers. The integer $ht(\alpha) := \sum_{i=1}^{\ell} n_i$ is called the *height* of α. For $k \in \mathbf{N}$, let Φ_k denote the set of all roots of height larger than k. It is clear that Φ_k is an admissible set of roots. We denote the corresponding ideal of $\mathfrak{b}_+$ by $\mathcal{I}_k$ and we call it a height k ideal. An alternative description of $\mathcal{I}_k$ is as $\mathrm{ad}^k_{\mathfrak{n}_+} \mathfrak{n}_+$. For $k = 1$, $\mathcal{I}_1 = [\mathfrak{n}_+, \mathfrak{n}_+]$ is the ideal which leads to the classical Toda lattice. We consider in the sequel mainly $\mathcal{I}_2 = [\mathfrak{n}_+, [\mathfrak{n}_+, \mathfrak{n}_+]]$ and the corresponding affine space $M_{\mathcal{I}_2}$.

Example 2. Consider a Lie algebra of type C_4. Take $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4\}$. It gives rise to a height 2 Toda system.

The Lax matrix is

$$L = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & a_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ c_1 & b_2 & a_3 & 1 & 0 & 0 & 0 & 0 \\ 0 & c_2 & b_3 & a_4 & 1 & 0 & 0 & 0 \\ 0 & 0 & c_3 & b_4 & -a_4 & -1 & 0 & 0 \\ 0 & 0 & 0 & c_3 & -b_3 & -a_3 & -1 & 0 \\ 0 & 0 & 0 & 0 & -c_2 & -b_2 & -a_2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -c_1 & -b_1 & -a_1 \end{pmatrix}.$$

The function

$$a_1 - a_2 + a_3 - a_4 + \frac{2b_1b_2c_3 + b_1c_2b_4 + b_3b_4c_1}{c_1c_3}$$

is a Casimir. We need five functions to establish integrability. Since $\det(L - \lambda I)$ is an even polynomial of the form $\lambda^8 + \sum_{i=0}^{3} f_i \lambda^{2i}$, we obtain four polynomial integrals f_0, f_1, f_2, f_3. Using an one-chop we obtain a characteristic polynomial of the form $A\lambda^2 + B$. The function $f_4 = B/A$ is the fifth integral.

3 Computation of the rank

In this section we compute the index of the Lie algebra $\mathfrak{b}_+/\mathcal{I}_2$ when $\mathfrak{b}_+$ is a Borel subalgebra of a simple Lie algebra of type A_ℓ, B_ℓ or C_ℓ. It yields the rank of the corresponding intermediate Kostant–Toda phase space (see Subsection 2.3). We firstly recall a few basic facts about stable linear forms, the index of a Lie algebra and the relation to the rank of the corresponding Lie–Poisson structure.

3.1 Stable linear forms

Let $\mathfrak{a}$ be any complex algebraic Lie algebra and $\mathfrak{a}^*$ its dual vector space. The *stabilizer* of a linear form $\varphi \in \mathfrak{a}^*$ is given by

$$\mathfrak{a}^\varphi := \{x \in \mathfrak{a} \mid \operatorname{ad}_x^* \varphi = 0\} = \{x \in \mathfrak{a} \mid \forall y \in \mathfrak{a},\ \langle \varphi, [x, y]\rangle = 0\}.$$

The integer $\min\{\dim \mathfrak{a}^\varphi \mid \varphi \in \mathfrak{a}^*\}$ is called the *index* of $\mathfrak{a}$ and is denoted by $\operatorname{Ind}(\mathfrak{a})$. Since the symplectic leaves of the canonical Lie–Poisson structure on $\mathfrak{a}^*$ are the coadjoint orbits, the codimension of the symplectic leaf through φ is the dimension of $\mathfrak{a}^\varphi$. It follows that the index of $\mathfrak{a}$ is the codimension of a symplectic leaf of maximal dimension, i.e., the rank of the canonical Lie–Poisson structure on $\mathfrak{a}^*$ is given by $\dim \mathfrak{a} - \operatorname{Ind}(\mathfrak{a})$; notice that, since the latter rank is always even, the index of $\mathfrak{a}$ and the dimension of $\mathfrak{a}$ have the same parity. A linear form $\varphi \in \mathfrak{a}^*$ is said to be *regular* if $\dim \mathfrak{a}^\varphi = \operatorname{Ind}(\mathfrak{a})$. Thus we can use regular linear forms to compute the index of $\mathfrak{a}$ and hence the rank of the canonical Lie–Poisson structure on $\mathfrak{a}^*$.

