

ALGEBRAIC INTEGRABILITY: A SURVEY

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CONTENTS

1. Introduction	1
2. Integrable systems on Poisson manifolds	2
2.1. Poisson manifolds	2
2.2. Liouville integrability	4
2.3. The Adler-Kostant-Symes Theorem	5
3. Algebraic integrability	7
3.1. A.c.i. systems	7
3.2. Proving a.c.i.	9
3.3. Lax equations with a parameter	16
3.4. Other notions of algebraic integrability	18
References	20

1. INTRODUCTION

The purpose of this paper is to give a short survey on algebraic integrability, with the purpose of making the ideas and techniques accessible to a larger community. We present the general theory, giving at each step an example that is simple, but non-trivial, highlighting the concept that has been introduced.

The paper is divided in two parts (Sections 2 and 3). In Section 2 we first recall the notion of a Poisson manifold and we recall the main examples, namely symplectic manifolds and the dual of a Lie algebra (Paragraph 2.1). We introduce in Paragraph 2.2 the notion of an integrable system on a Poisson manifold and we formulate the classical Liouville Theorem, adapted to the case of Poisson manifolds. The major tool for constructing integrable systems is the Adler-Kostant-Symes Theorem, given in Paragraph 2.3.

Section 3 deals with algebraic integrability. We mainly focus on algebraic complete integrability in the sense of Adler-van Moerbeke (a.c.i.). We present in detail the example of the periodic 5-particle Kac-van Moerbeke lattice in Paragraph 3.2. This paragraph is the most technical part of the paper, aimed at showing, on an example, how algebraic integrability can be proven, when only the functions in involution are given (no Lax equation or geometric data are given). It can be skipped on first reading; it is included because it contains many useful ideas and illustrates almost all difficulties that are generally encountered when proving algebraic integrability. We discuss in Paragraph 3.3 Lax equations with a parameter, a mayor source of a.c.i. systems. We finish the paper with some comments on the general notion of algebraic integrability and we illustrate the latter with two examples.

2. INTEGRABLE SYSTEMS ON POISSON MANIFOLDS

The phase space of the simplest mechanical systems has the structure of a symplectic manifold: a smooth manifold M which is equipped with a closed, non-degenerate 2-form, which allows one to identify the tangent and cotangent bundles to M , and hence to associate to any one-form on M a vector field on M . Using the energy (function) H of the system one associates through this to dH a vector field whose flow describes precisely the evolution of the mechanical system with energy H (see e.g. [5]). For more complex systems, still finite-dimensional, one uses the more general notion of a Poisson manifold (see [16]). For the infinite-dimensional case, which will not be discussed here, see [6].

2.1. Poisson manifolds.

Definition 2.1. Let M be a differentiable manifold and let $\{\cdot, \cdot\}$ be a Lie algebra structure on $\mathcal{F}(M)$, the algebra of smooth functions on M . One says that $\{\cdot, \cdot\}$ is a Poisson bracket on M when it is a derivation in each of its arguments (biderivation), i.e., for any $H \in \mathcal{F}(M)$ it satisfies the Leibniz rule

$$\forall F, G \in \mathcal{F}(M) \quad \{FG, H\} = F\{G, H\} + \{F, H\}G.$$

The derivation (vector field) \mathcal{X}_H , defined by $\mathcal{X}_H := \{\cdot, H\}$ is then called the Hamiltonian vector field associated to H . Functions H for which $\mathcal{X}_H = 0$ are called Casimirs.

It is clear that the notion of a Poisson manifold makes sense also for complex manifolds (taking for $\mathcal{F}(M)$ the algebra of holomorphic functions), for complex affine varieties, that are possibly singular (taking for $\mathcal{F}(M)$ the algebra of regular functions), and so on. This will be relevant later in this text.

It follows easily from the definition that the Hamiltonian vector fields form a Lie subalgebra $\text{Ham}(M)$ of the Lie algebra $\mathfrak{X}(M)$ of all vector fields on M and that the Casimirs form a subalgebra of $\mathcal{F}(M)$, denoted $\text{Cas}(M)$. Considering the bracket on $\mathcal{F}(\mathbf{R}^{2n})$, defined by the formula

$$(1) \quad \{F, G\} := \sum_{i=1}^n \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i} \frac{\partial F}{\partial p_i} \right),$$

where $q_1, p_1, \dots, q_n, p_n$ are coordinates on \mathbf{R}^{2n} we recover Hamilton's equations of motion

$$\begin{aligned} \dot{q}_i &= p_i, \\ \dot{p}_i &= -\partial V / \partial q_i, \end{aligned} \quad i = 1, \dots, n,$$

in their most symmetric form

$$\begin{aligned} \dot{q}_i &= \{q_i, H\}, \\ \dot{p}_i &= \{p_i, H\}, \end{aligned} \quad i = 1, \dots, n,$$

where $H : \mathbf{R}^{2n} \rightarrow \mathbf{R}$ is the Hamiltonian, $H = \sum_i p_i^2/2 + V(q)$ (the sum of the kinetic and potential energy, expressed in terms of position q and momentum p). It is clear that, in general, the constants of motion, i.e., the functions which are constant on the integral curves of \mathcal{X}_H , are the functions F for which $\{F, H\} = 0$. Finding such functions is important for integrating the equations of motion, as we will see.

Example 2.2. As suggested above, a symplectic manifold (M, ω) is a Poisson manifold. The Poisson bracket is defined for $F, G \in \mathcal{F}(M)$ by $\{F, G\} := \omega(\mathcal{X}_F, \mathcal{X}_G)$, where the vector field \mathcal{X}_H is defined for symplectic manifolds by $\omega(\mathcal{X}_H, \cdot) = \mathbf{d}H$, where $H \in \mathcal{F}(M)$. It follows easily that $\{\cdot, H\} = \mathcal{X}_H$, so that this definition of \mathcal{X}_H actually is consistent with the one given earlier. The Jacobi identity for $\{\cdot, \cdot\}$ is a consequence of the fact that ω is closed.

Example 2.3. The dual \mathfrak{g}^* of a (finite-dimensional) Lie algebra \mathfrak{g} has a natural Poisson structure that derives from the Lie bracket $[\cdot, \cdot]$. For smooth functions F, G on \mathfrak{g}^* the Poisson bracket is defined at $\xi \in \mathfrak{g}^*$ by

$$(2) \quad \{F, G\}(\xi) := \langle \xi, [\mathbf{d}F(\xi), \mathbf{d}G(\xi)] \rangle.$$

In this formula, $\mathbf{d}F(\xi)$ and $\mathbf{d}G(\xi)$, which belong to $(\mathfrak{g}^*)^*$, are interpreted as elements of \mathfrak{g} when computing the bracket $[\cdot, \cdot]$. The Jacobi identity for $\{\cdot, \cdot\}$ follows from the Jacobi identity for $[\cdot, \cdot]$. This Poisson structure is called the Lie-Poisson structure on \mathfrak{g}^* . When \mathfrak{g} is semi-simple one can use the Killing form to transfer it to get a Poisson structure on \mathfrak{g} .

The Hamiltonian vector fields define a generalized distribution, which is integrable, i.e., it admits a (unique) integral manifold through each point. These integrable manifolds carry a Poisson structure which is symplectic. In the case of the dual of a Lie algebra, for example, these leaves are precisely the coadjoint orbits.

Definition 2.4. For $m \in M$ the (even) dimension of $\{\mathcal{X}_H(m) \mid H \in \mathcal{F}(M)\}$ is denoted by $\text{Rk}_m \{\cdot, \cdot\}$ and is called the rank of $\{\cdot, \cdot\}$ at m . The rank of $\{\cdot, \cdot\}$, denoted by $\text{Rk} \{\cdot, \cdot\}$, is the maximum of all ranks $\text{Rk}_m \{\cdot, \cdot\}$ with $m \in M$. For $s \in \mathbf{N}$ we denote by $M_{(s)}$ the subset

$$M_{(s)} := \{m \in M \mid \text{Rk}_m \{\cdot, \cdot\} \geq 2s\}.$$

A Poisson manifold of constant rank is called a regular Poisson manifold.

It can be shown that each of the subsets $M_{(s)}$ is open and is invariant for the flows of all Hamiltonian vector fields \mathcal{X}_H .

Example 2.5. The Poisson structure that corresponds to a symplectic structure is regular. Indeed, its rank at any point is equal to the dimension of the symplectic manifold. Moreover, the latter property characterizes symplectic manifolds. For the Lie-Poisson structure on \mathfrak{g}^* the rank at a point m is the dimension of the coadjoint orbit that passes through m , so \mathfrak{g}^* is only a regular Poisson manifold when \mathfrak{g} is abelian.

If the rank is constant in the neighborhood of a point $m \in M$, say it is equal to $2r$, then there exist local coordinates $(q_1, \dots, q_r, p_1, \dots, p_r, z_1, \dots, z_s)$ around m , such that, on this neighborhood, the Poisson bracket takes the following canonical form,

$$\{q_i, q_j\} = \{p_i, p_j\} = \{q_i, z_k\} = \{p_i, z_k\} = \{z_k, z_l\} = 0, \quad \{q_i, p_j\} = \delta_{ij},$$

where $1 \leq i, j \leq r$ and $1 \leq k < l \leq s$. Such coordinates are called Darboux coordinates. For a symplectic manifold the existence of Darboux coordinates implies that a symplectic manifold has no local invariants. Clearly, this is not true for general Poisson manifolds.

2.2. Liouville integrability. We have seen that constants of motion are functions that Poisson-commute with the Hamiltonian (functions that Poisson-commute are often said to be in involution). Liouville's fundamental observation was that if one is given n independent functions on \mathbf{R}^{2n} which are (pairwise) in involution with respect to the Poisson bracket (1) then the equations of motion can be integrated by quadratures. In the general context of Poisson manifolds this is stated in the following theorem.

Theorem 2.6. *Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold of rank $2r$ and suppose that F_1, \dots, F_s are $s = \dim M - r$ functions in involution. Let $m \in M_{(r)}$ and suppose that we are given a system of coordinates around m . If the differentials dF_1, \dots, dF_s are independent at m then the integral curve, starting at m , of each of the Hamiltonian vector fields \mathcal{X}_{F_i} can be obtained locally by using only algebraic operations, the implicit function theorem and integration.*

Let us call an s -tuple of functions $\mathbf{F} = (F_1, \dots, F_s)$ independent when the open subset

$$\mathcal{U}_{\mathbf{F}} := \{m \in M \mid dF_1(m) \wedge \dots \wedge dF_s(m) \neq 0\}$$

is dense in M . Then the above theorem naturally leads to the following definition of integrability.

Definition 2.7. Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold of rank $2r$ and let $\mathbf{F} = (F_1, \dots, F_s)$ be involutive and independent, with $s = \dim M - r$. We say that \mathbf{F} is (Liouville) integrable and that $(M, \{\cdot, \cdot\}, \mathbf{F})$ is a (Liouville) integrable system. The vector fields \mathcal{X}_{F_i} are then called integrable vector fields and the map \mathbf{F} is called the momentum map.

The definition makes also sense when M is a complex manifold (such as a smooth affine variety), equipped with a Poisson structure on its algebra of holomorphic functions.

Example 2.8. For any fixed $\alpha, \beta \in \mathbf{R}$, consider the following two polynomial functions

$$\begin{aligned} H &= \frac{1}{2}(p_1^2 + p_2^2) + (q_1^2 + q_2^2)^2 + \alpha q_1^2 + \beta q_2^2, \\ F &= (q_1 p_2 - q_2 p_1)^2 + (\beta - \alpha)(p_1^2 + 2q_1^4 + 2q_1^2 q_2^2 + 2\alpha q_1^2), \end{aligned}$$

where p_1, \dots, q_2 are coordinates on \mathbf{R}^4 . Then obviously H and F are independent and it follows by direct computation that they are in involution with respect to the Poisson bracket $\{\cdot, \cdot\}$ defined in (1) (with $n = 2$). Letting $\mathbf{F} := (H, F)$ it follows that $(\mathbf{R}^4, \{\cdot, \cdot\}, \mathbf{F})$ is a Liouville integrable system. It describes the motion of an anisotropic harmonic oscillator in a central field. This system (actually its n -dimensional generalization) was first considered by Garnier in [10] and appeared as a by-product of his study of isomonodromic deformations of differential equations. It is nowadays referred to as the Garnier system.

If $(M, \{\cdot, \cdot\}, \mathbf{F})$ is a Liouville integrable system then the integrable vector fields define an integrable distribution D of rank r on the (non-empty) open subset $\mathcal{U}_{\mathbf{F}} \cap M_{(r)}$, where $2r$ denotes the rank of $\{\cdot, \cdot\}$. For m in this open subset we denote by \mathbf{F}'_m the leaf of D that passes through m . We call it the invariant manifold of \mathbf{F} , passing through m . It is the connected component of $\mathbf{F}^{-1}(\mathbf{F}(m)) \cap \mathcal{U}_{\mathbf{F}} \cap M_{(r)}$ that contains m . The Liouville Theorem gives, under some additional assumption, a precise description of the invariant manifolds.

Theorem 2.9 (Liouville). *Let $(M, \{\cdot, \cdot\}, \mathbf{F})$ be a real integrable system, where $\mathbf{F} = (F_1, \dots, F_s)$, and let $m \in \mathcal{U}_{\mathbf{F}} \cap M_{(r)}$, where $2r$ denotes the rank of $\{\cdot, \cdot\}$.*

- (1) *If \mathbf{F}'_m is compact then it is diffeomorphic to a torus $\mathbf{T}^r = (\mathbf{R}/\mathbf{Z})^r$.*
- (2) *If \mathbf{F}'_m is not compact, but the flow of each of the vector fields \mathcal{X}_{F_i} is complete on \mathbf{F}'_m then \mathbf{F}'_m is diffeomorphic to a cylinder $\mathbf{R}^{r-q} \times \mathbf{T}^q$ ($0 \leq q < r$).*

In both cases the diffeomorphism can be chosen in such a way that the vector fields $\mathcal{X}_{F_1}, \dots, \mathcal{X}_{F_s}$ are mapped to linear (i.e., translation invariant) vector fields.

In the case of a regular Poisson manifold M of rank r one has that $M_{(r)} = M$. For generic \mathbf{c} in the image of \mathbf{F} one has then that $\mathbf{F}^{-1}(\mathbf{c})$ is a disjoint union of smooth manifolds \mathbf{F}'_m , which support r everywhere independent commuting vector fields. If one of the functions F_i is proper then the fiber consists of a finite number of invariant manifolds \mathbf{F}'_m , each of which is an r -dimensional torus.

Example 2.10. If α and β are positive real constants then the energy H of the Garnier system is a proper map, hence all fibers of the momentum map \mathbf{F} are compact. If we take a regular value (h, f) of \mathbf{F} for which $\mathbf{F}^{-1}(h, f)$ is not empty then $\mathbf{F}^{-1}(h, f)$ is the union of a finite number of two-dimensional tori.

A stronger statement can be made around a torus that appears as an invariant manifold. Namely, the torus has a neighborhood which is Poisson diffeomorphic to the product of the torus and a ball of complementary dimension, where the Poisson structure on this product is such that the natural coordinates on it are Darboux coordinates. These coordinates are then called action-angle coordinates. It was shown by Duistermaat (see [8]) that in general *global* action-angle coordinates do not exist, due to monodromy.

2.3. The Adler-Kostant-Symes Theorem. The most powerful theorem for constructing integrable systems is the Adler-Kostant-Symes Theorem. To be precise, it is a theorem which allows one to construct a large family of functions in involution, associated to a splitting of a Lie algebra \mathfrak{g} ; in many cases it can be shown, by a separate argument, that this family of functions contains an independent subset that is large enough to insure integrability.

By a splitting of a Lie algebra \mathfrak{g} we mean a (vector space) direct sum decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, where \mathfrak{g}_+ and \mathfrak{g}_- are subalgebras of \mathfrak{g} ; the corresponding Lie groups will be denoted by \mathbf{G} , \mathbf{G}_+ and \mathbf{G}_- . In the version that we give here we assume that \mathfrak{g} is semi-simple, so that we can identify \mathfrak{g} with its dual by using the Killing form $\langle \cdot, \cdot \rangle$. Then the orthogonal complement \mathfrak{g}_\perp of \mathfrak{g}_- (with respect to $\langle \cdot, \cdot \rangle$) is isomorphic to \mathfrak{g}_+^* , hence it inherits a Poisson structure $\{\cdot, \cdot\}_+$ from the Lie-Poisson structure on \mathfrak{g}_+^* .

Consider the alternative Lie bracket on \mathfrak{g} , defined by

$$[X, Y]_R := [X_+, Y_+] - [X_-, Y_-],$$

where we have decomposed X and Y according to the above splitting of \mathfrak{g} . This yields another Lie-Poisson structure on \mathfrak{g}^* , and hence on \mathfrak{g} , denoted $\{\cdot, \cdot\}_R$. It can be shown that the restriction of $\{\cdot, \cdot\}_R$ to \mathfrak{g}_\perp is precisely $\{\cdot, \cdot\}_+$. For $H \in \mathcal{F}(\mathfrak{g})$ and $X \in \mathfrak{g}$ the Killing form allows us to view $dH(X) \in \mathfrak{g}^*$ as an element of \mathfrak{g} ; we will denote it by $\nabla H(X)$. For $\epsilon \in \mathfrak{g}$ and $H \in \mathcal{F}(\mathfrak{g})$ we denote by H_ϵ the function defined on \mathfrak{g}_\perp by $H_\epsilon(X) := H(X + \epsilon)$.

We are now ready to state the Adler-Kostant-Symes Theorem on \mathfrak{g} .

Theorem 2.11. *Let F and H be \mathbf{G} -invariant functions on \mathfrak{g} and suppose that $\epsilon \in \mathfrak{g}$ satisfies $[\epsilon, \mathfrak{g}_+] \in \mathfrak{g}_+^\perp$ and $[\epsilon, \mathfrak{g}_-] \in \mathfrak{g}_-^\perp$. Then*

- (1) $\{F, H\}_R = 0$ and $\{F_\epsilon, H_\epsilon\}_{\mathfrak{g}_\pm} = 0$;
- (2) *The Hamiltonian vector fields $\mathcal{X}_H := \{\cdot, H\}_R \in \text{Ham}(\mathfrak{g}, \{\cdot, \cdot\}_R)$ and $\mathcal{X}_{H_\epsilon} := \{\cdot, H_\epsilon\}_{\mathfrak{g}_+^*} \in \text{Ham}(\mathfrak{g}_+, \{\cdot, \cdot\}_+)$ are respectively given by*

$$\mathcal{X}_H(X) = \pm [X, (\nabla H(X))_{\mp}] \quad \text{and} \quad \mathcal{X}_{H_\epsilon}(Y) = \pm [Y, (\nabla H(Y))_{\mp}],$$

where $Y \in \mathfrak{g}_\pm^\perp + \epsilon$;

- (3) *For $X_0 \in \mathfrak{g}$ and for $|t|$ small, let $g_+(t)$ and $g_-(t)$ denote the smooth curves in \mathbf{G}_+ resp. \mathbf{G}_- which solve the factorisation problem*

$$\exp(t\nabla H(X_0)) = g_+(t)^{-1}g_-(t), \quad g_\pm(0) = e.$$

Then the integral curve of \mathcal{X}_H which starts at X_0 is given for $|t|$ small by

$$X(t) = \text{Ad}_{g_+(t)} X_0 = \text{Ad}_{g_-(t)} X_0.$$

The above equations for \mathcal{X}_H and for \mathcal{X}_{H_ϵ} are called Lax equations. There is also a more intrinsic version of the Adler-Kostant-Symes Theorem on \mathfrak{g}^* (it does not use the Killing form), but it does not lead to Lax equations.