We use the following proposition to compute the index of $\mathfrak{b}_+/\mathcal{I}_2$.

Proposition 3. *Let $\mathfrak{a}$ be a subalgebra of a semisimple complex Lie algebra $\mathfrak{g}$. Suppose that φ is a linear form on $\mathfrak{a}$ such that $\mathfrak{a}^\varphi$ is a commutative Lie algebra composed of semisimple elements. Then φ is regular so that the index of $\mathfrak{a}$ is given by* $\dim \mathfrak{a}^\varphi$.

Proof. A linear form $\varphi \in \mathfrak{a}^*$ is said to be *stable* if there exists a neighborhood U of φ in $\mathfrak{a}^*$ such that for every $\psi \in U$ the stabilizer $\mathfrak{a}^\psi$ is conjugate to $\mathfrak{a}^\varphi$ with respect to the adjoint group of $\mathfrak{a}$. According to [8] every stable linear form is regular. According to [7] and [8, Theorem 1.7, Corollary 1.8] φ is stable if and only if $[\mathfrak{a}, \mathfrak{a}^\varphi] \cap \mathfrak{a}^\varphi = \{0\}$. The latter equality holds when $\mathfrak{a}^\varphi$ is a commutative Lie algebra composed of semisimple elements (see [8, Lemma 2.6]). Thus φ is stable, hence regular. ■

3.2 Computation of the index

In this paragraph we compute the index of $\mathfrak{b}/\mathcal{I}$ under the following assumption on (the root system of) $\mathfrak{g}$:

(H) *The roots of height* 2 *of* $\mathfrak{g}$ *are given by* $\{\alpha_k + \alpha_{k+1} \mid 1 \le k \le \ell - 1\}$.

For classical Lie algebras the basis Π can be ordered such that this assumption occurs when $\mathfrak{g}$ is of type A_ℓ, B_ℓ or C_ℓ. Let $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta^+}(\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$ be the decomposition of $\mathfrak{g}$ according to the adjoint action of $\mathfrak{h}$. To each positive root α there corresponds a triple $(X_\alpha, X_{-\alpha}, H_\alpha)$ of elements of $\mathfrak{g}$, where $X_\alpha \in \mathfrak{g}_\alpha, X_{-\alpha} \in \mathfrak{g}_{-\alpha}, H_\alpha \in \mathfrak{h}$ and $(X_\alpha, X_{-\alpha}, H_\alpha)$ generates a subalgebra isomorphic to $\mathrm{sl}_2(\mathbf{C})$. We recall shortly how such a triple can be constructed. Let h_α be the unique element in $\mathfrak{h}$ such that $\langle \alpha, H\rangle = \langle h_\alpha \mid H\rangle$ for all $H \in \mathfrak{h}$. Define a scalar product on the real vector-space $\mathfrak{h}^*_{\mathbf{R}}$ by

$$\langle \alpha \mid \beta\rangle := \langle h_\alpha h_\beta\rangle = \langle \beta, h_\alpha\rangle = \langle \alpha, h_\beta\rangle$$

for all α and $\beta \in \Delta$. We set $H_\alpha := \dfrac{2}{\langle \alpha \mid \alpha\rangle} h_\alpha$. It is clear that $\langle \alpha, H_\alpha\rangle = 2$. Choose $X_\alpha \in \mathfrak{g}_\alpha, X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that

$$\langle X_\alpha \mid X_{-\alpha}\rangle = \frac{2}{\langle \alpha \mid \alpha\rangle}.$$

Then $(X_\alpha, X_{-\alpha}, H_\alpha)$ is the required triple. Moreover

$$\left[X_{\pm\alpha_k}, X_{\mp\alpha_k\mp\alpha_{k+1}}\right] = \epsilon_k^{\pm} X_{\mp\alpha_{k+1}}, \quad \left[X_{\pm\alpha_{k+1}}, X_{\mp\alpha_k\mp\alpha_{k+1}}\right] = \eta_k^{\pm} X_{\mp\alpha_k},$$

where each of the integers $\epsilon_k^{\pm}$ and $\eta_k^{\pm}$ is equal to 1 or to -1 depending upon $\mathfrak{g}$.

For all $\alpha, \beta \in \Pi$ let

$$C_{\alpha\beta} := \langle \beta, H_\alpha\rangle = 2\frac{\langle \alpha \mid \beta\rangle}{\langle \alpha \mid \alpha\rangle}.$$

The $\ell \times \ell$-matrix $C := (C_{ij}, 1 \le i, j \le \ell)$, where $C_{ij} := C_{\alpha_i\alpha_j}$, is invertible. It is called the *Cartan matrix* of $\mathfrak{g}$.

Proposition 4. *Consider the linear form φ on $\mathfrak{b}_+$ defined for $Z \in \mathfrak{b}_+$ by $\langle \varphi, Z\rangle := \langle X \mid Z\rangle$, where X is defined by*

$$X := \delta_\ell X_{-\alpha_\ell} + \sum_{i=1}^{\ell-1} X_{-\alpha_i-\alpha_{i+1}} \tag{10}$$

with $\delta_\ell := 1$ if ℓ is odd and $\delta_\ell := 0$ otherwise. Denote by $\bar\varphi$ the induced linear form on $\mathfrak{b}_+/\mathcal{I}_2$.

(1) $\bar\varphi$ is a regular linear form on $\mathfrak{b}_+/\mathcal{I}_2$;

(2) $\dim(\mathfrak{b}_+/\mathcal{I}_2)^{\bar\varphi} = 1 - \delta_\ell$;

(3) The index of $\mathfrak{b}_+/\mathcal{I}_2$ is 1 if the rank ℓ of $\mathfrak{g}$ is even and is 0 otherwise.

4 Integrability

We now get to the integrability of the intermediate Kostant–Toda lattice on $M_{\mathcal{I}_2} \subset \mathfrak{g}$ for any semisimple Lie algebra $\mathfrak{g}$ of type A_ℓ, B_ℓ or C_ℓ. Recall that this means that the Hamiltonian is part of a family of s independent functions in involution, where s is related to the dimension and the rank of the Poisson manifold $M_{\mathcal{I}_2}$ by the formula $\dim M_{\mathcal{I}_2} = \frac{1}{2}\mathrm{Rk}\, M_{\mathcal{I}_2} + s$. Since $\dim M_{\mathcal{I}_2} = 3\ell - 1$ and since the corank of $M_{\mathcal{I}_2}$ is 1 if ℓ is even and 0 otherwise (see item (3) in Proposition 4), we need $s = [3\ell/2]$ such functions. According to the Adler–Kostant–Symes Theorem the ℓ basic Ad-invariant polynomials provide already ℓ independent functions in involution. Thus one needs $[\ell/2]$ additional ones. As we see, they can be constructed by restricting certain chop-type integrals, except for the case of C_ℓ for which another integral (Casimir) is needed. We firstly recall from [5] the construction of the chop integrals on $M := \varepsilon + \mathfrak{b}_-$ in the case that $\mathfrak{g} = \mathrm{sl}_N(\mathbf{C})$ and explain why they are in involution. Since $M_{\mathcal{I}_2}$ is a Poisson submanifold of M, their restrictions to $M_{\mathcal{I}_2}$ are still in involution (but they may become trivial or dependent).

We consider $\mathfrak{g} = \mathrm{sl}_N(\mathbf{C})$ with the standard choice of $\mathfrak{h}$ and Π (see Subsection 2.1). Let k be an integer, $0 \le k \le [\frac{N-1}{2}]$. For any matrix X we denote by X_k the matrix obtained by removing the first k rows and last k columns from X. We denote by $\mathbf{G}_k$ the subgroup of $\mathbf{GL}_N(\mathbf{C})$ consisting of all $N \times N$ invertible matrices of the form

$$g = \begin{pmatrix} \Delta & A & B \\ 0 & D & C \\ 0 & 0 & \Delta' \end{pmatrix}, \tag{11}$$

where Δ and Δ' are arbitrary upper triangular matrices of size $k \times k$ and A, B, C and D are arbitrary[2]. The Lie algebra of $\mathbf{G}_k$ is denoted by $\mathfrak{g}_k$. A first, fundamental and nontrivial observation, due to [5], is that for all $g \in \mathbf{G}_k$, decomposed as in (11),

$$\det\left(gXg^{-1}\right)_k = \frac{\det \Delta'}{\det \Delta} \det X_k. \tag{12}$$