It follows from the theorem, or by a direct computation, that the spectral invariants of X are preserved by the flow of \mathcal{X}_H , where H is any \mathbf{G} -invariant function on \mathfrak{g} . One often speaks of isospectral flow. Lax equations become particularly interesting when the Lie algebra is a loop algebra, as we will see.

Example 2.12. Consider the Lie algebra splitting $\mathfrak{sl}_{n+1} = \Delta_{n+1}^{<} \oplus \Delta_{n+1}^{>}$, where $\Delta_{n+1}^{<}$ (resp. $\Delta_{n+1}^{>}$) denotes the Lie algebra of strictly lower triangular matrices (resp. of upper triangular matrices). Letting $\langle A|B \rangle = \text{Trace}(AB)$ we have that $(\Delta_{n+1}^{>})^\perp = \Delta_{n+1}^{>}$, the vector space of all strictly upper triangular matrices in \mathfrak{sl}_{n+1} . It can be shown that for any $p \leq n$ the set of p -band matrices in $\Delta_{n+1}^{>}$ forms a Poisson subspace of $\Delta_{n+1}^{>}$. Taking $p = 1$ and

$$\epsilon = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}$$

and $H : X \mapsto \frac{1}{3} \text{Trace } X^3$ we find the Hamiltonian vector field \mathcal{X}_{H_ϵ} , given by the Lax equation $\dot{X} = [X, Y_+] = -[X, Y_-]$, where

$$X := \begin{pmatrix} 0 & a_1 & & 0 \\ 1 & 0 & a_2 & \\ & \ddots & \ddots & \ddots \\ & & 1 & 0 & a_n \\ 0 & & & 1 & 0 \end{pmatrix}, \quad Y_+ := \begin{pmatrix} 0 & 0 & a_1 a_2 & & 0 \\ & & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & a_{n-1} a_n \\ & & & & 0 \\ 0 & \dots & & & 0 \end{pmatrix}$$

and where $Y_- = X^2 - Y_+$. The coefficients of the characteristic polynomial of X yield enough (independent) functions in involution to insure integrability.

3. ALGEBRAIC INTEGRABILITY

We now come to the notion of algebraic integrability. The idea is to consider complex integrable systems, whose (complex) geometry is the best possible analog of the (real) geometry that appears in the Liouville Theorem. Following the original idea of Adler and van Moerbeke we will mainly consider the case in which the fibers of the (complex) momentum map are affine parts of complex algebraic tori, leaving the general case to the end of this section.

3.1. A.c.i. systems. We first recall that an Abelian variety is a complex torus \mathbf{C}^r/Λ (Λ a lattice in \mathbf{C}^r) which is algebraic, which means that it admits an embedding in some projective space \mathbf{P}^N ; when $r > 1$ then most complex tori are not algebraic. The embedding can be done by using theta functions, which are quasi-periodic functions on \mathbf{C}^r (periodic in r directions, while periodic up to an exponential factor in r other directions). From the geometric point of view this means that one considers sections of a fixed (very ample) line bundle on the Abelian variety. From the analytic point of view it means that one considers functions with a fixed pole order along some (ample) divisor on the Abelian variety. The class of integrable systems that we will consider makes it possible to construct a basis of these functions, for the families of Abelian varieties that appear as the fibers of the momentum map; no other general method for constructing such a basis is known.

Definition 3.1. Let $(M, \{\cdot, \cdot\}, \mathbf{F})$ be an integrable system, where M is a non-singular affine variety and where $\mathbf{F} = (F_1, \dots, F_s)$. We say that $(M, \{\cdot, \cdot\}, \mathbf{F})$ is an algebraic completely integrable system or an a.c.i. system if for generic $m \in M$ the invariant manifold \mathbf{F}'_m is an affine part of an Abelian variety and if the Hamiltonian vector fields \mathcal{X}_{F_i} are translation invariant, when restricted to these tori.

We restrict ourselves to a.c.i. systems which are irreducible in the sense that the generic Abelian variety of the a.c.i. system does not contain a subtorus (an Abelian variety that contains a (non-trivial) subtorus is essentially a product of this subtorus with a complementary subtorus). It turns out that many (most) of the integrable systems that were known classically, turn out to be a.c.i., when complexified. This means that the powerful tools of the theory of Abelian varieties (dating mainly from the 19-th century) can be used to solve and to study these systems.

Example 3.2. Let us reconsider the Garnier system. \mathbf{R}^4 and the functions H and F are complexified in the obvious way. If $\alpha \neq \beta$ then for generic $(h, f) \in \mathbf{C}^2$ the fiber $\mathbf{F}^{-1}(h, f) \subset \mathbf{C}^4$ is an affine part of an Abelian surface of type (1, 4), which means that the lattice Λ that defines the torus is spanned by the columns of a matrix which has the form $\begin{pmatrix} 1 & 0 & a & b \\ 0 & 4 & b & c \end{pmatrix}$ with $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ positive definite. The divisor to be added to this affine part to complete it into an Abelian surface is a Riemann surface of genus 5. The commuting vector fields \mathcal{X}_H and \mathcal{X}_F extend to linear vector fields on these Abelian surfaces, hence the Garnier system is a.c.i. Notice that it follows that, if h and f are generic, then the affine variety defined by

$$\begin{aligned} h &= \frac{1}{2}(p_1^2 + p_2^2) + (q_1^2 + q_2^2)^2 + \alpha q_1^2 + \beta q_2^2, \\ f &= (q_1 p_2 - q_2 p_1)^2 + (\beta - \alpha)(p_1^2 + 2q_1^4 + 2q_1^2 q_2^2 + 2\alpha q_1^2), \end{aligned}$$

is an affine part of an Abelian surface of type (1, 4).

The vector fields of an a.c.i. system have good properties at infinity; after all, a linear vector field on a complex torus is the same along the divisor, which will happen to be absent in phase space, as on the rest of the torus. In fact, since every holomorphic function on a complex torus can be written as a quotient of theta functions, the integral curves (solutions) to any of the vector fields of an a.c.i. system can be written as a quotient of holomorphic functions. Intuitively speaking this means that we can consider not only Taylor solutions to the differential equations that describe these vector fields but also Laurent solutions, which will correspond to initial conditions at infinity (precisely: on the divisor that needs to be adjoined to the fibers of the momentum map to complete them into Abelian varieties). Moreover, these Laurent solutions must depend on $\dim M - 1$ free parameters, which corresponds to the freedom of choice of the initial condition at infinity. This idea is due to Kowalevski, who used it to find *her* top; her idea was further developed by Adler and van Moerbeke who applied it in several other cases of interest and who used the Laurent solutions to further explore the geometry of the invariant manifolds. A modern version of this idea is given in the following theorem (see [4]).

Theorem 3.3 (Kowalevski-Painlevé criterion). *Let $(\mathbf{C}^n, \{ \cdot, \cdot \}, \mathbf{F})$ be an irreducible, a.c.i. system, where $\mathbf{F} = (F_1, \dots, F_s)$ is a polynomial map, and let (x_1, \dots, x_n) be a system of linear coordinates on \mathbf{C}^n . Let \mathcal{V} be any one of the integrable vector fields $\mathcal{X}_{F_1}, \dots, \mathcal{X}_{F_s}$. For every $1 \leq i \leq n$ such that x_i is not constant along the integral curves of \mathcal{V} , i.e., $\dot{x}_i \neq 0$, there exists a Laurent solution $x(t) = (x_1(t), \dots, x_n(t))$, depending on $n - 1$ free parameters, for which $x_i(t)$ has a pole.*

This criterion has been used to single out from a family of Hamiltonian vector fields those that are integrable, and admit a good complexification (making them the most natural members of the family). Of course, in the end one still has to show that the Hamiltonians that have been selected lead indeed to an integrable system, resp. to an a.c.i. system. The most beautiful example is the following (see [3]).

Example 3.4. For $l \geq 1$ let e_0, \dots, e_l be linearly dependent vectors in an Euclidean vector space $(\mathbf{R}^{l+1}, \langle \cdot | \cdot \rangle)$, any l of which are linearly independent. Let us suppose that the non-zero real numbers p_0, p_1, \dots, p_l which satisfy $\sum_{i=0}^l p_i e_i = 0$ have a non-zero sum, i.e., $\sum_{i=0}^l p_i \neq 0$. Let A be the $(l+1, l+1)$ matrix, defined by

$$a_{ij} := \frac{2 \langle e_i | e_j \rangle}{\langle e_j | e_j \rangle}, \quad (0 \leq i, j \leq l).$$

Consider the vector field \mathcal{V} on $\mathbf{C}^{2(l+1)}$ which is given by

$$\begin{aligned} \dot{x} &= x \cdot y, \\ \dot{y} &= Ax, \end{aligned}$$

where $x, y \in \mathbf{C}^{l+1}$ and where $x \cdot y$ is defined by $(x \cdot y)_i = x_i y_i$, $i = 0, \dots, l$. Theorem 3.3 can be used to show that, if the vector field \mathcal{V} is one of the integrable vector fields of an irreducible a.c.i. system, then A is the Cartan matrix of a twisted affine Lie algebra.

An interesting feature of a.c.i. systems is that their solutions, with any initial conditions, are single-valued. The non single-valuedness of solutions to differential equations, in general, comes from the fact that analytic continuation of solutions usually depends on the (homotopy class of) the path along which the solution is analytically continued, and not just on the endpoints. This leads to a second

criterion for a.c.i. which was first used by Lyapounov. The proof of it is due to Luc Haine (see [13]).