This leads to (rational) $\mathbf{G}_k$-invariant functions on $\mathfrak{g}$ (and hence on M) which are constructed as follows. For $X \in \mathfrak{g}$ and for an arbitrary scalar l consider the so-called k-chop polynomial of X defined by $Q_k(X,\lambda) := \det(X - \lambda\,\mathrm{Id}_N)_k$. In view of (12) the coefficients of Q_k (as a polynomial in l) define polynomial functions on $\mathfrak{g}$, which transform under the action of $g \in \mathbf{G}_k$ with the same factor $\det \Delta'/\det \Delta$. We write

$$Q_k(X,\lambda) = \sum_{i=0}^{N-2k} E_{i,k}(X)\lambda^{N-2k-i}.$$

Each of the rational functions $E_{i,k}/E_{j,k}$ is $\mathbf{G}_k$-invariant. By restriction to M this yields $\mathbf{G}_k$-invariant elements of $\mathcal{F}(M)$. They are called k-chop integrals because they are integrals (constants of motion) for the full Kostant–Toda lattice. Notice that the constants of motion $H_i := \frac{1}{i}\operatorname{Tr} X^i$ are 0-chop integrals and that the Toda Hamiltonian is expressible in terms of them as $H = (H_1^2 - 2H_2)/2$.

We show that all chop integrals are in involution. To do this we let F be a k-chop integral and let $\tilde{F}$ denote its extension to a $\mathbf{G}_k$-invariant rational function

[2] With the understanding that, since X is supposed invertible, Δ, Δ' and D are invertible.

on $\mathfrak{g}$. Similarly let G be a l-chop integral with $\mathbf{G}_\ell$-invariant extension $\tilde{G}$. We may suppose that $k \leq \ell$. Infinitesimally the fact that $\tilde{F}$ is $\mathbf{G}_k$ invariant yields that

$$\left\langle X \left[\nabla_X \tilde{F}, Y \right] \right\rangle = 0 \tag{13}$$

for all $X \in \mathfrak{g}$ and for all $Y \in \mathfrak{g}_k$. Since $\mathfrak{b}_+ \subset \mathfrak{g}_k$, it follows that

$$\left\langle X \;\middle|\; \left[(\nabla_X \tilde{F})_+, \nabla_X \tilde{G} \right] \right\rangle = 0 = \left\langle X \;\middle|\; \left[\nabla_X \tilde{F}, (\nabla_X \tilde{G})_+ \right] \right\rangle$$

so that (6) can be rewritten for $X \in M$ as

$$\{F, G\}(X) = -\left\langle X \mid \left[\nabla_X \tilde{F}, \nabla_X \tilde{G} \right] \right\rangle . \tag{14}$$

We claim that $\nabla_X \tilde{G} \in \mathfrak{g}_\ell$. This follows from the construction of the function $\tilde{G} \in \mathcal{F}(\mathfrak{g})$: the rational function $\tilde{G}(X)$ depends only upon X_ℓ, the ℓ-chop of X, while, if an element Z of $\mathfrak{g}$ satisfies $\langle \mathfrak{g}_\ell \mid Z \rangle = 0$, then Z_ℓ is the zero matrix. Thus $\nabla_X \tilde{G} \in \mathfrak{g}_\ell \subset \mathfrak{g}_k$ so that (13) implies that the right hand side of (14) is zero for all $X \in M$. It follows that F and G have zero Poisson bracket.

Notice that in the case of the height 2 intermediate Kostant–Toda lattice all k-chops with $k > 1$ vanish and that only a few 1-chops survive. In what follows we consider separately the cases of A_ℓ, B_ℓ and C_ℓ.

4.1 The case of A_ℓ

We firstly consider $\mathfrak{g} = \mathrm{sl}_{\ell+1}(\mathbf{C})$, the Lie algebra of traceless matrices of size $N = \ell + 1$, and take for $\mathfrak{h}$, Π and ε the standard choices as before. A general element of $\mathcal{M}_{\mathcal{I}_2}$ is then of the form

$$X = \begin{pmatrix} a_1 & 1 & 0 & \dots & \dots & 0 \\ b_1 & a_2 & 1 & \ddots & & \vdots \\ c_1 & b_2 & a_3 & 1 & \ddots & \vdots \\ 0 & c_2 & b_3 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & c_{\ell-1} & b_\ell & a_{\ell+1} \end{pmatrix}$$