Theorem 3.5 (Lyapunov criterion). *Let $(\mathbf{C}^n, \{\cdot, \cdot\}, \mathbf{F})$ be an a.c.i. system and let F be an arbitrary element of \mathbf{F} . All solutions to the integrable vector field \mathcal{X}_F are single-valued. Moreover, if $\gamma : [0, 1] \rightarrow \mathbf{C}$ is any closed path and $x(t)$ is a solution to \mathcal{X}_F that is holomorphic in a neighborhood of the path, then the analytic continuation along γ of the solution to the variational equations which correspond to the $x(t)$, is single-valued.*

This theorem seems (at least!) as powerful as the above Kowalevski-Painlevé criterion but, up to now, it has only been used in one example (also due to Luc Haine, see [13]).

Example 3.6. Geodesic flow for a left invariant metric on a Lie group \mathbf{G} reduces to a Hamiltonian flow on its Lie algebra \mathfrak{g} . The corresponding vector field is described by the following Lax equation

$$\dot{X} = [X, \nabla H(X)],$$

where H is the quadratic form on \mathfrak{g} that describes the metric. It is a Hamiltonian vector field, with H as Hamiltonian, the Poisson structure being the Lie-Poisson structure on \mathfrak{g} (\mathfrak{g} and its dual are identified by using H). Suppose now that $\mathfrak{g} = \mathfrak{so}_n$ and that H is diagonal in the sense that $(\nabla H(X))_{ij} = \lambda_{ij} X_{ij}$ where $X \in \mathfrak{so}_n$ and $\lambda_{ij} = \lambda_{ji}$ for $1 \leq i < j \leq n$. Theorem 3.5 can be used to show that, if all λ_{ij} are distinct, then the vector field \mathcal{X}_H is a.c.i. if and only if there exists constants $a_1, \dots, a_n, b_1, \dots, b_n$ such that

$$\lambda_{ij} = \frac{b_i - b_j}{a_i - a_j},$$

i.e., H defines a Manakov metric on \mathfrak{so}_n .

The above criteria for algebraic integrability are related to the non-integrability theory developed by Morales-Ruiz and Ramis (see [15]), but the precise relationship still has to be worked out.

3.2. Proving a.c.i. The problem arises to show that a given integrable system is a.c.i. In exceptional cases this may follow from abstract arguments, for example the integrable system may have been constructed essentially from algebraic-geometric data. Prime examples of this are the Hitchin system and its generalizations (see [7] and [14]). The construction involves beautiful geometric constructions but it is not known how to realize these systems explicitly (e.g. to write down the functions in involution, the equations of motion, ...). What we wish to describe in this section is the other extreme, namely what to do when not much more is given than explicit formulas for the functions in involution and the Poisson structure (which gives the commuting vector fields). In particular it is not assumed that Lax equations for the vector field are known; they come indeed often only at the end, when the geometry of the problem has been revealed, which is done while (and by) proving a.c.i. The ideas that underlie the technique, based on the work of Adler and van Moerbeke (see [2] and [4]), will be illustrated on a non-trivial example, that we refer to as the KM5 lattice, which stands for *periodic 5-particle Kac-van Moerbeke lattice*. The latter is given by the quadratic vector field

$$\dot{x}_i = x_i(x_{i-1} - x_{i+1}), \quad (i = 1, \dots, 5),$$

where x_1, \dots, x_5 are coordinates on \mathbf{C}^5 and $x_{i+5} = x_i$ for $i \in \mathbf{Z}$. This vector field \mathcal{V} admits the following three independent constants of motion,

$$\begin{aligned} F_1 &= x_1 + x_2 + x_3 + x_4 + x_5, \\ F_2 &= x_1x_3 + x_2x_4 + x_3x_5 + x_4x_1 + x_5x_2, \\ F_3 &= x_1x_2x_3x_4x_5. \end{aligned}$$

A vector field \mathcal{W} that commutes with \mathcal{V} is given by

$$x'_i = x_i(x_{i+2}x_{i-1} - x_{i+1}x_{i-2}), \quad (i = 1, \dots, 5),$$

having the same constants of motion. In fact, \mathcal{V} and \mathcal{W} are both Hamiltonian with respect to the quadratic Poisson structure that is defined by

$$\{x_i, x_j\} = x_i x_j (\delta_{i,j+1} - \delta_{j,i+1}),$$

where $1 \leq i, j \leq 5$. Namely, we recover both vector fields by taking F_1 and F_2 as Hamiltonians, while F_3 is a Casimir. Notice also that both vector fields and the constants of motion are invariant with respect to the order 5 automorphism σ of \mathbf{C}^5 , which is defined by $\sigma(x_i) = x_{i+1}$. It follows that each invariant manifold (and its compactification) admits an automorphism of order 5, also denoted by σ . This has a pronounced implication for the geometry of these manifolds, as we will see.

The rank of $\{\cdot, \cdot\}$ is 0 on the five 2-planes, which are defined by three non-consecutive x_i being zero (i.e., the plane $x_1 = x_2 = x_4 = 0$ and its image, under all powers of σ); the rank is at most 2 on the ten 3-planes which are given by $x_i = x_j = 0$, where $1 \leq i < j \leq 5$. On the remaining dense open subset, the rank is 4. Letting $\mathbf{F} := (F_1, F_2, F_3)$ it follows that $(\mathbf{C}^5, \{\cdot, \cdot\}, \mathbf{F})$ is Liouville integrable.

In order to show that the KM5 lattice is a.c.i. we fix a generic $\mathbf{c} \in \mathbf{C}^3$ and we show that $\mathbf{F}^{-1}(\mathbf{c})$ is an affine part of an Abelian variety, on which \mathcal{V} and \mathcal{W} restrict to linear vector fields. To do this we verify that $\mathbf{F}^{-1}(\mathbf{c})$ satisfies the conditions of the following theorem, which is a complex analog of the Liouville Theorem.

Theorem 3.7 (Complex Liouville Theorem). *Let $\mathcal{A} \subset \mathbf{C}^s$ be a non-singular affine variety of dimension r which supports r holomorphic vector fields $\mathcal{V}_1, \dots, \mathcal{V}_r$ and let $\varphi : \mathcal{A} \rightarrow \mathbf{C}^N \subset \mathbf{P}^N$ be an embedding. We define $\Delta := \overline{\varphi(\mathcal{A})} \setminus \varphi(\mathcal{A})$ and we denote the union of all irreducible components of Δ of dimension $r - 1$ by Δ' . Suppose the following.*

- (1) *The vector fields commute pairwise, $[\mathcal{V}_i, \mathcal{V}_j] = 0$ for $1 \leq i, j \leq r$;*
- (2) *At every point $m \in \mathcal{A}$ the vector fields $\mathcal{V}_1, \dots, \mathcal{V}_r$ are independent;*
- (3) *The vector field $\varphi_* \mathcal{V}_1$ extends to a vector field $\overline{\mathcal{V}}_1$ which is holomorphic on a neighborhood of Δ' in \mathbf{P}^N ;*
- (4) *The integral curves of $\overline{\mathcal{V}}_1$ that start at points $m \in \Delta'$ go immediately into $\varphi(\mathcal{A})$.*

Then $\overline{\varphi(\mathcal{A})}$ is an Abelian variety of dimension r and the vector fields $\varphi_ \mathcal{V}_1, \dots, \varphi_* \mathcal{V}_r$ extend to holomorphic (hence linear) vector fields on $\overline{\varphi(\mathcal{A})}$. Moreover, $\Delta' = \Delta$.*

For our example, we take for \mathcal{A} the smooth manifold $\mathcal{A}_{\mathbf{c}} := \mathbf{F}^{-1}(\mathbf{c})$, where \mathbf{c} is a regular value of \mathbf{F} , satisfying $c_3 \neq 0$ (a generic $\mathbf{c} \in \mathbf{C}^3$ has these properties). Notice that if $\mathbf{c} = (c_1, c_2, c_3) \in \mathbf{C}^3$ is such that $c_3 \neq 0$ then $\mathbf{F}^{-1}(\mathbf{c})$ is entirely contained in $M_{(2)}$, so that each invariant manifold is a connected component of a fiber of the momentum map; in fact, since $\mathcal{A}_{\mathbf{c}}$ is irreducible it coincides with the invariant manifold through any of its points. Thus, $\mathcal{A}_{\mathbf{c}}$ is a non-singular affine variety of

dimension r , equipped with r holomorphic vector fields that satisfy conditions (1) and (2). It therefore suffices to construct an embedding $\varphi_{\mathbf{c}} : \mathcal{A}_{\mathbf{c}} \rightarrow \mathbf{C}^N \subset \mathbf{P}^N$ which satisfies (3) and (4) to conclude that the generic invariant manifold satisfies the conditions, given in the definition of a.c.i.

To do this, we use the Laurent solutions to \mathcal{V} . These are rather easily found because one can show that, in this case, each of the Laurent series $x_i(t)$ has a simple pole at worst. Substituting

$$x_i(t) = \frac{1}{t} \sum_{k=0}^{\infty} x_i^{(k)} t^k, \quad (i = 1, \dots, 5),$$

we find that the leading coefficients of the series satisfy

$$x_i^{(0)}(1 + x_{i-1}^{(0)} - x_{i+1}^{(0)}) = 0, \quad (i = 1, \dots, 5),$$

the so-called indicial equation. It has 10 solutions, namely the points

$$\begin{aligned} p_1 &:= (-1, 1, 0, 0, 0), \\ q_1 &:= (-2, 1, -1, 2, 0), \end{aligned}$$

and their images $p_i := \sigma^{i-1}(p_1)$, $q_i := \sigma^{i-1}(q_1)$ under repeated application of the order 5 automorphism σ . The other terms are then obtained recursively, but linearly, from the solutions to the indicial equation. We give the first few terms of the Laurent solution for which the leading term is given by p_1/t ,

$$\begin{aligned} x_1(t; p_1) &= -\frac{1}{t} + a - \frac{1}{3}(a^2 + 2b + c)t + O(t^2), \\ x_2(t; p_1) &= \frac{1}{t} + a + \frac{1}{3}(a^2 - b - 2c)t + O(t^2), \\ (3) \quad x_3(t; p_1) &= ct + O(t^2), \\ x_4(t; p_1) &= d + O(t^2). \\ x_5(t; p_1) &= bt + O(t^2), \end{aligned}$$

where the four free parameters have been denoted by a, \dots, d . Using the majoration method one shows that they are actually convergent. Using the automorphism σ we find in total 5 such Laurent solutions $x(t; p_i)$, depending on 4 free parameters. The Laurent solutions (depending on 4 free parameters) for \mathcal{W} (or any linear combination of \mathcal{V} and \mathcal{W}) can be computed from it, but they will not be needed (in certain examples, such as the periodic Toda lattice, these Laurent solutions are needed to prove a.c.i.). The points q_i lead to Laurent solutions that depend only on 3 free parameters; they are not used in what follows.

Since the functions F_i are constants of motion, a direct substitution of $x(t; p_1)$ in each of them yields a Laurent series in t , of which only the term in t^0 survives. Clearly this term can be computed by using only the first terms of the series $x(t; p_1)$. In the present case the terms that are given, suffice to do this for each of the functions F_i . We find three algebraic relations between the free parameters a, \dots, d and the values of the constants of motion c_1, c_2, c_3 , to wit,

$$\begin{aligned} 2a + d &= c_1, \\ b - c + 2ad &= c_2, \\ bcd &= -c_3. \end{aligned}$$

This defines an affine algebraic curve, denoted $\Gamma_{\mathbf{c}}^{(1)}$, in \mathbf{C}^4 . Clearly, $\Gamma_{\mathbf{c}}^{(1)}$ is isomorphic to the non-singular plane curve

$$\Gamma_{\mathbf{c}} : bd(b - c_2 + d(c_1 - d)) + c_3 = 0.$$

By Riemann-Hurwitz the genus of $\Gamma_{\mathbf{c}}$ is two. It is easy to see that $\Gamma_{\mathbf{c}}$ has three points at infinity, corresponding to $d = 0$ or $d = \infty$; they will be denoted by $\infty_{\mathbf{c}}$, $\infty'_{\mathbf{c}}$ and $\infty''_{\mathbf{c}}$. In terms of a local parameter ς a neighborhood of these points is parametrized as follows.

$$(4) \quad \begin{array}{ll} \infty_{\mathbf{c}} : & d = \varsigma^{-1}, \quad b = \varsigma^{-2} - c_1\varsigma^{-1} + c_2 - c_3\varsigma^3 + O(\varsigma^4), \\ \infty'_{\mathbf{c}} : & b = \varsigma^{-1}, \quad d = -c_3\varsigma^2 - c_2c_3\varsigma^3 - c_2^2c_3\varsigma^4 + O(\varsigma^5), \\ \infty''_{\mathbf{c}} : & d = \varsigma^{-1}, \quad b = c_3\varsigma^3 + c_1c_3\varsigma^4 + O(\varsigma^5). \end{array}$$

It is clear that, if we use any other principal balance $x(t; p_i)$ then we get an affine curve $\Gamma_{\mathbf{c}}^{(i)}$, which is also isomorphic to $\Gamma_{\mathbf{c}}$. Thus, considering all 5 Laurent solutions that depend on 4 free parameters (they are obtained from the given one by using the automorphism σ) we get 5 isomorphic curves of genus two. Each of them is compactified into a Riemann surface by adding three points at infinity.

We now construct the embedding $\varphi_{\mathbf{c}} : \mathcal{A}_{\mathbf{c}} \rightarrow \mathbf{P}^N$. The idea of the construction is based on the following fact. If KM5 is indeed an irreducible a.c.i. system then a divisor \mathcal{D} can be added to a Zariski open subset of phase space \mathbf{C}^5 , having the effect of compactifying all fibers $\mathcal{A}_{\mathbf{c}}$, where \mathbf{c} is generic; the divisor that is added to $\mathcal{A}_{\mathbf{c}}$ will be denoted by $\mathcal{D}_{\mathbf{c}}$ and the resulting torus by $\mathbf{T}_{\mathbf{c}}^2$. The KM5 vector fields \mathcal{V} and \mathcal{W} extend to linear (hence holomorphic) vector field $\overline{\mathcal{V}}$ and $\overline{\mathcal{W}}$ on this partial compactification of \mathbf{C}^5 , hence we may consider the integral curves of \mathcal{V} , starting from any irreducible component $\mathcal{D}_{\mathbf{c}}^{(i)}$ of this divisor. This gives a Laurent solution, depending on 4 free parameters, hence it must coincide with one of the Laurent solutions $x(t; p_i)$ that we have computed. Let f be a polynomial function on \mathbf{C}^n . Since $\overline{\mathcal{V}}$ is transversal to the divisor \mathcal{D} the pole order of the Laurent series $f(t; p_i)$ (which can be obtained by substituting the series $x(t; p_i)$ in f) equals the pole order of $f|_{\mathbf{T}_{\mathbf{c}}^2}$ along the divisor $\mathcal{D}_{\mathbf{c}}^{(i)}$, where $f|_{\mathbf{T}_{\mathbf{c}}^2}$ is by definition $f|_{\mathcal{A}_{\mathbf{c}}}$, viewed as a meromorphic function on $\mathbf{T}_{\mathbf{c}}^2$. Since the third power of an ample divisor on an Abelian variety is very ample we look for all the polynomials which have a triple pole at most when $x(t; p_1)$ is substituted in them, and no pole at all when $x(t; p_i)$ for $2 \leq i \leq 5$ is substituted in them. Precisely, we look for a maximal independent set of such functions which are independent when restricted to $\mathcal{A}_{\mathbf{c}}$. One finds easily, besides $z_0 := 1$ the following independent functions,

$$\begin{array}{ll} z_1 := x_1x_2, & z_5 := x_1x_2^2x_4, \\ z_2 := x_1x_2x_4, & z_6 := x_1x_2x_4((x_3 + x_4)x_1 - (x_4 + x_5)x_2), \\ z_3 := x_1x_2(x_1 + x_5), & z_7 := x_1^2x_2^2x_4x_5, \\ z_4 := x_1^2x_2x_4, & z_8 := x_1x_2^2x_4((x_4 + x_5)^2 + x_3x_4). \end{array}$$

In the sequel we think of these functions as being restricted to $\mathcal{A}_{\mathbf{c}}$, and we consider the map

$$(5) \quad \begin{array}{ll} \varphi_{\mathbf{c}} : & \mathcal{A}_{\mathbf{c}} \subset \mathbf{C}^5 \rightarrow \mathbf{P}^8 \\ & (x_1, \dots, x_6) \mapsto (1 : z_1 : \dots : z_8). \end{array}$$

Since $c_3 \neq 0$, $\mathcal{A}_{\mathbf{c}}$ does not intersect any of the hyperplanes $x_i = 0$, so that $\varphi_{\mathbf{c}}$ is birational on its image, namely

$$x_1 = z_4/z_2, \quad x_2 = z_5/z_2, \quad x_4 = z_2/z_1, \quad x_5 = z_7/(z_1z_2),$$

while x_3 is recovered from $F_1 = c_1$. In particular, $\varphi_{\mathbf{c}}$ is an embedding. The closure of $\varphi_{\mathbf{c}}(\mathcal{A}_{\mathbf{c}})$ (with respect to the Zariski topology, or, what is the same, with respect to the complex topology) is denoted by $\overline{\varphi_{\mathbf{c}}(\mathcal{A}_{\mathbf{c}})}$. Also, we denote by $\Delta'_{\mathbf{c}}$ the union of all irreducible components of $\overline{\varphi_{\mathbf{c}}(\mathcal{A}_{\mathbf{c}})} \setminus \varphi_{\mathbf{c}}(\mathcal{A}_{\mathbf{c}})$ of dimension $r - 1 = 1$, as in Theorem 3.7.

The leading terms of the above Laurent series $z_i(t; p_1)$ lead to a map $\varphi_{\mathbf{c}}^{(1)} : \Gamma_{\mathbf{c}} \rightarrow \mathbf{P}^8$, given by

$$(6) \quad \varphi_{\mathbf{c}}^{(1)} : (b, d) \mapsto (0 : 0 : 0 : 1 : d : -d : 2d^2 : bd : -d^3),$$

which is obviously an embedding (recall that $b \neq 0$ on $\Gamma_{\mathbf{c}}$). Similarly, the Taylor (!) series $z(t; p_2), \dots, z(t; p_5)$ lead to four different embeddings of this curve, to wit

$$(7) \quad \begin{aligned} \varphi_{\mathbf{c}}^{(2)} : (b, d) &\mapsto (1 : -b : 0 : -bd : 0 : bc : c_3 : 0 : bc(c + d^2)), \\ \varphi_{\mathbf{c}}^{(3)} : (b, d) &\mapsto (1 : 0 : bd : 0 : bd^2 : 0 : bd(2ad - b) : 0 : 2ab^2d), \\ \varphi_{\mathbf{c}}^{(4)} : (b, d) &\mapsto (1 : 0 : -cd : cd : 0 : -cd^2 : cd(c + 2ad) : \\ &\quad -(cd)^2 : -cd^2(4a^2 - b)), \\ \varphi_{\mathbf{c}}^{(5)} : (b, d) &\mapsto (1 : c : 0 : 2ac : bc : 0 : -c_3 : -bc^2 : bc^2), \end{aligned}$$

where we recall that $a = (c_1 - d)/2$ and $c = b - c_2 - d(c_1 - d)$. Since $bcd = -c_3 \neq 0$ we see at once that the five image curves are disjoint. However, these images are not complete, so we check if maybe their closures intersect. In order to do this, we compute the image of $\infty_{\mathbf{c}}, \infty'_{\mathbf{c}}$ and $\infty''_{\mathbf{c}}$ under each of the above five embeddings. This is done by substituting the above local parametrizations (4) in the embeddings $\varphi_{\mathbf{c}}^{(i)}$; taking an extra term in the computation we can also determine the tangent line to the image of $\infty_{\mathbf{c}}, \infty'_{\mathbf{c}}$ and $\infty''_{\mathbf{c}}$ to deduce from it whether these closures are non-singular and how they intersect.