with $\sum_{i=1}^{\ell+1} a_i = 0$. The 1-chop matrix of X is given by

$$(X - \lambda\, \mathrm{Id}_{\ell+1})_1 = \begin{pmatrix} b_1 & a_2^\lambda & 1 & 0 & \dots & 0 \\ c_1 & b_2 & a_3^\lambda & 1 & \ddots & \vdots \\ 0 & c_2 & b_3 & \ddots & \ddots & 0 \\ \vdots & \ddots & c_3 & b_4 & \ddots & 1 \\ \vdots & & \ddots & \ddots & \ddots & a_\ell^\lambda \\ 0 & \dots & \dots & 0 & c_{\ell-1} & b_\ell \end{pmatrix},$$

where a_i^λ is a shorthand for $a_i - \lambda$. We also use the matrix $X(\lambda,\alpha)$, defined by

$$X(\lambda,\alpha) = \begin{pmatrix} b_1 & a_2^\lambda & \alpha_{13} & \dots & \dots & \alpha_{1\ell} \\ c_1 & b_2 & a_3^\lambda & \alpha_{24} & & \vdots \\ 0 & c_2 & b_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & c_3 & b_4 & \ddots & \alpha_{\ell-2,\ell} \\ \vdots & & \ddots & \ddots & \ddots & a_\ell^\lambda \\ 0 & \dots & \dots & 0 & c_{\ell-1} & b_\ell \end{pmatrix}.$$

Proposition 5. *The polynomials* $\det(X - \lambda\,\mathrm{Id}_{\ell+1})_1$ *and* $\det X(\lambda,\alpha)$ *have degree* $d := [\frac{\ell}{2}]$ *in* λ.

4.2 The case of B_ℓ

A Lie algebra of type B_ℓ can be realized as the Lie algebra $\mathfrak{g}$ of all square matrices of size $N = 2\ell+1$, satisfying $XJ + JX^t = 0$, where J is the matrix of size $2\ell+1$, all of whose entries are zero except for the entries on the antidiagonal, which are all equal to one. Clearly X satisfies $XJ + JX^t = 0$ if and only if X is skew-symmetric with respect to its antidiagonal. It follows for such X that $\det(X - \lambda\,\mathrm{Id}_{\ell+1}) = (-1)^N \det(X + \lambda\,\mathrm{Id}_{\ell+1})$ so that the characteristic polynomial is an odd polynomial in λ. The 1-chop matrix X_1 satisfies the same relation $X_1 J + J X_1^t = 0$ so that its determinant is an even polynomial in λ. As a Cartan subalgebra of $\mathfrak{g}$ one can take the diagonal matrices in $\mathfrak{g}$ and one can take as a basis for Δ^+ the matrices $E_{i,i+1} - E_{2\ell-i,2\ell-i+1}$ for $i = 1,\dots,\ell$. If one finally chooses ε to be the matrix $\sum_{i=1}^{\ell}(E_{i,i+1} - E_{2\ell-i,2\ell-i+1})$, then the height 2 phase space is given by all matrices of the form

$$\begin{pmatrix} a_1 & 1 & & & & & & & \\ b_1 & \ddots & \ddots & & & & & & \\ c_1 & \ddots & \ddots & 1 & & & & & \\ & \ddots & b_{n-1} & a_n & 1 & & & & \\ & & c_{n-1} & b_n & 0 & -1 & & & \\ & & & 0 & -b_n & -a_n & \ddots & & \\ & & & & -c_{n-1} & -b_{n-1} & \ddots & \ddots & \\ & & & & & \ddots & \ddots & & -1 \\ & & & & & -c_1 & -b_1 & -a_1 & \end{pmatrix}.$$

In this case $N = 2\ell+1$, the 1-chop polynomial is even and so the 1-chop polynomial is degree ℓ when ℓ is even and of degree $\ell-1$ when ℓ is odd. This yields $\frac{\ell}{2}$ integrals when ℓ is even and $\frac{\ell-1}{2}$ when ℓ is odd. Therefore the number of integrals is correct in each case.