TABLE 1. The image of the points at infinity on $\Gamma_{\mathbf{c}}$ under the five embeddings $\varphi_{\mathbf{c}}^{(i)}$ of $\Gamma_{\mathbf{c}}$, given by (6) and (7).

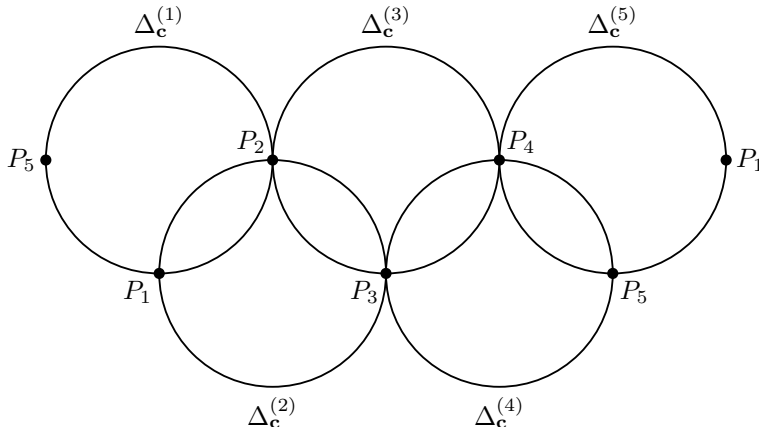
	$\varphi_{\mathbf{c}}^{(1)}$	$\varphi_{\mathbf{c}}^{(2)}$	$\varphi_{\mathbf{c}}^{(3)}$	$\varphi_{\mathbf{c}}^{(4)}$	$\varphi_{\mathbf{c}}^{(5)}$
$\infty_{\mathbf{c}}$	P_5	P_1	P_2	P_3	P_4
$\infty'_{\mathbf{c}}$	P_1	P_2	P_3	P_4	P_5
$\infty''_{\mathbf{c}}$	P_2	P_3	P_4	P_5	P_1

In this table the points P_i are the following points in \mathbf{P}^8 .

$$\begin{aligned} P_1 &= (0 : 0 : 0 : 1 : 0 : 0 : 0 : 0 : 0), \\ P_2 &= (0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 1), \\ P_3 &= (1 : 0 : 0 : 0 : 0 : 0 : c_3 : 0 : -c_1c_3), \\ P_4 &= (1 : 0 : 0 : 0 : 0 : 0 : -c_3 : 0 : 0), \\ P_5 &= (0 : 0 : 0 : 0 : 0 : 0 : 0 : 1 : -1). \end{aligned}$$

Denoting $\overline{\varphi_{\mathbf{c}}^{(i)}(\Gamma_{\mathbf{c}})}$ by $\Delta_{\mathbf{c}}^{(i)}$, we find that $\Delta_{\mathbf{c}}^{(i)}$ contains the points P_{i-1}, P_i and P_{i+1} and that each $\Delta_{\mathbf{c}}^{(i)}$ intersects its neighbor $\Delta_{\mathbf{c}}^{(i+1)}$ in two different points P_i and P_{i+1} , while being tangent to the divisors $\Delta_{\mathbf{c}}^{(i-2)}$ and $\Delta_{\mathbf{c}}^{(i+2)}$. The resulting divisor is depicted in Figure 1. We need to show that $(\varphi_{\mathbf{c}})_*\mathcal{V}$ extends to a holomorphic vector field on a neighborhood of $\Delta'_{\mathbf{c}}$ in \mathbf{P}^8 (condition (3) of Theorem 3.7). Notice

FIGURE 1. The divisor $\cup_{i=1}^5 \Delta_{\mathbf{c}}^{(i)}$ consists of five genus two curves that intersect in five points, each of which is a triple point of the divisor. To make the picture exact one has to identify the two points labeled P_1 , as well as the two points labeled P_5 in such a way that the curves $\Delta_{\mathbf{c}}^{(2)}$ and $\Delta_{\mathbf{c}}^{(5)}$ are tangent, as well as the curves $\Delta_{\mathbf{c}}^{(1)}$ and $\Delta_{\mathbf{c}}^{(4)}$. As we will show later, $\Delta_{\mathbf{c}} = \cup_{i=1}^5 \Delta_{\mathbf{c}}^{(i)}$.



that we do not know yet whether $\Delta'_{\mathbf{c}}$ coincides with $\cup_{i=1}^5 \Delta_{\mathbf{c}}^{(i)}$, so we do not really have control over $\Delta'_{\mathbf{c}}$. In fact, we will be able to show that $(\varphi_{\mathbf{c}})_* \mathcal{V}$ extends to a holomorphic vector field on *all* of \mathbf{P}^8 , which implies in particular holomorphicity in a neighborhood of $\Delta'_{\mathbf{c}}$, whatever the latter may be. The construction of the extension is based on the following theorem, which says that, in appropriate projective coordinates, any holomorphic vector field on an Abelian variety is a quadratic vector field, hence it is globally defined and holomorphic (on all of \mathbf{P}^N).

Theorem 3.8. *Let \mathcal{L} be an ample line bundle on an irreducible Abelian variety \mathbf{T}^r and let \mathcal{V} be a holomorphic vector field on \mathbf{T}^r . Denoting by $\varphi : \mathbf{T}^r \rightarrow \mathbf{P}^N$ the Kodaira embedding that corresponds to $\mathcal{L}^{\otimes 3}$, there exists a quadratic vector field $\overline{\mathcal{V}}$ on \mathbf{P}^N , such that $\varphi_* \mathcal{V} = \overline{\mathcal{V}}$.*

Thus, even if we do not know yet that our example is a.c.i., we look for quadratic differential equations on \mathbf{P}^8 , whose restriction to $\varphi_{\mathbf{c}}(\mathcal{A}_{\mathbf{c}})$ describes $(\varphi_{\mathbf{c}})_* \mathcal{V}$. This can be done in an algorithmic way, because \mathcal{V} and all invariants are homogeneous. In the chart $Z_0 \neq 0$ of \mathbf{P}^8 we arrive at the following¹ result.

$$\begin{aligned} \dot{z}_1 &= z_2 + 2z_3 - c_1 z_1, \\ \dot{z}_2 &= z_4 - z_5, \end{aligned}$$

¹The result is not unique because one can always add to the right hand side any quadratic polynomial that vanishes on the Abelian variety; we have used this fact to keep the formulas as simple as possible.

$$\begin{aligned}
\dot{z}_3 &= z_4 + 2z_5 - 3z_1^2 + c_1z_3 - 2c_2z_1, \\
\dot{z}_4 &= -3z_1z_2 + c_1z_4 + \frac{1}{2}(c_3 - c_2z_2 - z_6), \\
\dot{z}_5 &= 3z_1z_2 - c_1z_5 - \frac{1}{2}(c_3 - c_2z_2 + z_6), \\
\dot{z}_6 &= (2z_1 + c_2)(z_4 + z_5) - 4z_2^2 - 2c_1z_1z_2, \\
\dot{z}_7 &= 3z_2z_5 + c_1z_7 + \frac{1}{2}(3z_1z_6 - 3c_2z_1z_2 - c_3z_1), \\
\dot{z}_8 &= -3z_2(z_4 + z_5 - c_1z_2) + c_3z_1 + \frac{1}{2}c_2(z_6 - c_2z_2 + c_3).
\end{aligned}$$

In the same way one easily finds the differential equations in another chart. In view of Hartog's Theorem we may conclude that the vector field $(\varphi_{\mathbf{c}})_*\mathcal{V}$ extends to a holomorphic vector field on \mathbf{P}^8 , which we denote by $\bar{\mathcal{V}}$. Thus, we have verified that $\varphi_{\mathbf{c}}$ satisfies (3) in the Complex Liouville Theorem.

The final thing to be shown is that the integral curves of $\bar{\mathcal{V}}$ that start at points $m \in \Delta'_{\mathbf{c}}$ go immediately into $\varphi_{\mathbf{c}}(\mathcal{A}_{\mathbf{c}})$. We repeat that we do not know if $\Delta'_{\mathbf{c}} = \cup_{i=1}^5 \Delta_{\mathbf{c}}^{(i)}$. Therefore, there are three types of points that we need to consider.

- (1) Points in the image $\varphi_{\mathbf{c}}^{(i)}(\Gamma_{\mathbf{c}})$, where $i = 1, \dots, 5$;
- (2) Points in $\Delta_{\mathbf{c}}^{(i)} \setminus \varphi_{\mathbf{c}}^{(i)}(\Gamma_{\mathbf{c}})$, where $i = 1, \dots, 5$;
- (3) Points in $\Delta'_{\mathbf{c}} \setminus \cup_{i=1}^5 \Delta_{\mathbf{c}}^{(i)}$.

For (1) one uses that the embedding functions z_i are polynomials in the phase variables x_1, \dots, x_5 and the fact that the Laurent series are convergent to conclude that, for small $|t|$, all series $z_i(t; p_1)$ are finite, hence the flow starting from the points as in (1) does not belong to the hyperplane $z_0 = 0$. Similarly all quotients $z_i/z_j(t; p_1)$ are finite, so we do not flow into any of the other $\Delta_{\mathbf{c}}^{(i)}$. For (2), we do the check for $P_1 = \varphi_{\mathbf{c}}^{(1)}(\infty'_{\mathbf{c}})$. Since the only non-zero entry corresponds to z_3 we substitute the parametrization (4) of a neighborhood of $\infty'_{\mathbf{c}}$ into $1/z_3(t; p_1)$. Since $z_3(t; p_1) = \frac{1}{t^3} - \frac{a}{t^2} + O(1)$ we have that

$$(8) \quad z_3^{-1}(t; p_1) = t^3(1 + at + O(t^2)) = t^3 \left(1 + \frac{1}{2}(c_1 + c_3\varsigma^2 + O(\varsigma^3))t + O(t^2) \right),$$

which yields, in the limit $\varsigma \rightarrow 0$, the series $t^3(1 + c_1t/2 + O(t^2))$, which is different from zero. This means that, starting from P_1 , we do not flow into the divisor $\Delta_{\mathbf{c}}^{(1)}$. In order to show that we flow into the affine $\varphi_{\mathbf{c}}(\mathcal{A}_{\mathbf{c}})$ we show that $z_1/z_3(t; p_1)$ and $z_2/z_3(t; p_1)$ have also a non-zero limit, as $\varsigma \rightarrow 0$. For the first one, this is trivial since

$$\frac{z_1}{z_3}(t; p_1) = \left(-\frac{1}{t^2} + O(1) \right) (t^3 + O(t^4)) = -t + O(t^2),$$

as follows from (3). For the second one, the computation is longer, since the first terms of the series $z_2/z_3(t; p_1)$ vanish when taking the limit; besides this fact, which only makes the computation longer, the calculation is trivial and we only state the final formula,

$$\lim_{\varsigma \rightarrow 0} \frac{z_2}{z_3}(t; p_1) = -\frac{c_3}{4}t^5 + O(t^6).$$

This shows that we flow into the affine, starting from P_1 .

Finally, we turn to the points (3). We show that there are no such points. Since $\mathcal{A}_{\mathbf{c}}$ is irreducible and since $\varphi_{\mathbf{c}}$ is regular the divisor $\Delta'_{\mathbf{c}}$ is connected. Therefore, if $\Delta'_{\mathbf{c}}$ contains irreducible components that are different from the $\Delta_{\mathbf{c}}^{(i)}$ then at least one of the former ones must intersect one of the latter ones. Moreover, this must happen at points where the series break down, i.e., at the points (2). We can however compute the degree of $\Delta'_{\mathbf{c}}$ at these points. We do this for the point P_1 . Letting $y_0 := 1/z_3$ we have that y_0 is a defining function for the divisor around P_1 and since \mathcal{V} is transversal to $\Delta_{\mathbf{c}}^{(1)}$ at P_1 we can read off from (8) that the degree of $\Delta_{\mathbf{c}}$ equals 3. Since $\overline{\varphi_{\mathbf{c}}^{(1)}(\Gamma)}$ passes through P_1 and since this divisor is taken with multiplicity 3, the divisor that we have found already account for the degree three and there cannot be another divisor, passing through P_1 . For the points P_2, \dots, P_5 one proceeds in the same way, thereby concluding the proof that KM5 is a.c.i.

Since the torus $\mathbf{T}_{\mathbf{c}}^2$ contains a curve $\Gamma_{\mathbf{c}}$ of genus two it is the Jacobian of this curve and this curve is a translate of the theta divisor. Notice that three times the theta divisor induces on the Jacobian a polarization of type (3, 3), which leads to an embedding of the Jacobian in \mathbf{P}^8 ; the above embedding is a concrete realization of this. For an alternative characterization of these tori in terms of hyperelliptic Prym varieties, and for an alternative proof of algebraic integrability of KM5 (in fact of all KM), see [9]. For more elaborate examples and a justification of the facts that are used in the above proof, see [4].

3.3. Lax equations with a parameter. The integrable vector fields of most a.c.i. systems can be written down in the form of a Lax equation with a parameter, defined as follows.

Definition 3.9. Let M be a finite dimensional affine subspace of the Lie algebra $\mathfrak{gl}_N[\mathfrak{h}]$, let $\{\cdot, \cdot\}$ be a Poisson structure on M and let $H \in \mathcal{F}(M)$. If the Hamiltonian vector field \mathcal{X}_H on M can be written in the form

$$(9) \quad \dot{X}(\mathfrak{h}) = [X(\mathfrak{h}), Y(\mathfrak{h})],$$

where $Y(\mathfrak{h})$ is a function on M with values in $\mathfrak{gl}_N[\mathfrak{h}, \mathfrak{h}^{-1}]$ then (9) is called a Lax equation with parameter for \mathcal{X}_H .

As in the case of Lax equations (without a parameter) one has that the functions q_{ij} which are defined by the coefficients of the characteristic polynomial of $X(\mathfrak{h})$,

$$Q(\mathfrak{h}, \mathfrak{s}) = \det(\mathfrak{s} \text{Id}_N - X(\mathfrak{h})) = \mathfrak{s}^N + \sum_{(i,j) \in I} q_{ij} \mathfrak{h}^i \mathfrak{s}^j.$$

are constants of motion of (9). Therefore, if we associate to each $X(\mathfrak{h})$ the plane algebraic curve

$$\Gamma_X := \{(\mathfrak{h}, \mathfrak{s}) \in \mathbf{C} \times \mathbf{C} \mid \det(\mathfrak{s} \text{Id}_N - X(\mathfrak{h})) = 0\}$$

we have that Γ_X is preserved by the flow of (9). Similarly, for each $X(\mathfrak{h})$ the variety of matrices

$$\mathcal{A}_X := \{X'(\mathfrak{h}) \mid X(\mathfrak{h}) \text{ and } X'(\mathfrak{h}) \text{ have the same characteristic polynomial}\}$$

is preserved by the flow of (9). Each of the curves Γ_X is called a spectral curve and each of the varieties \mathcal{A}_X is an isospectral variety; when \mathcal{A}_X is smooth we also use the name isospectral manifold. Let us suppose that for generic $X(\mathfrak{h}) \in M$ the

affine curve Γ_X is non-singular and that for generic $(\mathfrak{h}, \mathfrak{s}) \in \Gamma_X$, the eigenspace of $X(\mathfrak{h})$ with eigenvalue \mathfrak{s} is one-dimensional. Let (possibly after some relabeling)

$$\xi(\mathfrak{h}, \mathfrak{s}) = (\xi_1(\mathfrak{h}, \mathfrak{s}), \dots, \xi_N(\mathfrak{h}, \mathfrak{s}))^\top$$

be the eigenvector which is normalized at $\xi_1(\mathfrak{h}, \mathfrak{s}) = 1$. Then each $\xi_i(\mathfrak{h}, \mathfrak{s})$ is a rational function in $(\mathfrak{h}, \mathfrak{s})$, whose coefficients are functions on M . When $X(\mathfrak{h})$ flows according to (9), the normalized eigenvector $\xi(\mathfrak{h}, \mathfrak{s})$ also evolves, satisfying the autonomous equation

$$\dot{\xi} + Y(\mathfrak{h})\xi = \lambda(\mathfrak{h}, \mathfrak{s})\xi.$$

where $\lambda = \lambda(\mathfrak{h}, \mathfrak{s})$ is also a rational function whose coefficients are functions on M .

The isospectral manifolds and isospectral curves are linked by a map that we introduce next. For simplicity we will assume that \mathfrak{c} is chosen such that $\Gamma_{\mathfrak{c}}$ is non-singular and such that $\mathcal{A}_{\mathfrak{c}}$ is connected. Since $\Gamma_{\mathfrak{c}}$ is non-singular it can be completed into a compact Riemann surface $\bar{\Gamma}_{\mathfrak{c}}$ by adding a few points, which will be denoted by p_1, \dots, p_d . Denote for $i = 1, \dots, N$ and for U an open subset of $\bar{\Gamma}_{\mathfrak{c}}$ by $(\xi_i)_U$ the divisor of zeros and poles of ξ_i , restricted to U . Since ξ_i depends in a holomorphic way on t (for $|t|$ small) we have that for a generic $X(\mathfrak{h}) \in \mathcal{A}_{\mathfrak{c}}$, with corresponding normalized eigenvector ξ , the minimal effective divisor \mathcal{D}_X on $\Gamma_{\mathfrak{c}}$ which satisfies

$$(\xi_i)_{\Gamma_{\mathfrak{c}}} \geq -\mathcal{D}_X, \quad i = 1, \dots, N,$$

has a degree d which is independent of $X = X(\mathfrak{h})$. This leads, by continuity, to an effective divisor \mathcal{D}_X of degree d in $\bar{\Gamma}_{\mathfrak{c}}$ for any $X = X(\mathfrak{h}) \in \mathcal{A}_{\mathfrak{c}}$. The resulting map

$$\begin{aligned} \iota_{\mathfrak{c}} : \mathcal{A}_{\mathfrak{c}} &\rightarrow \text{Div}^d(\bar{\Gamma}_{\mathfrak{c}}) \\ X(\mathfrak{h}) &\mapsto \mathcal{D}_X \end{aligned}$$

is called the divisor map. When $X(\mathfrak{h})$ evolves according to (9), the image of $X(\mathfrak{h}, t)$ under $\iota_{\mathfrak{c}}$ evolves on $\text{Div}^d(\bar{\Gamma}_{\mathfrak{c}})$; we will denote $\iota_{\mathfrak{c}}(X(\mathfrak{h}, t))$ by $\mathcal{D}_{X(t)}$. Choose a basis $(\omega_1, \dots, \omega_g)$ of $\bar{\Gamma}_{\mathfrak{c}}$, where g is the genus of $\bar{\Gamma}_{\mathfrak{c}}$, and let $\vec{\omega} := (\omega_1, \dots, \omega_g)^\top$. The map $J_{X(0)}$, defined by

$$(10) \quad \begin{aligned} J_{X(0)} : U &\rightarrow \text{Jac}(\Gamma_{\mathfrak{c}}) \\ t &\mapsto \int_{\mathcal{D}_{X(0)}}^{\mathcal{D}_{X(t)}} \vec{\omega}, \end{aligned}$$

is called the linearizing map, starting at $X(0)$ (with respect to $\vec{\omega}$). The following theorem gives a criterion to check whether this map linearizes the isospectral flow.