4.3 The case of C_ℓ

A Lie algebra of type C_ℓ can be realized as the Lie algebra $\mathfrak{g}$ of all square matrices of size $N = 2\ell$, satisfying $XJ + JX^t = 0$, where J is the matrix of size 2ℓ, given by

$$J = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}.$$

It follows for such X that $\det(X - \lambda\,\mathrm{Id}_{\ell+1}) = (-1)^{2l}\det(X + \lambda\,\mathrm{Id}_{\ell+1})$ so that the characteristic polynomial is an even polynomial in λ. The 1-chop matrix X_1 satisfies the same relation $X_1 J + JX_1^t = 0$ so that its determinant is an even polynomial in λ. As a Cartan subalgebra of $\mathfrak{g}$ one can take the diagonal matrices in $\mathfrak{g}$ and one can take as a basis for Δ^+ the matrices $E_{i,i+1} - E_{2\ell-1-i,2\ell-i}$ for $i = 1, \dots, \ell$. The height 2 phase space for C_ℓ is given by all matrices of the form

$$\begin{pmatrix}
a_1 & 1 & & & & & & & \\
b_1 & a_2 & \ddots & & & & & & \\
c_1 & b_2 & \ddots & 1 & & & & & \\
 & \ddots & \ddots & a_n & 1 & & & & \\
 & & c_{n-1} & b_n & -a_n & -1 & & & \\
 & & & c_{n-1} & -b_{n-1} & \ddots & \ddots & & \\
 & & & & -c_{n-2} & \ddots & \ddots & \ddots & \\
 & & & & & \ddots & \ddots & & -1 \\
 & & & & & -c_1 & -b_1 & -a_1 &
\end{pmatrix}.$$

In this case, $N = 2\ell$, the 1-chop polynomial is even so that we get $\frac{l}{2} - 1$ integrals from the 1-chop when l is even and $\frac{l-2}{2}$ integrals when l is odd. Therefore the odd case gives the correct number of integrals. For the even case there exists a Casimir function which does not arise from the method of chopping and we describe it as follows:

The Casimir f has the form $f = A + B/C$, where

$$A = \sum_{i=1}^{\ell-1}(a_i - a_{i+1}), \quad B = \sum_{i,j} d_{ij} m_{ij}, \quad \text{and} \quad C = \prod_{i=1}^{\ell-1} c_{2i-1}.$$

The term m_{ij} in B is determined as follows: We associate the variables b_1, b_2, $\dots$, b_l to the simple roots $\alpha_1, \alpha_2, \dots, \alpha_l$ and the variables c_1, c_2, $\dots$, c_{l-1} to the height 2 roots $\alpha_1 + \alpha_2$, $\alpha_2 + \alpha_3$, $\dots$, $\alpha_{l-1} + \alpha_l$.

Take simple roots α_i and α_j (with corresponding variables b_i, b_j) such that i is odd and j is even. The remaining variables correspond to the height two roots $\alpha_k + \alpha_{k+1}$, where $k \neq i, i-1$, $k \neq j, j-1$. The term m_{ij} is a product of b_i, b_j and $\frac{l-1}{2}$ c variables.

The coefficient d_{ij} is 2 if m_{ij} includes the term c_{l-1} (corresponding to the root $\alpha_{l-1}+\alpha_l$) and is equal to 1 otherwise.

Example 3. $l = 6$.

$$f = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \frac{b_5 b_6 c_1 c_3 + 2b_1 b_4 c_2 c_5 + b_3 b_6 c_1 c_4 + 2b_1 b_2 c_3 c_5 + 2b_3 b_4 c_1 c_5 + b_1 b_6 c_2 c_4}{c_1 c_3 c_5}.$$

4.4 The case of D_ℓ

We conclude with some comments on the case of D_ℓ. A Lie algebra of type D_ℓ can be realized as the Lie algebra $\mathfrak{g}$ of all square matrices of size $N = 2\ell$ satisfying $XJ + JX^t = 0$, where J is the matrix of size 2ℓ, given by

$$J = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}.$$

As in the case of C_ℓ the characteristic polynomial is an even polynomial. On the other hand the 1-chop polynomial is odd so that the degree of this polynomial is $\ell-1$ when ℓ is even. However, when ℓ is odd the degree of the 1-chop polynomial is again ℓ. This gives $\frac{\ell}{2}-1$ integrals when ℓ is even and $\frac{\ell-1}{2}$ integrals when ℓ is odd. In the even case we need an additional function, i.e. a Casimir, but at this point we do not have an explicit formula. There is no stable form in this case, but we can produce a form which gives a lower bound for the rank and this lower bound is good enough, once we have the Casimir.

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