Theorem 3.10 (Linearization Criterion). *Suppose that \mathfrak{c} is chosen such that $\Gamma_{\mathfrak{c}}$ is smooth, and let $X(0)$ be an element of $\mathcal{A}_{\mathfrak{c}}$. The linearizing map $J_{X(0)}$ linearizes the isospectral flow defined by (9) if and only if there exists a meromorphic function ϕ on Γ with $(\phi) \geq -N(\mathfrak{h})_\infty$, and such that for all p_i*

$$(11) \quad \frac{d}{dt}(\text{Laurent tail of } \lambda \text{ at } p_i) = (\text{Laurent tail of } \phi \text{ at } p_i).$$

For a list of examples that satisfy the linearization criterion, see [1, Theorem 4.3, page 302]. Our example 2.12 is easily adapted to write the KM5 vector field as a

Lax equation (with parameter) $\dot{X}(\mathfrak{h}) = [X(\mathfrak{h}), Y(\mathfrak{h})]$, by taking

$$X(\mathfrak{h}) := \begin{pmatrix} 0 & x_1 & 0 & 0 & \mathfrak{h}^{-1} \\ 1 & 0 & x_2 & 0 & 0 \\ 0 & 1 & 0 & x_3 & 0 \\ 0 & 0 & 1 & 0 & x_4 \\ x_5 \mathfrak{h} & 0 & 0 & 1 & 0 \end{pmatrix}.$$

In this case all Laurent tails are constant at infinity, so KM5 satisfies the linearization criterion.

3.4. Other notions of algebraic integrability. For certain examples of interest the generic invariant manifold is not an affine part of an Abelian variety, but still the flow of the integrable vector fields induces on them the action of a local group and the group law is algebraic. In order to cover these cases, we propose here the following definition that generalizes Definition 3.1.

Definition 3.11. Let $(M, \{\cdot, \cdot\}, \mathbf{F})$ be a complex integrable system, where M is a (non-singular) affine variety, and where $\mathbf{F} = (F_1, \dots, F_s)$. We say that $(M, \{\cdot, \cdot\}, \mathbf{F})$ is a generalized a.c.i. system if for generic $m \in M$ the integrable vector fields $\mathcal{X}_{F_1}, \dots, \mathcal{X}_{F_s}$ define the local action of an algebraic group on \mathbf{F}'_m .

Notice that the algebraic group is necessarily commutative because the vector fields \mathcal{X}_{F_i} commute pairwise. The case of an a.c.i. system is a particular example: in this case the algebraic group is a compact algebraic group, i.e., an Abelian variety. We give two examples where the group is not compact.

Example 3.12. Let Σ be a compact oriented topological surface of genus $g \geq 1$ with fundamental group $\pi_1(\Sigma)$ and let \mathbf{G} be a reductive algebraic group. Then $\text{Hom}(\pi_1(\Sigma), \mathbf{G})$ is an affine variety on which \mathbf{G} acts by conjugation, more precisely if $\rho : \pi_1(\Sigma) \rightarrow \mathbf{G}$ and $g \in \mathbf{G}$ then $g \cdot \rho$ is the homomorphism $\pi_1(\Sigma) \rightarrow \mathbf{G}$ defined by

$$g \cdot \rho(C) = g(\rho(C))g^{-1},$$

for $C \in \pi_1(\Sigma)$. It turns out (see [12]) that the quotient

$$M := \text{Hom}(\pi_1(\Sigma), \mathbf{G})/\mathbf{G}$$

(which is an affine variety since \mathbf{G} is reductive) has a natural Poisson structure which can very explicitly be described for the classical groups. For simplicity let us consider the case $\mathbf{G} = SL(n)$ in the standard representation. For a curve $C \in \pi_1(\Sigma)$ the function

$$f_C : M \rightarrow \mathbf{G} : \rho \mapsto \text{Trace}(\rho(C))$$

is a well-defined regular function on M and these functions generate $\mathcal{F}(M)$. It was shown by Goldman that on such functions a Poisson bracket of maximal rank is given by

$$(12) \quad \{f_C, f_{C'}\} = \sum_{p \in C \# C'} \epsilon(p; C, C') \left(f_{C_p C'_p} - \frac{1}{n} f_C f_{C'} \right).$$

The sum runs over the intersection points of C and C' (one may suppose that the curves intersect transversally) and $\epsilon(p; C, C')$ is a sign which is determined by the way the (oriented) curves C and C' intersect at p , upon using the orientation of Σ . Finally, $C_p C'_p$ is the curve on Σ , based at p , which is obtained by first following C and then following C' .

Let us consider the case $\mathbf{G} = SL(2)$. Since $\pi_1(\Sigma)$ has a system of $2g$ generators, which are bound by one relation, $\text{Hom}(\pi_1(\Sigma), SL(2))$ has dimension $6g - 3$, hence M has dimension $6g - 6$. Since the rank of the Poisson structure is maximal we need to find $3g - 3$ independent functions on M that are in involution. Now Σ can be decomposed (in an infinite number of different ways) into so-called trinions; a trinion, also called a pair of pants, is just a three-holed sphere and such a decomposition will consist of $2g - 2$ trinions. Each trinion being bounded by three curves (which are identified two by two) one gets $3g - 3$ curves on Σ and what is important here is that they are non-intersecting. Calling these curves C_1, \dots, C_{3g-3} we find from Goldman's formula (12) that the functions $f_{C_1}, \dots, f_{C_{3g-3}}$ are in involution. They can be shown to be independent, hence they lead to an integrable system on M . The fibers of the momentum map are in this case toric varieties and (for a generic point in M) the non-compact algebraic group is \mathbf{C}^{3g-3} .

In the following example the algebraic group is a mixture of the two types of algebraic groups considered up to now, namely it is an extension of an Abelian variety with several copies of \mathbf{C}^* .

Example 3.13. The Lagrange top is by definition a rigid body with a fixed point ("spinning top") which is symmetric with respect to an axis passing through the center of gravity and the fixed point. It admits a complexification, which is the integrable system, defined by the Hamiltonian

$$H := \frac{1}{2}(\Omega_1^2 + \Omega_2^2 + (1 + m)\Omega_3^2) - \Gamma_3,$$

where $\Omega_1, \dots, \Omega_3, \Gamma_1, \dots, \Gamma_3$ are coordinates on \mathbf{C}^6 . The Poisson structure, which comes from the Lie-Poisson structure of $\mathfrak{so}(3)$, the Lie algebra of the group of Euclidean motions in three-space, is given by

$$\{I_i \Omega_i, I_j \Omega_j\} = -\epsilon_{ijk} I_k \Omega_k, \quad \{I_i \Omega_i, \Gamma_j\} = -\epsilon_{ijk} \Gamma_k, \quad \{\Gamma_i, \Gamma_j\} = 0,$$

where $1 \leq i, j, k \leq 3$ and where ϵ_{ijk} is the totally anti-symmetric tensor for which $\epsilon_{123} = 1$. The constants I_1, \dots, I_3 are the moments of inertia and they satisfy $I_1 = I_2$; the constant m is given by $m := I_3/I_1 - 1$. It follows that the Hamiltonian vector field \mathcal{X}_H is given, up to a constant, by

$$\begin{aligned} \dot{\Omega}_1 &= -m\Omega_2\Omega_3 - \Gamma_2, & \dot{\Gamma}_1 &= \Gamma_2\Omega_3 - \Gamma_3\Omega_2, \\ \dot{\Omega}_2 &= m\Omega_3\Omega_1 + \Gamma_1, & \dot{\Gamma}_2 &= \Gamma_3\Omega_1 - \Gamma_1\Omega_3, \\ \dot{\Omega}_3 &= 0, & \dot{\Gamma}_3 &= \Gamma_1\Omega_2 - \Gamma_2\Omega_1. \end{aligned}$$

One shows by direct computation that, besides the Hamiltonian, one has the following three constants of motion, making the Lagrange top integrable:

$$\begin{aligned} F_1 &:= \Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2, \\ F_2 &:= \Omega_1\Gamma_1 + \Omega_2\Gamma_2 + (1 + m)\Omega_3\Gamma_3, \\ F_3 &:= \Omega_3. \end{aligned}$$

It is shown in [11] that, in this form, the Lagrange top is a generalized a.c.i. system. The following characterization is given: the generic level of the momentum map is isomorphic to an affine part of the generalized Jacobian of an elliptic curve, which two points (at infinity) identified. Thus, the algebraic group is in this case a (non-trivial) \mathbf{C}^* -extension of an elliptic curve (which is a one-dimensional Abelian variety).

Further examples of a.c.i. systems and generalized a.c.i. systems are the periodic and non-periodic Toda lattices, the periodic and non-periodic Kac-van Moerbeke lattices, the Mumford system and some of its generalizations, the Hitchin system and its generalizations, the classical integrable tops (with the exception of the Goryachev-Chaplygin top), three families of geodesic flows on $\mathbf{SO}(4)$, and many, many others. See [4] and the references therein. For a large family of polynomial integrable systems, associated to an arbitrary algebraic curve, but which are not (generalized) a.c.i. see [17].

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